Eisenstein Cohomology and p-adic L-Functions.

Duff Campbell* and Glenn Stevens**

*Department of Mathematics, Hendrix College, Conway, AR 72032

**Department of Mathematics, Boston University, Boston, MA 02215

§0. Introduction.

In this paper, we define the module $\tilde{\mathcal{D}}(V)$ of distributions with rational poles on a finite dimensional rational vector space a V. This is an infinite dimensional vector space over $\mathbf{Q}$ endowed with a natural action of the reductive group $G_V := \mathrm{Aut}(V)$. Indeed, this action extends to a natural action of the adelic group $G_V(\mathbf{A_Q})$. For each prime p, we define we define the notion of p-adic continuity of elements of $\tilde{\mathcal{D}}(V)$ and explain how p-adically continuous distributions with rational poles give rise to p-adic L-functions.

In section 2, we construct a special element $\xi \in \tilde{\mathcal{D}}(\mathbf{Q})$, and explain its connection to the Kubota-Leopoldt p-adic L-functions. The key feature of our construction is that ξ is a *global* distribution, which is p-adically continuous for every prime p and therefore gives rise to p-adic L-functions for every p. In other words, we obtain a global object that specializes to the p-adic Kubota-Leopoldt p-adic L-functions.

In section 3, we extend the constructions of section 2 and construct a $GL(n)$-symbol ψ_n for every $n \geq 1$. When $n \geq 1$, ψ_n is a $GL_n(\mathbf{Q})$-invariant linear map from the $(n-1)$-dimensional cohomology of the Borel-Serre boundary of the symmetric space for $GL(n)$ to $\tilde{\mathcal{D}}(\mathbf{Q}^n)$. When $n = 1$, ψ_1 is determined by its value on the fundamental class, and this value is just the Kubota-Leopoldt distribution. When $n = 2$, we show that the coboundary of ψ_2 vanishes identically and therefore gives rise to a classical modular symbol over $GL_2(\mathbf{Q})$.

In section 4 (not yet included), we will explain how to associate p-adic L-functions to ξ_2 and explain how these can be viewed as p-adic L-functions attached to Eisenstein series over the weight space. We will also use ξ_2 to construct explicit overconvergent modular symbols, as defined by R. Pollack and G. Stevens. These are modular symbols taking values in the space of rigid analytic distributions on the p-adic upper half-plane.

§1. p-adic Distributions on Rational Vector Spaces.

Let V be a $\mathbf{Q}$-vector space of finite dimension n. A lattice in V is a finitely generated $\mathbf{Z}$-submodule that spans V over $\mathbf{Q}$. By an affine lattice, we mean a coset $v + L$ where $v \in V$ and L is a lattice. A subset U is said to be *uniform* if there is a lattice L such that $v + L \subseteq U$ whenever $v \in U$. In this case we also say U is *uniform* with respect to L. We endow V with the lattice topology, in which the collection of affine lattices forms a basis of open sets. Thus every uniform set is open (but not conversely). A subset $U \subseteq V$ will be called *bounded* if U is contained in some lattice.

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§1.1: Locally polynomial and locally analytic distributions.

Recall that a function $f : V \longrightarrow W$, where W is an m-dimensional $\mathbf{Q}$-vector space, is called a polynomial function if for some, hence any, choice of linear isomorphisms $V \cong \mathbf{Q}^n$, $W \cong \mathbf{Q}^m$, the induced map $F : \mathbf{Q}^n \longrightarrow \mathbf{Q}^m$ is a polynomial function with rational coefficients (i.e. each of the m coordinate functions of F is defined by a polynomial in n variables with rational coefficients). More generally, for a subset U of V, we say a function $f : U \longrightarrow W$ is polynomial if f is the restriction to U of a polynomial function on V.

A function $f : V \longrightarrow \mathbf{Q}$ is said to be *polynomial mod* L, where L is a lattice in V, if the restriction of f to each L-coset is a polynomial function. In this case we define the support of f to be the set $supp(f) \subseteq V$ defined as the union of all L-cosets on which f is not the zero polynomial. We note that this definition of $supp(f)$ does not depend on the choice of L. If U is an L-uniform subset of V, then we define

$$\mathcal{A}(U : L) := \{f : V \longrightarrow \mathbf{Q} \bigm| f \text{ is polynomial mod } L \text{ and } supp(f) \subseteq U\}.$$

A function $f : V \longrightarrow \mathbf{Q}$ is said to be locally polynomial if f is polynomial mod L for some lattice L. If U is a uniform subset of V then we define

$$\mathcal{A}(U) := \{\, f : V \longrightarrow \mathbf{Q} \bigm| f \text{ is locally polynomial and } supp(f) \subseteq U\,\}.$$

Finally, we define

$$\mathcal{A} := \{\, f : V \longrightarrow \mathbf{Q} \bigm| f \text{ is locally polynomial and } supp(f) \text{ is bounded}\,\}.$$

Dually, we have the notion of locally polynomial distributions. For an arbitrary $\mathbf{Q}$-vector space W, let $W^* := \mathrm{Hom}_{\mathbf{Q}}(W, \mathbf{Q})$ be the space of linear functionals on W. We define the "distribution spaces"

$$\mathcal{D} := \mathcal{A}^*, \qquad \mathcal{D}(U) := \mathcal{A}(U)^*, \qquad \text{and} \qquad \mathcal{D}(U : L) := A(U : L)^*$$

for any lattice L and any L-uniform set U. We will sometimes use the notation of "integration" and write

$$\int_U f(v) d\mu(v)$$

for the value of μ on the function $c_U \cdot f \in \mathcal{A}(U)$ where f is any locally polynomial function, U is a bounded uniform set, and c_U is the characteristic function of U.

§1.2. Differential Operators.

For each $x \in V$, we let D_x be the derivative in the direction x acting on each of the spaces $\mathcal{A}(U : L)$, $\mathcal{A}(U)$, and $\mathcal{A}$. By duality we also let D_x act on $\mathcal{D}(U : L)$, $\mathcal{D}(U)$, and $\mathcal{D}$. Since the collection of operators D_x, $x \in V$, all commute with one another, the linear map

$$\begin{aligned} V &\longrightarrow \mathrm{End}_{\mathbf{Q}}(\mathcal{D}) \\ x &\longmapsto D_x \end{aligned}$$

extends naturally to a $\mathbf{Q}$-algebra homomorphism

$$\begin{aligned}\mathbf{Q}[V] &\longrightarrow \mathrm{End}_{\mathbf{Q}}(\mathcal{D})\\ P &\longmapsto D_P\end{aligned}$$

where $\mathbf{Q}[V]$ is the symmetric algebra on V.

§1.3. Group actions.

We let $G := \mathrm{Aut}(V)$ be the group of *right* automorphisms of V. For $v \in V$ and $\gamma \in G$, we let $v\gamma$ denote the result of applying γ to v. By functoriality, this action induces a left action of G on $\mathcal{A}$ and a right action of G on $\mathcal{D}$. For any $f \in \mathcal{A}$, $\mu \in \mathcal{D}$, and $\gamma \in G$ we write γf, respectively $\mu|\gamma$, for the action of γ on f, respectively on μ. Thus, by definition, we have the identities

$$\int f(v) \cdot d\mu|\gamma(v) = \int \gamma f(v) \cdot d\mu(v) = \int f(v\gamma) \cdot d\mu(v).$$

We also note that G acts naturally on the right on $\mathbf{Q}[V]$ and that the action of G on $\mathcal{D}$ is semilinear with respect to the action of $\mathbf{Q}[V]$. Thus, for any $P \in \mathbf{Q}[V]$, $\mu \in \mathcal{D}$ and any $\gamma \in G$ we have the identity

$$(D_P(\mu))|\gamma = D_{P|\gamma}(\mu|\gamma).$$

§1.4. Distributions with rational poles.

Let $S \subseteq \mathbf{Q}[V]$ be the multiplicative subset of $\mathbf{Q}[V]$ generated by V and let $\mathbf{Q}[V]_S$ denote the localization of $\mathbf{Q}[V]$ with respect to S.

Definition. (Distributions with rational poles). We define

$$\tilde{\mathcal{D}} := \mathcal{D} \otimes_{\mathbf{Q}[V]} \mathbf{Q}[V]_S.$$

We will refer to the elements of $\tilde{\mathcal{D}}$ as *distributions with rational poles.*

Proposition. *The canonical map $\mathcal{D} \longrightarrow \tilde{\mathcal{D}}$ is injective.*

Proof. It suffices to show that for arbitrary $x \in V$, the differential operator D_x is injective on $\mathcal{D}$. But it is clear that the map $D_x : \mathcal{A} \longrightarrow \mathcal{A}$ is surjective. Hence the dual map $D_x : \mathcal{D} \longrightarrow \mathcal{D}$ is injective. This completes the proof.

We will regard $\tilde{\mathcal{D}}$ as a $\mathbf{Q}[V]_S$-module. From the above proposition we see that $\tilde{\mathcal{D}}$ is generated by $\mathcal{D}$ as a $\mathbf{Q}[V]_S$-module. Moreover, since S is preserved by G acting on $\mathbf{Q}[V]$, we see that the action of G on $\mathcal{D}$ extends to a $\mathbf{Q}[V]_S$-semilinear action on $\tilde{\mathcal{D}}$.

§1.5. Homogeneity.

Let $\chi : \mathbf{Q}^\times \longrightarrow \mathbf{Q}^\times$ be an arbitrary character. For each $\lambda \in \mathbf{Q}^\times$, let $z_\lambda \in G$ denote scalar multiplication by λ on V.

Definition. We say a distribution $\mu \in \tilde{\mathcal{D}}$ is homogeneous of degree χ if $\chi(\lambda) \cdot \mu | z_\lambda = \mu$ for all $\lambda \in \mathbf{Q}^\times$. We define $\tilde{\mathcal{D}}_\chi$ by

$$\tilde{\mathcal{D}}_\chi := \{\mu \in \tilde{\mathcal{D}} \bigm| \mu \text{ is homogeneous of degree } \chi\}.$$

§1.6. The Fourier transform.

Now let $\mathrm{Symm}^k(V) \subseteq \mathbf{Q}[V]$ be the space of homogeneous polynomials of degree k. The map $V \longrightarrow \mathrm{Symm}^k(V)$ defined by $v \mapsto v^k$ is a polynomial function. Thus for any bounded uniform set U we have a natural map $\mathcal{F}_{k,U} : \mathcal{D}(U) \longrightarrow \mathrm{Symm}^k(V)$ defined by

$$\mathcal{F}_{k,U}(\mu) := \int_U v^k \cdot d\mu(v).$$

Since $\prod_{k \geq 0} \mathrm{Symm}^k(V) = \mathbf{Q}[[V]]$, we define a map

$$\begin{aligned} \mathcal{F}_U : \mathcal{D} &\longrightarrow \mathbf{Q}[[V]] \\ \mu &\longmapsto \sum_{k=0}^{\infty} \frac{\mathcal{F}_{k,U}(\mu)}{k!} =: \int_U \exp(v) \cdot d\mu(v). \end{aligned}$$

Letting U vary, we obtain a map

$$\mathcal{F} : \mathcal{D} \longrightarrow D\bigl(V, \mathbf{Q}[[V]]\bigr).$$

We will call $\mathcal{F}$ the Fourier transform.

We regard the space on the right of the last display as a $\mathbf{Q}[V]$-module via the action of $\mathbf{Q}[V]$ on $\mathbf{Q}[[V]]$ given by multiplication of polynomials by power series. We also let G act on this space by functoriality. Thus, G acts $\mathbf{Q}[V]$-semilinearly. The proof of the following proposition is straightforward from these definitions.

Proposition. *The Fourier transform is an isomorphism of $\mathbf{Q}$-spaces. Moreover $\mathcal{F}$ commutes with the action of G and with the action of $\mathbf{Q}[V]$. More precisely, we have*

$$\mathcal{F}(\mu|\gamma) = \mathcal{F}(\mu)|\gamma \qquad \text{and} \qquad \mathcal{F}(D_P\mu) = P \cdot \mathcal{F}(\mu)$$

for every $\mu \in \mathcal{D}$, $\gamma \in G$, and $P \in \mathbf{Q}[V]$.

It follows from this proposition that the Fourier transform extends to an isomorphism

$$\mathcal{F} : \tilde{\mathcal{D}} \longrightarrow D(V, \mathbf{Q}[[V]]_S)$$

which is both G-equivariant and $\mathbf{Q}[V]_S$-equivariant.

§2. The Kubota-Leopoldt Distribution.

Let $V = \mathbf{Q}$ and let $T \in V$ denote the basis vector $1 \in \mathbf{Q}$. Thus we have canonically

$$\mathbf{Q}[[V]] = \mathbf{Q}[[T]].$$

Also let L be the lattice $L := \mathbf{Z}T \subseteq V$ and $\epsilon : \mathbf{Q}^\times \longrightarrow \mathbf{Q}^\times$ be the character given by

$$\epsilon(\lambda) = sign(\lambda) := \frac{\lambda}{|\lambda|}.$$

Theorem. *There is a unique distribution* $\xi \in \tilde{\mathcal{D}}_\epsilon$ *such that*

$$\mathcal{F}_L(\xi) = \frac{1}{e^T - 1} + \frac{1}{2} \in \mathbf{Q}[[T]].$$

Moreover, the Fourier transform of ξ *is given by*

$$\mathcal{F}(\xi) = \frac{1}{T} \cdot \sum_{n=0}^{\infty} \mathbf{b}_n T^n$$

where the $\mathbf{b}_n$*,* $n \geq 0$*, are the Bernoulli distributions defined in Duff's thesis.*

Proof. This is just a reformulation of well-known properties of the Bernoulli polynomials. Indeed, we will produce a locally analytic distribution $\mu \in \mathcal{D}_\chi$ with χ defined by $\chi(\lambda) = |\lambda|^{-1}$, for which

$$\mathcal{F}_L(\mu) = \frac{T}{e^T - 1} + \frac{T}{2}$$

and then define $\xi := T^{-1} \cdot \mu \in \tilde{\mathcal{D}}_\epsilon$.

We begin by defining an element $\mu_{\mathbf{Z}} \in \mathcal{D}(\mathbf{Q} : \mathbf{Z})$. Note that any $f \in \mathcal{A}(\mathbf{Q} : \mathbf{Z})$ has a Taylor expansion at each $t \in V$ of the form

$$f(z) = \sum_{n=0}^{d_f} c_{n,f}(t) \cdot (z - t)^n.$$

Moreover, each coset has a unique representative t with $0 \leq t < 1$. We may therefore define an element $\mu_{\mathbf{Z}} \in \mathcal{D}(\mathbf{Q} : \mathbf{Z})$ the integration formula

$$\int f \cdot \mu_{\mathbf{Z}} := \sum_{\substack{t \in \mathbf{Q} \\ 0 \leq t < 1}} \sum_{n=0}^{d_f} c_{n,f}(t) \cdot \mathbf{B}_n$$

where the $\mathbf{B}_n$ are defined by their exponential generating function

$$\frac{T}{e^T - 1} + \frac{T}{2} = \sum_{n=0}^{\infty} \mathbf{B}_n \cdot \frac{T^n}{n!}.$$

(Note: $\mathbf{B}_1 = 0$, and for $n \neq 1$, $\mathbf{B}_n$ is the nth Bernoulli number.)

Next, for any natural number $m \in \mathbf{N}$ we define $\mu_{m\mathbf{Z}} \in \mathcal{D}(\mathbf{Q} : m\mathbf{Z})$ by

$$\mu_{m\mathbf{Z}} := m^{-1} \cdot \mu_{\mathbf{Z}}|z_m.$$

Finally, we must note that the system of distributions $\{\mu_{m\mathbf{Z}}\}_m$ is coherent under the canonical map

$$\mathcal{D}(\mathbf{Q} : m_1\mathbf{Z}) \longrightarrow \mathcal{D}(\mathbf{Q} : m_2\mathbf{Z})$$

whenever $m_2|m_1$. For this it suffices to show that the natural map

$$\varphi_{m\mathbf{Z},\mathbf{Z}} : \mathbf{D}(\mathbf{Q} : m\mathbf{Z}) \longrightarrow \mathbf{D}(\mathbf{Q} : \mathbf{Z})$$

sends $\mu_{m\mathbf{Z}}$ to $\mu_{\mathbf{Z}}$ which amounts to proving the identity

$$\int_{t+\mathbf{Z}} x^n \cdot \mu_{m\mathbf{Z}}(x) = \int_{t+\mathbf{Z}} x^n \cdot \mu_{\mathbf{Z}}(x)$$

for all $n \geq 0$, $t \in \mathbf{Q}$ with $0 \leq t < 1$. By definition, the right hand side is equal to

$$\int_{t+\mathbf{Z}} x^n \cdot \mu_{\mathbf{Z}}(x) = \int_{t+\mathbf{Z}} ((x-t)+t)^n \cdot \mu_{\mathbf{Z}}(x) = \sum_{i=0}^{n} \binom{n}{i} t^i \mathbf{B}_{n-i} = \mathbf{B}_n(t).$$

On the other hand, the left hand side is given by

$$\begin{aligned}
\int_{t+\mathbf{Z}} x^n \cdot \mu_{m\mathbf{Z}} &= m^{-1} \cdot \int_{t+\mathbf{Z}} x^n \cdot \mu_{\mathbf{Z}}|m \\
&= m^{-1} \cdot \int_{\frac{t}{m}+\frac{1}{m}\mathbf{Z}} (mx)^n \cdot \mu_{\mathbf{Z}} \\
&= m^{n-1} \cdot \sum_{a=0}^{m-1} \int_{\frac{t+a}{m}+\mathbf{Z}} x^n \cdot \mu_{\mathbf{Z}} \\
&= m^{n-1} \cdot \sum_{a=0}^{m-1} \mathbf{B}_n\left(\frac{t+a}{m}\right) \\
&= \mathbf{B}_n(t).
\end{aligned}$$

Thus the family $\mu_{m\mathbf{Z}}$, $m \in \mathbf{N}$ is coherent as claimed.

We therefore obtain a well-defined element $\mu \in \mathcal{D}$. Moreover, from its definition we see that μ is homogeneous of degree χ and has the stated fourier transform. We define $\xi := T^{-1}\mu$ and thus obtain an element $\xi \in \tilde{\mathcal{D}}(V)$ with the desired properties. This proves existence of a distribution with the stated properties.

Uniqueness follows at once from the homogeneity condition. This completes the proof of the theorem.

§3. Eisenstein Symbols on $GL(n)$

§3.1. Symbols on $GL_n(\mathbf{Q})$.

Now let $V = \mathbf{Q}^n$ with $n \geq 1$. Let T be the standard torus in $G := GL_n(\mathbf{Q})$ and $N := N(T)$ be the normalizer of T in G. For $\gamma \in GL_n(\mathbf{Q})$ we define $sign(\gamma) := sign(\det(\gamma))$. Now let $\Gamma \subseteq GL_n(\mathbf{Q})$ be a subgroup and M be a (right) Γ-module, then a function

$$\psi : GL_n(\mathbf{Q}) \longrightarrow M$$

will be called an M-valued Γ-symbol if $\psi(\gamma x w)|\gamma = sign(w) \cdot \psi(x)$ for every $x \in GL_n(\mathbf{Q})$, $\gamma \in \Gamma$, and $w \in N$.

If $\psi : G \longrightarrow M$ is a Γ-symbol, then for each ordered basis $v_1, \ldots, v_n$ of V, we let $x \in G$ be the unique element whose columns are $v_i, \ldots, v_n$ and define

$$\psi(v_1, \ldots, v_n) := \psi(x).$$

Then the condition that ψ is a symbol is equivalent to the conditions

$$\psi(\lambda_1 v_1, \ldots, \lambda_n v_n) = sign(\lambda_1 \cdots \lambda_n) \cdot \psi(v_1, \ldots, v_n),$$
$$\psi(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = sign(\sigma) \cdot \psi(v_1, \ldots, v_n)$$

whenever $\lambda_1, \ldots, \lambda_n \in \mathbf{Q}^\times$ and $\sigma \in S_n$ is a permutation.

If ψ is a Γ-symbol and $v_0, v_1, \ldots, v_n \in V$ is any set of vectors in general position (i.e. any subset of n of these vectors spans V), then we define

$$(\partial\psi)(v_0, v_1, \ldots, v_n) := \sum_{i=0}^{n} (-1)^i \cdot \psi(v_0, \ldots, \hat{v}_i, \ldots, v_n).$$

We will say that the Γ-symbol ψ is a *modular symbol* if

$$(\partial\psi)(v_0, v_1, \ldots, v_n) = 0$$

for every sequence of vectors $v_0, v_1, \ldots, v_n$ in general position.

We remark that Ash and Rudolph have defined a universal modular G-symbol

$$[\ \ ] : G \longrightarrow H_{n-1}(X, \partial X; \mathbf{Z})$$

which maps onto the $(n-1)$-dimensional homology of the Borel-Serre completion X of the symmetric space of $GL(n)_{/\mathbf{Q}}$ relative to the Borel-Serre boundary. This homology group has a natural action by the group $G = GL_n(\mathbf{Q})$.

§3.2. The Eisenstein Symbol.

As before, we let $V = \mathbf{Q}^n$ and $G := GL_n(\mathbf{Q})$. Define $\Psi_n : G \longrightarrow \tilde{\mathcal{D}}$ by

$$\Psi_n(\gamma) := sign(\gamma) \cdot (\xi \otimes \xi \otimes \ldots \otimes \xi)|\gamma^{-1} \in \tilde{\mathcal{D}}(V).$$

It follows immediately from the properties of ξ that Ψ_n is a G-symbol. It is natural to wonder if Ψ_n is a modular symbol. For $n = 1$, every symbol is a modular symbol, so there is nothing more to prove in this case. In fact, it can be seen that for $n = 2, 3$, Ψ_n is not a modular symbol.

Question. Can we give a simple closed formula for $\partial\Psi_n$ for any n?

We will answer this question in the affirmative for $n = 2$.

§3.3. The boundary of the Eisenstein symbol on $GL(2)$.

Let $V = \mathbf{Q}^2$, $G = GL_2(\mathbf{Q})$ and let $X := (0,-1) \in V$ and $Y := (1,0) \in V$ so that a typical vector $v = (x,y) \in V$ is given by $v = xY - yX$. Then $\mathbf{Q}[[V]] = \mathbf{Q}[[X,Y]]$ and the action of an element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ on a polynomial $P = P(X,Y) \in \mathbf{Q}[[V]]$ is given by

$$(P|\gamma)(X,Y) = P((X,Y)\gamma^*)$$

where $\gamma^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is the adjugate of γ. Now define

$$\Phi := \mathcal{F} \circ \Psi_2 : G \longrightarrow D(V, \mathbf{Q}[[X,Y]]).$$

Since $\mathcal{F}$ is an isomorphism, to give a formula for $\partial\Psi_2$ is equivalent to giving a formula for $\partial\Phi$. For any $v_0, v_1, v_2 \in V$ in general position, we define

$$\epsilon(v_0,v_1,v_2) := sign\big(\det(v_0,v_1)\cdot\det(v_0,v_2)\cdot\det(v_1,v_2)\big).$$

Theorem. *Let $V := \mathbf{Q}^2$ and $\Phi := \mathcal{F} \circ \Psi_2$. Then*

$$\Phi(I) = \frac{1}{XY} \cdot \sum_{r,s=0}^{\infty} (-1)^{s+1} \cdot \mathbf{b}_r \times \mathbf{b}_s \cdot \frac{X^s}{s!}\frac{Y^r}{r!}$$

and for any triple $v_0, v_1, v_2 \in V$ in general position, we have

$$\partial\Phi(v_0,v_1,v_2) = \epsilon(v_0,v_1,v_2) \cdot \frac{1}{2} \cdot \delta_0.$$

Proof. To prove the first identity it suffices to calculate the value $\Phi_U(I)$ of $\Phi(I)$ on an arbitrary factorizable uniform bounded set $U = U_1 \times U_2 \subseteq V$. For such U we have

$$\begin{aligned}
\Phi_U(I) &= -\frac{1}{XY} \cdot \int_U \exp(v) \cdot d(\xi \times \xi)(v) \\
&= -\frac{1}{XY} \cdot \sum_{k=0}^{\infty} \int_U \frac{v^k}{k!} \cdot d(\xi \times \xi)(v) \\
&= -\frac{1}{XY} \cdot \sum_{k=0}^{\infty} \int_{U_2}\int_{U_1} \frac{(xY - yX)^k}{k!} \cdot d\xi(x) \cdot d\xi(y) \\
&= -\frac{1}{XY} \cdot \sum_{r,s=0}^{\infty} \int_{U_2}\int_{U_1} (-1)^s x^r y^s \cdot \frac{Y^r}{r!}\frac{X^s}{s!} \cdot d\xi(x) \cdot d\xi(y) \\
&= -\frac{1}{XY} \cdot \sum_{r,s=0}^{\infty} (-1)^s \left(\int_{U_1} x^r \cdot d\xi(x)\right) \cdot \left(\int_{U_2} y^s \cdot d\xi(y)\right) \cdot \frac{X^s}{s!}\frac{Y^r}{r!} \\
&= -\frac{1}{XY} \cdot \sum_{r,s=0}^{\infty} (-1)^s \mathbf{b}_r(U_1) \cdot \mathbf{b}_s(U_2) \cdot \frac{X^s}{s!}\frac{Y^r}{r!} \\
&= -\frac{1}{XY} \cdot \sum_{r,s=0}^{\infty} (-1)^s (\mathbf{b}_r \times \mathbf{b}_s)(U) \cdot \frac{X^s}{s!}\frac{Y^r}{r!}.
\end{aligned}$$

This proves the first identity of the theorem.

§4. p-Adic L-functions and overconvergent modular symbols

In this section, we will describe Φ in terms of periods of Eisenstein series. The proof of the formula for $\partial\Phi$ will then follow from an application of Stokes' theorem. We will explain how the p-adic L-functions of Φ interpolate the critical L-values associated to Eisenstein series.

§5. p-Adic Locally Analytic Distributions

§5.1. p-Adic Continuity and the Kubota-Leopoldt Distribution.

Fix a positive rational prime p and let $\mathbf{C}_p$ be a completion of an algebraic closure of $\mathbf{Q}_p$. Let $\mathbf{O}_p$ be the ring of integers in $\mathbf{C}_p$. As always, V is finite dimensional $\mathbf{Q}$-vector space. We embed V in $V_{\mathbf{C}_p} := V \otimes_{\mathbf{Q}} \mathbf{C}_p$ in the obvious way. For each lattice L in V and each $v \in V$, we let

$$B_L := L \otimes \mathbf{O}_p \qquad \text{and} \qquad B_L(v) := v + B_L \subseteq V_{\mathbf{C}_p}.$$

Then $B_L(v)$ is a $\mathbf{Q}_p$-affinoid polydisk containing $v + L$. We let $A_p(v+L) := A_p\big(B_L(v)\big)$ be the $\mathbf{Q}_p$-affinoid algebra of $B_L(v)$. A function $f : V \longrightarrow \mathbf{Q}_p$ is said to be p-adic analytic on the affine lattice $v + L$ if there is an $F \in A_L(v)$ whose restriction to $v + L$ agrees with f on $v + L$. More generally, we say the function $f : V \longrightarrow \mathbf{Q}_p$ is p-adic analytic modulo L if it is p-adic analytic on every L-coset. If U is an L-uniform subset of V we define

$$\mathcal{A}_p(U:L) = \big\{ f : V \longrightarrow \mathbf{Q}_p \;\big|\; f \text{ is } p\text{-adic analytic modulo } L \text{ and } \mathit{supp}(f) \subseteq U \big\}.$$

If U is bounded and L-uniform then the canonical map $\mathcal{A}_p(U:L) \longrightarrow \bigoplus_{v \in U/L} A_p(v+L)$ is an isomorphism. We endow $\mathcal{A}_p(U:L)$ with the norm $\|\cdot\|_{p,L}$ induced from the affinoid norms on the factors.

If $M \subseteq L$ then any function that is p-adic analytic modulo L is also p-adic analytic modulo M. Thus we have a natural continuous inclusion $\mathcal{A}_p(U:L) \hookrightarrow \mathcal{A}_p(U:M)$. We define

$$\mathcal{A}_p(U) := \varinjlim_{L} \mathcal{A}_p(U:L) \qquad \text{and} \qquad \mathcal{A}_p := \mathcal{A}_p(V) := \varinjlim_{U} \mathcal{A}_p(U)$$

each endowed with the inductive limit topology. We refer to the elements of $\mathcal{A}_p(U)$ and $\mathcal{A}_p$ as *p-adic locally analytic functions.*

Definition. (Locally Analytic Distributions). Let L be a lattice and U be a bounded L-uniform subset of V. We define the spaces $\mathcal{D}_p(U:L)$, $\mathcal{D}_p(U)$, and $\mathcal{D}_p := \mathcal{D}_p(V)$ to be the spaces of continuous linear functionals on $\mathcal{A}_p(U:L)$, $\mathcal{A}_p(U)$, and $\mathcal{A}_p$ respectively. We endow each of these spaces with the strong topology. The elements of $\mathcal{D}_p(U:L)$, $\mathcal{D}_p(U)$, and $\mathcal{D}_p$ will be called *p-adic locally analytic distributions.*

For each lattice L and every L-uniform set U we have natural inclusions

$$\mathcal{A}(U:L) \hookrightarrow \mathcal{A}_p(U:L), \qquad \mathcal{A}(U) \hookrightarrow \mathcal{A}_p(U), \qquad \text{and} \qquad \mathcal{A} \hookrightarrow \mathcal{A}_p.$$

Moreover, the images of these inclusions are dense. The topologies on $\mathcal{A}(U:L)$, $\mathcal{A}(U)$, and $\mathcal{A}$ induced by these inclusions will be called the p-adic topologies. Thus, in particular, the p-adic topology on $\mathcal{A}(U:L)$ is the topology induced by the norm $\|\cdot\|_{p,L}$.

Definition. A distribution $\mu \in \mathcal{D}$ is said to be p-adically continuous if the map $\mu : \mathcal{A} \longrightarrow \mathbf{Q}$ is continuous with respect to the p-adic topologies on $\mathcal{A}$ and $\mathbf{Q}$.

By duality, we have canonical maps

$$\mathcal{D}_p(U:L) \hookrightarrow \mathcal{D}(U:L) \otimes \mathbf{Q}_p, \qquad \mathcal{D}_p(U) \hookrightarrow \mathcal{D}(U) \otimes \mathbf{Q}_p, \qquad \text{and} \qquad \mathcal{D}_p \hookrightarrow \mathcal{D} \otimes \mathbf{Q}_p.$$

These maps are injective by the density of locally polynomial functions in the space of p-adic locally analytic functions. We note that a global distribution $\mu \in \mathcal{D}$ is in the image of p-adic locally analytic distributions if and only if μ is p-adically continuous.

The space $\mathcal{D}_p$ of p-adic locally analytic distributions inherits all of the structures we endowed on $\mathcal{D}$. Namely, the group $G = \mathrm{Aut}(V)$ acts the right on $\mathcal{D}_p$. Indeed, this action extends to a continuous action of the group $G_p := \mathrm{Aut}(V_p)$ of continuous $\mathbf{Q}_p$-linear automorphisms of $V_p := V \otimes \mathbf{Q}_p$. Also the action of the ring of differential operators $\mathbf{Q}[V]$ on $\mathcal{D}$ extends to an action of $\mathbf{Q}_p[V]$. Moreover, the elements of S act injectively. Thus we may localize $\mathcal{D}_p$ with respect to S and define the space of locally analytic distribtuions with rational poles to be the space

$$\tilde{\mathcal{D}}_p := \mathcal{D}_p \otimes_{\mathbf{Q}_p[V]} \mathbf{Q}_p[V]_S.$$

We have a natural inclusion $\tilde{\mathcal{D}}_p \subseteq \tilde{\mathcal{D}} \otimes \mathbf{Q}_p$. We will say that an element $\mu \in \tilde{\mathcal{D}}$ is p-adically continuous if the element $\mu \otimes 1 \in \mathcal{D} \otimes \mathbf{Q}_p$ lies in $\tilde{\mathcal{D}}_p$. We note that for any element $\mu \in \mathcal{D}$, μ is p-adically continuous as an element of $\mathcal{D}$ if and only if μ is p-adically continuous as an element of $\tilde{\mathcal{D}}$. (This requires proof. It's not totally obvious).

Theorem. *The Kubota-Leopoldt distribution $\xi \in \tilde{\mathcal{D}}(\mathbf{Q})$ is p-adically continuous for every prime p.*

Proof. We use the notation of the previous section on Kubota-Leopoldt. Namely we write T for $1 \in \mathbf{Q}$. Hence $\mu := D_T \xi$ is in $\mathcal{D}$. Now fix a prime p. It suffices to show μ is p-adically continuous. For this we need to show

$$\mu : \mathcal{A}(t + m\mathbf{Z}) \longrightarrow \mathbf{Q}$$

is p-adically continuous for every affine lattice $t + m\mathbf{Z}$ with m a positive integer.

So we fix a positive integer m and, without loss of generality, we may then suppose $t \in \mathbf{Q}$ satisfies the inequality $0 \leq t < m$. Then the sequence of polynomials $\left(\frac{z-t}{m}\right)^k$, $k = 0, 1, 2, \ldots$, is an orthonormal basis for $\mathcal{A}_p(t + m\mathbf{Z})$. We have the identity

$$\int_{t+m\mathbf{Z}} \left(\frac{z-t}{m}\right)^k \cdot d\mu(z) = \frac{1}{m}\mathbf{B}_k$$

for all $k \geq 0$. Hence

$$\|\mu\|_{p,m\mathbf{Z}} = \sup_k |\mathbf{B}_k/m|_p = |mp|_p^{-1}$$

by the Clausen von Staudt theorem. Thus μ is p-adically continuous and the theorem is proved.

The following corollary is an immediate consequence of the theorem.

Corollary. *For every $n \geq 1$, the distribution $\xi \otimes \xi \otimes \cdots \otimes \xi \in \tilde{\mathcal{D}}(\mathbf{Q}^n)$ is p-adically continuous.*

§5.2. The Adelic Point of View.

Let $\hat{V}$ be the completion of V with respect to the lattice topology and for each prime q, let $V_q := V \otimes \mathbf{Q}_q$ be the q-adic completion of V. For an arbitrary lattice $L \subseteq V$, we let $L_q \subseteq V_q$ be the closure of L in V_q. Then the restricted product $\prod_q' V_q$ with respect to the sublattices $L_q \subseteq V_q$ is well-defined independent of L and we have a canonical isomorphism

$$\hat{V} \xrightarrow{\sim} \prod_q{}' \, V_q.$$

We let $\hat{L} \subseteq \hat{V}$ be the closure of L in $\hat{V}$ and note that the canonical map

$$\hat{L} \xrightarrow{\sim} \prod_q L_q$$

is an isomorphism. A subset $\hat{U} \subseteq \hat{V}$ is said to be $\hat{L}$-uniform if $\hat{U} + \hat{L} = \hat{L}$. If $\hat{U}$ is compact and $\hat{L}$-uniform, then $\hat{U}$ is equal to a finite disjoint union of $\hat{L}$-cosets in $\hat{V}$. In particular, such a set is both compact and open. If we let $U = \hat{U} \cap V$ then it is not hard to see that U is a bounded L-uniform subset of V and moreover that $\hat{U}$ is the closure of U in $\hat{V}$ (thus justifying our notation $\hat{U}$).

We will say that a function $f : \hat{V} \longrightarrow \mathbf{Q}_p$ is p-adic "analytic" modulo $\hat{L}$ if for each coset $v + \hat{L} \subseteq \hat{V}$ $(v \in V)$ there is a p-adic rigid analytic function $F \in A_p(v + L)$ such that for every $\hat{\ell} = (\ell_q)_q \in \hat{L}$ we have $f(v + \hat{\ell}) = F(v + \ell_p)$. For a compact L-uniform subset $\hat{U} \subseteq \hat{V}$ we then define

$$\mathcal{A}_p(\hat{U} : \hat{L}) := \{ f : \hat{V} \longrightarrow \mathbf{Q}_p \,|\, f \text{ is } p\text{-adic rigid analytic mod } \hat{L} \text{ and } \mathit{supp}(f) \subseteq \hat{U} \, \}.$$

The following proposition is an immediate consequence of the definitions.

Proposition. *For every L-uniform bounded subset $U \subseteq V$ the canonical restriction map*

$$\mathcal{A}_p(\hat{U} : \hat{L}) \longrightarrow \mathcal{A}_p(U : L)$$

is an isomorphism.

Thus p-adic locally analytic functions on V extend (uniquely) to p-adic local analytic functions on $\hat{V}$.

We will say that a bounded L-uniform set $U \subseteq V$ is "factorizable" if $\hat{U} = \prod_q U_q$ where each U_q is a (necessarily compact and open) subset of V_q. By a simple compactness argument, it then follows that $U_q = L_q$ for almost all q. For each q we let c_{L_q} be the characteristic function of L_q and let $\bigotimes'_q \mathcal{A}_p(U_q, L_q)$ be the restricted tensor product with respect to the functions c_{L_q}. The canonical map

$$\bigotimes_q{}' \mathcal{A}_p(U_q, L_q) \longrightarrow \mathcal{A}_p(\hat{U} : \hat{L})$$

$$\otimes_q f_q \longmapsto \left((v_q)_q \mapsto \prod_q f_q(v_q) \right)$$

is an isomorphism, as one easily confirms. Composing this map with the isomorphism of the last proposition, we obtain an isomorphism

$$\bigotimes_q{}' \mathcal{A}_p(U_q, L_q) \cong \mathcal{A}_p(U : L).$$

A p-adic locally analytic function $f \in \mathcal{A}_p(U : L)$ will be called factorizable if it corresponds under this isomorphism to a factorizable element $\otimes_q f_q \in \bigotimes'_q \mathcal{A}_p(U_q : L_q)$ and in this case $\otimes_q f_q$ will be called a factorization of f.

§5.3. p-Adic Locally Analytic Dirichlet characters.

For a prime q, a p-adic locally analytic function $\chi_q \in \mathcal{A}_p(\mathbf{Z}_q)$ will be called a Dirichlet character on $\mathbf{Z}_q$ if χ_q satisfies the following two properties:

(1) χ_q is multiplicative on $\mathbf{Z}_q$, i.e. for all $a, b \in \mathbf{Z}_q$ we have $\chi_q(ab) = \chi_q(a) \cdot \chi_q(b)$;

and

(2) the restriction of χ_q to $\mathbf{Z}_q^\times$ is a multiplicative character $\chi_q : \mathbf{Z}_q^\times \longrightarrow K^\times$.

If, moreover, χ_q is rigid modulo q^n, then q^n will be called a modulus for χ_q. The level $M(\chi_q)$ of χ_q is defined to be the smallest modulus of χ_q.

We note that a Dirichlet character χ_q has level one if and only if either $q \neq p$ and $\chi_q = c_{\mathbf{Z}_q}$, the characteristic function of $\mathbf{Z}_q$, or $q = p$ and there is a non-negative integer $k \geq 0$ such that χ_p is given by

$$\chi_p(t) := \begin{cases} t^k & \text{if } t \in \mathbf{Z}_p; \\ \\ 0 & \text{if } t \notin \mathbf{Z}_p. \end{cases}$$

Definition. A p-adic Dirichlet character is a factorizable function $\chi \in \mathcal{A}_p(\mathbf{Q})$ admitting a factorization of the form $\otimes_q \chi_q$ where each χ_q is a Dirichlet character on $\mathbf{Z}_q$. Then $M(\chi_q) = 1$ for almost all q, so we may (and do) define the level of χ to be the product $M(\chi) := \prod_q M(\chi_q)$ of the levels of the χ_q.

We note that

$$U_m := \{\, a \in \mathbf{Z} \,|\, (a, m) = 1 \,\}.$$

A p-adic locally analytic function

$$\chi : V \longrightarrow K$$

will be called a (p-adic locally analytic) Dirichlet character if χ satisfies the following two properties:
(1) there is a natural number m such that $\chi \in \mathcal{A}_p(U_m : m\mathbf{Z})$; and
(2) χ is multiplicative (i.e. $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in \hat{\mathbf{Z}}$).

If χ is a Dirichlet character, then any natural number m for which $\chi \in \mathcal{A}_p(U_m : m\mathbf{Z})$ will be called a modulus for χ. We define the "level" M_χ of χ to be the smallest modulus of χ. It's not hard to see that a natural number m is a modulus of χ if and only if m is divisible by M_χ and has the same prime divisors. We define

$$\mathcal{X}_p[m] := \{\, \chi : V \longrightarrow \mathbf{C}_p \,|\, \chi \text{ is a Dirichlet character and } m \text{ is a modulus of } \chi \,\}.$$

If $d|m$ then $U_m \subseteq U_d$ and restriction from U_d to U_m induces a homomorphism

$$\mathcal{X}_p[d] \longrightarrow \mathcal{X}_p[m].$$

We say a pair of Dirichlet characters $\chi_i \in \mathcal{X}_p[m_i]$, $i = 1, 2$, are associated if they have the same image in $\mathcal{X}_p[m_1 m_2]$. As in the classical theory of Dirichlet characters it's not hard to see that the relation of being associated is an equivalence relation on the set of Dirichlet characters. Moreover, the associate class of any Dirichlet character χ contains exactly one character whose level divides the level of every other character in the associate class. This minimal level will be called the *conductor* of χ and denoted m_χ. If $m_\chi = M_\chi$ we will say χ is primitive of conductor m_χ.

§5.4. p-adic L-functions.

Theorem. *Let $\xi \in \tilde{\mathcal{D}}(\mathbf{Q})$ be the Kubota-Leopoldt distribution. Then for every prime p and every primitive arithmetic Dirichlet character χ we have*

$$L_p(\xi, \chi) = L_\infty(\chi).$$

If χ' is a character of level M associated to χ then

$$L_p(\xi, \chi') = \left(\prod_{\ell} (1 - \chi(\ell)) \right) \cdot L_\infty(\chi)$$

where the product is over all primes ℓ that divide M but not m_χ.