Computations with overconvergent modular symbols

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2.1 Basic definitions

Let $\Delta_0 := \mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q}))$ be the set of degree zero divisors on $\mathbf{P}^1(\mathbf{Q})$. Then Δ_0 (the Steinberg module) has the structure of a left $\mathbf{Z}[\mathrm{GL}_2(\mathbf{Q})]$-module with $\mathrm{GL}_2(\mathbf{Q})$ acting via standard linear fractional transformations.

Let Γ be a finite index subgroup of $\mathrm{PSL}_2(\mathbf{Z})$ and let V be a right $\mathbf{Z}[\Gamma]$-module. We endow the set of additive homomorphisms $\mathrm{Hom}(\Delta_0, V)$ with the structure of a right Γ-module by defining

$$(\varphi\big|\gamma)(D) := \varphi(\gamma D)\big|\gamma$$

for $\varphi : \Delta_0 \to V$, $D \in \Delta_0$ and $\gamma \in \Gamma$. We say that φ is a *V-valued modular symbol on* Γ if $\varphi\big|\gamma = \varphi$ for all $\gamma \in \Gamma$ and denote the space of all V-valued modular symbols by $\mathrm{Symb}_\Gamma(V)$. Thus for an additive homomorphism $\varphi : \Delta_0 \to V$,

$$\varphi \in \mathrm{Symb}_\Gamma(V) \iff \varphi\big|\gamma = \varphi \text{ for all } \gamma \in \Gamma.$$

In the main examples of this paper, we will take Γ to be the image of $\Gamma_0(Np)$ in $\mathrm{PSL}_2(\mathbf{Z})$ where p is a prime not dividing N. The modules V we consider will have the addition structure of a right action by $S_0(p)$ where

$$S_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}) \text{ such that } (a,p) = 1, p \mid c \text{ and } ad - bc \neq 0 \right\}.$$

One can then define Hecke operators T_ℓ for $\ell \nmid Np$ and U_q for $q|Np$ on $\mathrm{Symb}_\Gamma(V)$. For example,

$$\varphi|T_\ell = \varphi|\left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right) + \sum_{a=0}^{l-1} \varphi|\left(\begin{smallmatrix} 1 & a \\ 0 & l \end{smallmatrix}\right) \quad \text{and} \quad \varphi|U_p = \sum_{a=0}^{p-1} \varphi|\left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right).$$

2.2 The Manin relations

In this section, we review the Manin relations which give a description of Δ_0 as a $\mathbf{Z}[\Gamma]$-module in terms of generators and relations. In subsequent sections, we will "solve the Manin relations" to a give a presentation of Δ_0 that involves only one relation (in the case that Γ is torsion-free).

If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbf{Q})$, let us denote by $[g]$ the singular 1-chain in the extended upper half-plane $\mathcal{H}^*$ represented by the geodesic path joining $\frac{a}{c}$ to $\frac{b}{d}$. We will call any such 1-chain a *modular path* and any finite formal sum of such modular paths, a *modular* 1*-chain*. The $\mathbf{Z}$-module of all such modular chains will be denoted by

$$Z_1 = Z_1(\mathcal{H}^*, \mathbf{P}^1(\mathbf{Q})),$$

which we regard as a module of 1-cycles in $\mathcal{H}^*$ relative to the boundary $\mathbf{P}^1(\mathbf{Q})$ of $\mathcal{H}^*$.

The group $PGL_2^+(\mathbf{Q})$ acts on Z_1 via standard fractional linear transformation on $\mathcal{H}^*$; hence Z_1 is naturally a $PGL_2^+(\mathbf{Q})$-module. If $\beta, \gamma \in GL_2^+(\mathbf{Q})$ then we have

$$\beta \cdot [\gamma] = [\beta\gamma].$$

The boundary map gives us a surjective $PGL_2^+(\mathbf{Q})$-morphism

$$\partial : Z_1 \to \Delta_0.$$

We say two modular chains c, c' are homologous if $\partial c = \partial c'$. Thus ∂ induces a $PGL_2^+(\mathbf{Q})$-isomorphism from the one-dimensional relative homology of the pair $(\mathcal{H}^*, \mathbf{P}^1(\mathbf{Q}))$ to the Steinberg module Δ_0:

$$\partial : H^1(\mathcal{H}^*, \mathbf{P}^1(\mathbf{Q}); \mathbf{Z}) \xrightarrow{\cong} \Delta_0.$$

Let $G = PSL_2(\mathbf{Z})$. A modular path of the form $[\gamma]$ with $\gamma \in G$ is called a *unimodular path* and any finite formal sum of such unimodular paths is called a *unimodular 1-chain*. Using continued fractions it is easy to see (and is a well-known result of Manin [3]) that every modular chain is homologous to a unimodular chain. Moreover, G acts transitively on the unimodular paths. Indeed, the map

$$\begin{aligned} G &\to Z_1 \\ \gamma &\mapsto [\gamma] \end{aligned}$$

is a bijection from G to the set of unimodular paths in Z_1. Extending by linearity, we obtain a G-morphism $\mathbf{Z}[G] \to Z_1$, and composing with the boundary map ∂ we obtain a surjective G-morphism

$$e : \mathbf{Z}[G] \to \Delta_0.$$

Let $\mathcal{U}_1$ be the image of $\mathbf{Z}[G]$ in Z_1 under $[\cdot]$; that is, $\mathcal{U}_1$ is the collection of all unimodular 1-chains. The map $[\cdot]$ is equivariant with respect to the left action of $\mathbf{Z}[G]$. Furthermore, by transport of structure, it induces a right $\mathbf{Z}[G]$-module structure on $\mathcal{U}_1$ (from the right action of $\mathbf{Z}[G]$ on itself). In the following lemma, we describe this right action on $\mathcal{U}_1$ by certain torsion elements of G.

Lemma 2.1.

1. *If σ is the two-torsion element $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and u is the unimodular path connecting r to s then $u\sigma$ is the unimodular path connecting s to r.*

2. *If τ is the three-torsion element $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ and u is an oriented unimodular path then u, $u\tau$ and $u\tau^2$ form three sides of an ideal triangle oriented counter-clockwise.*

Proof. This is a straightforward computation. □

The above lemma identifies some obvious elements in the kernel of e; namely, since the images of $g+g\sigma$ and $g+g\tau+g\tau^2$ in Z_1 form loops, e kills these elements. It is a result of Manin [3] that these relations actually generate the kernel of e; that is, $\ker(e)$ is the left ideal

$$I = \mathbf{Z}[G](1+\tau+\tau^2) + \mathbf{Z}[G](1+\sigma).$$

Thus, e induces an isomorphism $\Delta_0 \cong \mathbf{Z}[G]/I$.

Using this isomorphism, we can now give a description of Δ_0 as a $\mathbf{Z}[\Gamma]$-module in terms of generators and relations. First note that $\mathbf{Z}[G]$ is a free $\mathbf{Z}[\Gamma]$-module; a basis is given by a complete set of right coset representatives for $\Gamma\backslash G$, say $g_1, \cdots, g_r$. The ideal I then gives $\mathbf{Z}[\Gamma]$-relations between these generators. For instance,

$$g_i(1+\sigma) = g_i + g_i\sigma = g_i + \gamma_{ij} g_j \in I$$

for some j and some $\gamma_{ij} \in \Gamma$. The triangle relations similarly yield three term $\mathbf{Z}[\Gamma]$-relations on the g_i.

2.3 Fundamental domains

Let Γ be a finite index subgroup of $G := \mathrm{PSL}_2(\mathbf{Z})$ without any three-torsion. We will describe an explicit method for producing a fundamental domain $\mathcal{F}$ for the action of Γ on $\mathcal{H}^*$ which has a particularly nice shape. Namely, all of the vertices of $\mathcal{F}$ will be cusps and its boundary will be a union of unimodular paths. This precise description of $\mathcal{F}$ will enable us to prove that the Manin relations are essentially equivalent to the single relation $\partial\mathcal{F} = 0$ when Γ is torsion free (see section 2.4).

Definition 2.2. For $\Gamma \subseteq G$, a *fundamental domain* for the action of Γ on $\mathcal{H}^*$ is a region $\mathcal{F} \subseteq \mathcal{H}^*$ such that

1. $\mathcal{F}$ is closed and $\mathcal{F}^\circ$ (the interior of $\mathcal{F}$) is connected,
2. for each $\gamma \in \Gamma$, $\gamma\mathcal{F}^\circ \cap \mathcal{F}^\circ = \emptyset$
3. $\bigcup_{\alpha\in\Gamma} \alpha\mathcal{F} = \mathcal{H}^*$

If the region $\mathcal{F}$ only satisfies the first two conditions then we call this region a *potential fundamental domain* for Γ.

Consider the ideal triangle T whose vertices are 0, ρ and ∞ where $\rho = (1+\sqrt{-3})/2$. It is standard that this is a fundamental domain for G. The matrix $\tau = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ is an order three matrix that fixes ρ, sends 0 to 1 and 1 to ∞. Thus, $R = T \cup \tau T \cup \tau^2 T$ is an ideal triangle with vertices 0, 1 and ∞. The translates of R under G tile $\mathcal{H}^*$ (since R contains a fundamental domain for G). We will build a fundamental domain for Γ out of translates of R.

Lemma 2.3. *Let Γ be a subgroup of G without any three-torsion, let R be the ideal triangle with vertices 0, 1 and ∞ and let $\mathcal{F} = \alpha_1 R \cup \cdots \cup \alpha_r R \subseteq \mathcal{H}^*$ be a potential fundamental domain for $\Gamma \subseteq G$. Then for $\alpha \in G$, the following are equivalent:*

1. *$\mathcal{F} \cup \alpha R$ is not a fundamental domain for Γ.*
2. *Every point in αR is Γ-equivalent to some point in $\mathcal{F}$.*
3. *For some i and r, $\alpha_i \tau^r \alpha^{-1} \in \Gamma$.*

Proof. The proof of this lemma is straightforward. One needs to assume that Γ has no three-torsion because the element $\alpha\tau\alpha^{-1}$ has order three and fixes a point in the interior of αR (namely $\alpha\rho$). □

Theorem 2.4. *If Γ is a finite index subgroup of $\mathrm{PSL}_2(\mathbf{Z})$ without any three-torsion, then there exists a fundamental domain $\mathcal{F}$ for the action of Γ on $\mathcal{H}^*$ of the form*

$$\mathcal{F} = \alpha_1 R \cup \cdots \cup \alpha_n R.$$

In particular, there exists a fundamental domain all of whose vertices are cusps and whose boundary is a union of unimodular paths.

Proof. Note that since Γ has no three-torsion, R is a potential fundamental domain for Γ. Furthermore, since Γ has finite index in G, an infinite union of translates of R cannot be a potential fundamental domain. In particular, there exists some *maximal* potential domain for Γ of the form $\mathcal{F} := \alpha_1 R \cup \cdots \cup \alpha_n R$.

Consider now any triangle αR that abuts the region $\mathcal{F}$. Since $\mathcal{F}$ is maximal, $\mathcal{F} \cup \alpha R$ is not a potential fundamental domain. Thus, by part (2) of Lemma 2.3, every point of αR is Γ-equivalent to some point in $\mathcal{F}$. In particular, there is some open neighborhood $U \subseteq \mathcal{H}$ of $\mathcal{F} \cap \mathcal{H}$ such that each point of U is Γ-equivalent to some point in $\mathcal{F}$.

Let $Z = \bigcup_{\gamma \in \Gamma} \gamma\mathcal{F} \cap \mathcal{H}$. Then, Z is closed in $\mathcal{H}$ since it is the union of closed sets. But, we also have that $Z = \bigcup_{\gamma \in \Gamma} \gamma U$ and, thus, Z is open. Since Z is both open and closed, we can deduce that $Z = \mathcal{H}$. Finally, since $\bigcup_{\gamma \in \Gamma} \gamma\mathcal{F}$ is closed in $\mathcal{H}^*$ and contains $\mathcal{H}$, it must be all of $\mathcal{H}^*$. Thus, $\mathcal{F}$ is a fundamental domain for Γ. □

Remark 2.5. The proof of Theorem 2.4 shows that any maximal potential fundamental domain for Γ of the form $\alpha_1 R \cup \cdots \cup \alpha_n R$ is actually a fundamental domain. Using Lemma 2.3, one can explicitly form such a fundamental domain.

We illustrate this for $\Gamma := \mathrm{im}(\Gamma_0(5) \to \mathrm{PSL}_2(\mathbf{Z}))$ (which has no three-torsion since 5 is a prime congruent to 2 mod 3.)

Starting with R, note that the two triangles abutting R to the left and right are Γ-equivalent to R since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0(5)$. The triangle directly below R equals αR with $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. One verifies that $\tau^r \alpha^{-1} \notin \Gamma_0(5)$ for $r = 0, 1, 2$ and, thus, by Lemma 2.3, $R \cup \alpha R$ is a potential fundamental domain.

The two triangles directly below $R \cup \alpha R$ are given by βR and δR where $\beta = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ and $\delta = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. One checks that $\alpha\tau\beta^{-1}$ and $\alpha\tau^2\gamma^{-1}$ are both in $\Gamma_0(5)$. Thus, $R \cup \alpha R$ is a maximal potential fundamental domain and thus a fundamental domain for Γ.

2.4 Solving the Manin relations

In this section, we will use the explicit description of a fundamental domain for the action of Γ on $\mathcal{H}^*$ given in Theorem 2.4 to completely solve the Manin relations for Γ a finite index subgroup of $\mathrm{PSL}_2(\mathbf{Z})$ without any three torsion.

Let $\mathcal{F} = \alpha_1 R \cup \cdots \cup \alpha_n R$ be a fundamental domain for Γ. Assume (for convenience) that Γ contains the element $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and fix $\mathcal{F}$ to be a fundamental domain with its left (resp. right) boundary to be the path connecting ∞ to -1 (resp. to 0).

Let $\mathcal{E}$ be the set composed of all oriented unimodular paths passing through the interior of $\mathcal{F}$ together with all the oriented unimodular paths contained in $\partial\mathcal{F}$ where $\partial\mathcal{F}$ is given the counter-clockwise orientation. Thus, each interior path is counted twice (once for each orientation) and each boundary edge is only counted once.

Lemma 2.6. *Any oriented unimodular path is* Γ*-equivalent to a unique element of* $\mathcal{E}$*.*

Proof. To prove this lemma, one simply uses the fact that $\mathcal{F}$ is a fundamental domain for the action of Γ. □

From section 2.2, we have an exact sequence

$$0 \to I \to \mathbf{Z}[G] \to \Delta_0 \to 0.$$

of left $\mathbf{Z}[\Gamma]$-modules. If we set J to be the image of I in Z_1 under $[\cdot]$, we then have

$$0 \to J \to \mathcal{U}_1 \to \Delta_0 \to 0.$$

(Recall that $\mathcal{U}_1$ is the collection of all unimodular 1-chains.) Thus, Δ_0 is given by $\mathcal{U}_1$ modulo the relations that the three sides of a triangle sum to zero and that reversing the orientation of a path introduces a minus sign. We now give a simple presentation of Δ_0 as a left $\mathbf{Z}[\Gamma]$-module taking advantage of its more geometric description as $\mathcal{U}_1/J$.

By Lemma 2.6, we have that $\mathcal{U}_1$ is freely generated as a $\mathbf{Z}[\Gamma]$-module by the set of $e \in \mathcal{E}$. Denote by $\partial\mathcal{E}$ the set of all paths in $\mathcal{E}$ that are contained in $\partial\mathcal{F}$. We first claim that the elements of $\partial\mathcal{E}$ generate $\mathcal{U}_1/J$ over $\mathbf{Z}[\Gamma]$. Indeed, if

$v \in \mathcal{E} - \partial\mathcal{E}$ is oriented from right to left (resp. left to right), then v is equivalent mod J to the sum of the paths in $\partial\mathcal{E}$ sitting below (resp. above) v since these paths taken together with v form a loop.

The elements in $\partial\mathcal{E}$ are not independent over $\mathbf{Z}[\Gamma]$. Indeed, one obvious relation that they satisfy is

$$\rho := \sum_{e \in \partial\mathcal{E}} e \in J. \tag{1}$$

Also, $e + e\sigma \in J$ for any $e \in \partial\mathcal{E}$; by Lemma 2.6, $e\sigma$ is Γ-equivalent to some element $e' \in \partial\mathcal{E}$. (It is possible that $e = e'$.)

The right action of σ on $\partial\mathcal{E}$ (up to Γ-equivalence) breaks $\partial\mathcal{E}$ up into a certain number of two element orbits and a certain number of fixed points. Note that $\{0, \infty\}$ and $\{\infty, -1\}$ form an orbit of size two since

$$\{0, \infty\}\sigma = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\{\infty, -1\}.$$

We set $e_\infty = \{0, \infty\}$ and $e_\infty^* = \{\infty, -1\}$. Similarly, we enumerate the other two element orbits and write e_i and e_i^* for the elements of the i-th orbit of size two. For the elements in $\partial\mathcal{E}$ fixed by σ, we label them as $e'_1, \cdots, e'_s$.

In this notation, $e_i\sigma = \gamma_i e_i^*$ for some $\gamma_i \in \Gamma$ and hence

$$e_i^* + \gamma_i^{-1} e_i \in J.$$

Also, $e'_i\sigma = \gamma'_i e'_i$ for some $\gamma'_i \in \Gamma$ and hence

$$(\gamma'_i + 1)e_i \in J.$$

Note that γ'_i has order two since σ has order two. Also, one can check that the number of terms e'_i that appear is equal to the number of conjugacy classes of elements of order two in Γ; in particular, if Γ has no two-torsion then these additional relations do not appear.

The relations $e_i^* + \gamma_i^{-1} e_i \in J$ implies that $\mathcal{U}_1/J$ is generated by $e_1, \ldots, e_t, e_\infty, e'_1, \ldots, e'_s$ over $\mathbf{Z}[\Gamma]$. Moreover, the relation ρ given by $\partial\mathcal{F} = 0$, written in terms of these generators, takes the form

$$\rho = \sum_{e \in \partial\mathcal{E}} e = \left(1 - \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)\right) e_\infty + \sum_{i=1}^{t}(1 - \gamma_i^{-1})e_i + \sum_{i=1}^{s} e'_i.$$

For i between 1 and t, let $D_i = \partial(e_i)$ and let $D_\infty = -\partial(e_\infty)$. Also, set $D'_i = -\partial(e'_i)$ for i between 1 and s. We then have the following theorem.

Theorem 2.7. *Let Γ be a finite index subgroup of $\mathrm{PSL}_2(\mathbf{Z})$ without any three-torsion. Then Δ_0 is generated as a $\mathbf{Z}[\Gamma]$-module by $D_1, \ldots, D_t, D_\infty, D'_1, \ldots, D'_s$. Moreover, the relations satisfied by these divisors are generated by*

$$\sum_{i=1}^{t}(\gamma_i^{-1} - 1)D_i + \sum_{i=1}^{s} D'_i = \left(\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) - 1\right) D_\infty$$

and

$$(1+\gamma_i')D_i' = 0$$

for i between 1 and s.

Proof. We have already verified that these divisors generate and satisfy these relations since we verified the analogous statements in $\mathcal{U}_1/J$. Moreover, working backwards through the proceeding arguments, one can see that these relations are equivalent to the relations occurring in J. $\square$

This complete description of Δ_0 as a $\mathbf{Z}[\Gamma]$-module yields the following corollary.

Corollary 2.8. *Let Γ be a finite index subgroup of $\mathrm{PSL}_2(\mathbf{Z})$ without three-torsion and let V be an arbitrary right Γ-module. Let $v_1, \cdots, v_t, v_\infty, v_1', \cdots, v_s'$ be elements of V such that*

$$v_\infty \big| \Delta = \sum_{i=1}^{t} v_i \big| (\gamma_i - 1) + \sum_{i=1}^{s} v_i'$$

and, for i between 1 and s,

$$v_i' \big| (1+\gamma_i') = 0.$$

Here $\Delta = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) - 1$ is the difference operator. Then there is a unique modular symbol $\varphi \in \mathrm{Symb}_\Gamma(V)$ such that for each i

$$\varphi(D_i) = v_i \text{ and } \varphi(D_i') = v_i'.$$

Conversely, every modular symbol $\varphi \in \mathrm{Symb}_\Gamma(V)$ arises in this manner.

2.5 The case of three-torsion

In constructing the fundamental domain $\mathcal{F}$ we needed to assume that Γ has no elements of order three. We now sketch what occurs when such torsion elements are present in $\Gamma \subsetneq G$.

Recall that R is the ideal triangle with vertices 0, 1 and ∞. The key fact that we used when building $\mathcal{F}$ out of translates of R was that for $\alpha \in \mathrm{PSL}_2(\mathbf{Z})$, we have that αR is a potential fundamental domain if and only if $\alpha\tau\alpha^{-1} \notin \Gamma$. Now when the three torsion element $\alpha\tau\alpha^{-1}$ is present in Γ, we will not want to include all of αR, but instead only "one-third" of it.

Indeed, recall that R is the union of T, τT and $\tau^2 T$ where T is the ideal triangle with vertices 0, 1 and ρ. Assume we have a potential fundamental domain of the form $\alpha_1 R \cup \cdots \cup \alpha_n R$ and that we are considering whether or not to extend this domain past the edge e to the triangle αR. If $\alpha\tau\alpha^{-1} \notin \Gamma$ then we extend our fundamental domain by including all of αR. However, if $\alpha\tau\alpha^{-1} \in \Gamma$ then we extend our potential fundamental domain by only including the unique triangle of αT, $\alpha\tau T$ and $\alpha\tau^2 T$ that contains the edge e. Continuing in this manner would lead to a domain

$$\mathcal{F} = \alpha_1 R \cup \cdots \cup \alpha_n R \cup \beta_1 T \cup \cdots \cup \beta_r T$$

that is a maximal potential fundamental domain of this form. As in Theorem 2.4, we have that such a domain is necessarily a fundamental domain for the action of Γ. These considerations yield the following theorem.

Theorem 2.9. *Let Γ any finite index subgroup of $\mathrm{PSL}_2(\mathbf{Z})$. Then there exists a fundamental domain $\mathcal{F}$ for the action of Γ on $\mathcal{H}^*$ of the form*

$$\mathcal{F} = \alpha_1 R \cup \cdots \alpha_n R \cup \beta_1 T \cup \cdots \beta_r T$$

for $\alpha_i, \beta_i \in \mathrm{PSL}_2(\mathbf{Z})$.

One can check that the triangles $\beta_i T$ are in one-to-one correspondence with pairs of conjugacy classes of three-torsion elements; two conjugacy classes are paired if the inverse of any element of one class appears in the other class.

We now consider how this change in the shape of the fundamental domain $\mathcal{F}$ affects our solution of the Manin relations. We first need to make an alteration to the definition of $\partial\mathcal{E}$; namely, $\partial\mathcal{E}$ should not only contain all elements of $\mathcal{E}$ that are contained in $\partial\mathcal{F}$, but also the unique edge of the each triangle $\beta_i T$ that is not contained in $\partial\mathcal{F}$. These addition edges should be oriented from left to right.

If e_i'' is the exceptional edge that corresponds to $\beta_i T$, then the three term relation involving e_i'' takes the form $e_i'' + e_i''\tau + e_i''\tau^2$ which is the sum of the three edges of $\beta_i R$. However, the matrix $\gamma_i'' = \beta_i \tau \beta_i^{-1}$ fixes a point in the interior of $\beta_i R$ and thus,

$$e_i'' + e_i''\tau + e_i''\tau^2 = e_i'' + \gamma_i'' e_i'' + (\gamma_i'')^2 e_i'' = (1 + \gamma_i'' + (\gamma_i'')^2) e_i'' \in J.$$

Set $D_i'' = -\partial(e_i'')$ for i between 1 and r.

Following the same arguments as in section 2.2 yields the following two results.

Theorem 2.10. *Let Γ be a proper finite index subgroup of $\mathrm{PSL}_2(\mathbf{Z})$. Then Δ_0 is generated as a $\mathbf{Z}[\Gamma]$-module by $D_1, \cdots, D_t, D_\infty, D_1', \cdots, D_s', D_1'', \cdots, D_r''$. Moreover, the relations satisfied by these divisors are generated by*

$$\sum_{i=1}^{t} (\gamma_i^{-1} - 1) D_i + \sum_{i=1}^{s} D_i' + \sum_{i=1}^{r} D_i'' = \left(\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix} \right) - 1 \right) D_\infty,$$

for i between 1 and s,

$$(1 + \gamma_i') D_i' = 0,$$

and, for i between 1 and r,

$$(1 + \gamma_i'' + (\gamma_i'')^2) D_i'' = 0.$$

Proof. We point out that the condition that Γ be proper is needed to ensure that the divisors D_i' and D_j'' are all distinct. Indeed, if two such divisors were equal, this would imply that Γ contains $\alpha\sigma\alpha^{-1}$ and $\alpha\tau\alpha^{-1}$ for some $\alpha \in G$. But then Γ is conjugate to $\mathrm{PSL}_2(\mathbf{Z})$ and thus equal to $\mathrm{PSL}_2(\mathbf{Z})$. □

Corollary 2.11. *Let Γ be a finite index subgroup of $\mathrm{PSL}_2(\mathbf{Z})$ and let V be an arbitrary right Γ-module. Let $v_1, \cdots, v_t, v_\infty, v_1', \cdots, v_s', v_1'', \cdots, v_r'' \in V$ be such that*

$$v_\infty \big| \Delta = \sum_{i=1}^{t} v_i \big| (\gamma_i - 1) + \sum_{i=1}^{s} v_i' + \sum_{i=1}^{r} v_i'',$$

for i between 1 and s

$$v_i' \big| (1 + \gamma_i') = 0$$

and for i between 1 and r

$$v_i'' \big| (1 + \gamma_i' + (\gamma_i'')^2) = 0.$$

Then there is a unique modular symbols $\varphi \in \mathrm{Symb}_\Gamma(V)$ such that

$$\varphi(D_i) = v_i, \ \varphi(D_i') = m_i' \text{ and } \varphi(D_i'') = m_i''.$$

Conversely, every modular symbol $\varphi \in \mathrm{Symb}_\Gamma(V)$ arises in this manner.

2.6 Computing with modular symbols in practice

The results of the previous sections can used in practice to compute efficiently with V-valued modular symbols for V some right Γ-module as long as one can efficiently compute the action of Γ on V. Namely, Corollary 2.8 tells us that a modular symbol $\varphi \in \mathrm{Symb}_\Gamma(M)$ is determined by its value on an explicit finite set of divisors. We will now describe how if one is given the values for φ on these divisors, one can compute the value of φ on any degree zero divisor.

First note that since Δ_0 is generated by divisors of the form $\left\{\frac{a}{c}\right\} - \left\{\frac{b}{d}\right\}$, it suffices to work with these two term divisors. Then Manin's continued fraction algorithm (see [3]) allows us to write such a divisor as a sum of divisors arising from unimodular paths. Therefore, it suffices to be able to compute φ on divisors arising from unimodular paths.

Let $[g]$ be some unimodular path with $g \in \mathrm{PSL}_2(\mathbf{Z})$. This path is Γ-equivalent to some path in $\mathcal{E}$ by Lemma 2.6. Equivalently, if $\mathcal{E} = \{[g_1], \ldots, [g_r]\}$ with $g_i \in \mathrm{PSL}_2(\mathbf{Z})$, then the g_i form a complete set of right coset representatives for Γ/G. Thus, for some i (which is easily computable), we have that $g = \gamma g_i$ with $\gamma \in \Gamma$. We then have

$$\varphi([g]) = \varphi(\gamma[g_i]) = \varphi([g_i]) \big| \gamma^{-1}$$

and, hence, to compute φ on any divisor it suffices to be able to compute the values of φ on elements of $\mathcal{E}$. (By evaluating φ on a unimodular path, we really mean evaluating on the boundary on this path.)

Finally, the constructions that lead up to Theorem 2.7, give an explicit way of writing any $e \in \mathcal{E}$ as a linear combination of our generators over $\mathbf{Z}[\Gamma]$. Since we are storing the value of φ on these generators, the value of $\varphi(e)$ is readily computed.

3 Modules of interest

3.1 Classical case

For $k \geq 0$ an integer, consider

$$L_k = \{F(Z) \in \mathbf{Q}_p[Z] \text{ such that } \deg(F) \leq k\}.$$

Then L_k has the structure of a right $\mathrm{GL}_2(\mathbf{Q}_p)$-module by

$$(F\big|_k\gamma)(Z) = (d - cZ)^k \, F\left(\frac{-b + aZ}{d - cZ}\right)$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $F \in L_k$.

By Eichler-Shimura theory, L_k-valued modular symbols correspond to classical modular forms.

3.2 Distributions

For each $r \in |\mathbf{C}_p^\times|$, let

$$B[\mathbf{Z}_p, r] = \{x \in \mathbf{C}_p \mid \text{there exists some } a \in \mathbf{Z}_p \text{ with } |x - a| \leq r\}.$$

Then $B[\mathbf{Z}_p, r]$ is a finite union of closed discs. For example, if $r \geq 1$ then $B[\mathbf{Z}_p, r]$ is the closed disc in $\mathbf{C}_p$ of radius r around 0. If $r = \frac{1}{p}$ then $B[\mathbf{Z}_p, r]$ is the disjoint union of the p discs of radius $\frac{1}{p}$ around the points $0, 1, \ldots, p-1$.

Let $A[\mathbf{Z}_p, r]$ denote the $\mathbf{Q}_p$-Banach algebra of $\mathbf{Q}_p$-affinoid functions on $B[\mathbf{Z}_p, r]$. For example, if $r \geq 1$

$$A[\mathbf{Z}_p, r] = \left\{ \sum_{n=0}^{\infty} a_n x^n \in \mathbf{Q}_p[[x]] \text{ such that } \{|a_n| \cdot r^n\} \to 0 \right\}.$$

The norm on $A[\mathbf{Z}_p, r]$ is given by the supremum norm. That is, if $f \in A[\mathbf{Z}_p, r]$, then

$$||f||_r = \sup_{x \in B[\mathbf{Z}_p, r]} |f(x)|.$$

For $r_1 \geq r_2$, there is a natural restriction map $A[\mathbf{Z}_p, r_1] \to A[\mathbf{Z}_p, r_2]$ that is injective, completely continuous and has dense image. We define

$$\mathcal{A}(\mathbf{Z}_p) = \varinjlim_{s>0} A[\mathbf{Z}_p, s] \text{ and } \mathcal{A}^\dagger(\mathbf{Z}_p, r) = \varinjlim_{s>r} A[\mathbf{Z}_p, s].$$

(It should be pointed out that these direct limits are taken over sets with no smallest element and therefore are not vacuous.) We endow each of these spaces with the inductive limit topology. Then $\mathcal{A}(\mathbf{Z}_p)$ is naturally identified with the space of locally analytic $\mathbf{Q}_p$-valued functions on $\mathbf{Z}_p$ while $\mathcal{A}^\dagger(\mathbf{Z}_p, r)$ is identified with the space of $\mathbf{Q}_p$-overconvergent functions on $B[\mathbf{Z}_p, r]$. Note that there are natural continuous inclusions

$$\mathcal{A}^\dagger(\mathbf{Z}_p, r) \hookrightarrow A[\mathbf{Z}_p, r] \hookrightarrow \mathcal{A}(\mathbf{Z}_p).$$

Moreover, the image of each of these maps is dense in its target space.

We now define our distributions modules as dual to these topological vector spaces. Namely, set $\mathcal{D}(\mathbf{Z}_p)$, $D[\mathbf{Z}_p, r]$ and $\mathcal{D}^\dagger(\mathbf{Z}_p, r)$ to be the space of continuous $\mathbf{Q}_p$-linear functionals on $\mathcal{A}(\mathbf{Z}_p)$, $A[\mathbf{Z}_p, r]$, and $\mathcal{A}^\dagger(\mathbf{Z}_p, r)$ respectively, each endowed with the strong topology. Equivalently,

$$\mathcal{D}(\mathbf{Z}_p) = \varprojlim_{s>0} D[\mathbf{Z}_p, s] \text{ and } \mathcal{D}^\dagger(\mathbf{Z}_p, r) = \varprojlim_{s>r} D[\mathbf{Z}_p, s],$$

each endowed with the projective limit topology.

Note that $D[\mathbf{Z}_p, r]$ is a Banach space under the norm

$$||\mu||_r = \sup_{\substack{f \in A[\mathbf{Z}_p, r] \\ f \neq 0}} \frac{|\mu(f)|}{||f||_r}.$$

for $\mu \in D[\mathbf{Z}_p, r]$. On the other hand, $\mathcal{D}(\mathbf{Z}_p)$ (resp. $\mathcal{D}^\dagger(\mathbf{Z}_p, r)$) has its topology defined by the family of norms $\{|| \cdot ||_s\}$ for $s \in |\mathbf{C}_p^\times|$ with $s > 0$ (resp. $s > r$). By duality, we have continuous linear injective maps

$$\mathcal{D}(\mathbf{Z}_p) \hookrightarrow D[\mathbf{Z}_p, r] \hookrightarrow \mathcal{D}^\dagger(\mathbf{Z}_p, r).$$

Let

$$\Sigma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}_p) \text{ such that } p \nmid a, p \mid c \text{ and } ad - bc \neq 0 \right\}$$

be the p-adic version of $S_0(p)$. We now define an action of $\Sigma_0(p)$ on these spaces of distributions. As in the classical case, we will incorporate a weight into this action.

Fix k a non-negative integer. Let $\Sigma_0(p)$ act on $A[\mathbf{Z}_p, r]$ on the left by

$$(\gamma \cdot_k f)(x) = (a + cx)^k \cdot f\left(\frac{b + dx}{a + cx}\right)$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Sigma_0(p)$ and $f \in A[\mathbf{Z}_p, r]$. Then $\Sigma_0(p)$ acts on $D[\mathbf{Z}_p, r]$ on the right by

$$(\mu\big|_k \gamma)(f) = \mu(\gamma \cdot_k f).$$

where $\mu \in D[\mathbf{Z}_p, r]$.

These two actions then induce actions on $\mathcal{A}(\mathbf{Z}_p)$, $\mathcal{A}^\dagger(\mathbf{Z}_p, r)$, $\mathcal{D}(\mathbf{Z}_p)$ and $\mathcal{D}^\dagger(\mathbf{Z}_p, r)$. To emphasis the role of k in this action, we will sometimes write k in the subscript, i.e. $\mathcal{A}_k(\mathbf{Z}_p)$, $\mathcal{D}_k(\mathbf{Z}_p)$, etc.

3.3 Log-differentials on Wide Open Subspaces

In this section, we give an alternate description of these spaces of distributions in terms of log-differentials. For each $r \in |\mathbf{C}_p^\times|$, let

$$W_r = W(\mathbf{Z}_p, r) = \mathbf{P}^1(\mathbf{C}_p) - B[\mathbf{Z}_p, r].$$

The space W_r is the standard example of a wide open subspace of $\mathbf{P}^1(\mathbf{C}_p)$. The ring of $\mathbf{Q}_p$-rigid analytic functions $A(W_r)$ on W_r is a topological $\mathbf{Q}_p$-algebra and the space $\Omega(W_r)$ of Kahler differentials on W_r is an $A(W_r)$-module.

Note that $1/z \in A(W_r)$ and thus $dz/z^2 \in \Omega(W_r)$. However, $dz/z \notin \Omega(W_r)$ as it has a pole at infinity.

Proposition 3.1. *Let $r \in |\mathbf{C}_p^\times|$ be greater than or equal to 1. Then we have the following descriptions of $A(W_r)$ and $\Omega(W_r)$:*

1. *Every function $f \in A(W_r)$ has a unique representation in the form*

$$f = \sum_{j=0}^{\infty} a_j z^{-j}$$

   with each $a_j \in \mathbf{Q}_p$.

2. *Every $\omega \in \Omega(W_r)$ has a unique representation in the form*

$$\omega = \sum_{n=1}^{\infty} a_j z^{-j} \frac{dz}{z}$$

   with each $a_j \in \mathbf{Q}_p$.

3. *Conversely, an expression of the form (1) (resp. (2)) represents an element of $A(W_r)$ (resp. $\Omega(W_r)$) if and only if for every real number $t > r$ the coefficient a_j satisfy*

$$|a_j|_p \text{ is } o(t^n) \text{ as } n \to \infty.$$

Proof. **Proof or reference needed here.** □

For our applications, we will need to consider a slightly larger space $\Omega_{\log}(W_r)$ of log-differentials that contains $\Omega(W_r)$ with codimension one. Let $\widetilde{\mathbf{P}}$ denote two-dimensional affine space with the origin deleted. Thus, for an arbitrary field K, we have $\widetilde{\mathbf{P}}(K) = K^2 \backslash \{0\}$. We have a natural morphism $\pi : \widetilde{\mathbf{P}} \to \mathbf{P}^1$, whose fibers are copies of the multiplicative group. Let $\widetilde{W}_r$ be the full preimage in $\widetilde{\mathbf{P}}(\mathbf{C}_p)$ under π of W_r. Thus, the fibers of the natural map $\pi : \widetilde{W}_r \to W_r$ are copies of $\mathbf{C}_p^\times$. The space $\widetilde{W}_r$ has a natural structure as a $\mathbf{Q}_p$-rigid analytic space. The action of $\mathbf{Q}_p^\times$ on the fibers of π induces an action of $\mathbf{Q}_p^\times$ on $A(\widetilde{W}_r)$ and $\Omega(\widetilde{W}_r)$.

Define $\Omega_0(W_r) \subseteq \Omega(\widetilde{W_r})$ to be the subspace on which $\mathbf{Q}_p^\times$ acts trivially. Notice that $\Omega_0(W_r)$ contains $\Omega(W_r)$ but this inclusion is far from an equality. Indeed, if L_1, L_2 are linearly independent $\mathbf{Q}_p$-linear forms on $\mathbf{Q}_p^2$, whose zeros are in $\mathbf{Z}_p$, then every element $\omega \in \Omega_0(W_r)$ can be expressed uniquely in the form

$$\omega = f(z)\frac{dL_1}{L_1} + g(z)\frac{dL_2}{L_2}$$

where $f, g \in A(W_r)$. We have $\omega \in \Omega(W_r)$ if and only if $f + g = 0$. We define

$$\Omega_{\log}(W_r) := \left\{ \omega = f(z)\frac{dL_1}{L_1} + g(z)\frac{dL_2}{L_2} \;\middle|\; f + g \text{ is a constant} \right\}.$$

For each $s \in \mathbf{Z}_p$ we set

$$\delta_s := \frac{dL}{L} \in \Omega_{\log}(W_r)$$

where L is any non-zero linear form that vanishes at s. For example,

$$\delta_\infty = \frac{dX}{X} \text{ and } \delta_0 = \frac{dY}{Y}.$$

The set of all δ_s where s ranges over $\mathbf{Z}_p$ is invariant under the action of G. Indeed, $\delta_s|\gamma = \delta_{s\gamma}$. Moreover, these elements are all the same modulo the space of holomorphic forms: if $s, t \in \mathbf{Z}_p$ are given in $\mathbf{P}^1(\mathbf{Q}_p)$ by $s = [a, b]$ and $t = [c, d]$, then

$$\delta_t - \delta_s = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{dz}{(az - b)(cz - d)} \in \Omega(W_r).$$

We therefore have a natural exact sequence of G-modules

$$0 \to \Omega(W_r) \to \Omega_{\log}(W_r) \xrightarrow{\rho} \mathbf{Q}_p \to 0$$

where the first map is the canonical inclusion and the ρ is the "residue" map, which vanishes on $\Omega(W_r)$ and takes the value 1 on δ_s for every $s \in \mathbf{Z}_p$.

Proposition 3.2. *Let $r \in |\mathbf{C}_p^\times|$ be greater than or equal to 1. Every $\omega \in \Omega_{\log}(W_r)$ has a unique representation in the form*

$$\omega = a_0\delta_0 + \sum_{n=1}^{\infty} a_j z^{-j} \frac{dz}{z}$$

with each $a_j \in \mathbf{Q}_p$.

Conversely, an expression in this form represents an element of $\Omega_{\log}(W_r)$ if and only if for every real number $t > r$ the coefficient a_j satisfy

$$|a_j|_p \text{ is } o(t^n) \text{ as } n \to \infty.$$

Proof. **Proof or reference needed here.** □

We now describe the weight zero right action of $\Sigma_0(p)$ on $\Omega_{\log}(W_r)$. Let $H \subseteq \mathrm{PGL}_2(\mathbf{Q}_p)$ be any semi-group that preserves W_r. Then the natural right action of H on W_r induces a left action on $A(W_r)$ and thus a left action on $\Omega(W_r)$. Explicitly, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in H$ and $f \in A(W_r)$ then

$$(\gamma \cdot f)(z) = (\gamma \cdot_0 f)(z) = f\left(\frac{b + dz}{a + cz}\right)$$

which in turn induces a left action on $\Omega(W_r)$ by pull-back. Since spaces of distributions are naturally right $\Sigma_0(p)$-modules, we express this action as a right action; that is, we write

$$\omega\big|_0\gamma := \gamma^{-1} \cdot \omega$$

for $\omega \in \Omega(W_r)$ and $\gamma \in \Sigma_0(p)$. (Note that if $\gamma \in \Sigma_0(p)$ then γ^{-1} preserves W_r.)

To define the higher weight action, we first introduce a truncation operation. Namely, define

$$\mathrm{trunc}\left(\sum_{j=-\infty}^{\infty} a_j z^{-j} \frac{dz}{z}\right) = \sum_{j=0}^{\infty} a_j z^{-j} \frac{dz}{z}.$$

We then define the weight k action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Sigma_0(p)$ on $\omega \in \Omega(W_r)$ by

$$\omega\big|_k\gamma = \mathrm{trunc}\left((a + cz)^k \cdot \omega\right)\big|_0\gamma.$$

3.4 Relation between log-differentials and distributions

In this section, we give a $\Sigma_0(p)$-equivariant isomorphism between spaces of log-differentials and spaces of overconvergent distributions.

Set $W := W_1 = W(\mathbf{Z}_p, 1)$. For $\omega \in \Omega_{\log}(W)$, define a distribution $\mu_\omega \in \mathcal{D}^\dagger(\mathbf{Z}_p, 1)$ by

$$\int_{\mathbf{Z}_p} f d\mu_\omega := \rho_{\partial W}(fw)$$

for each $f \in A^\dagger(\mathbf{Z}_p, 1)$. Here $\rho_{\partial W}$ is the residue around the unit disc of $\mathbf{C}_p$. Taking the residue of $f\omega$ is a valid operation because f is overconvergent and, hence, defined on some disc of radius strictly larger than 1.

Theorem 3.3. *The map*

$$\mu : \Omega_{\log}(W(\mathbf{Z}_p, 1)) \to \mathcal{D}^\dagger(\mathbf{Z}_p, 1)$$
$$\omega \mapsto \mu_\omega$$

is an isomorphism. Moreover, for each integer $k \geq 0$ we have that

$$(\mu_\omega)\big|_k\gamma = \mu_{(\omega|_k\gamma)}$$

for $\gamma \in \Sigma_0(p)$. That is, μ is a $\Sigma_0(p)$-equivariant map with respect to the weight k action.

Proof. **Proof or reference needed here.** □

Under this isomorphism between distributions and log-differentials, the moments of a distribution correspond to the coefficients of the associated log-differential.

Corollary 3.4. *If* $\omega = a_0\delta_0 + \sum_{j=1}^{\infty} a_j z^{-j} \frac{dz}{z} \in \Omega_{\log}(W)$ *then* $\mu_\omega(x^j) = a_j$.

Proof. We have that

$$\mu_\omega(x^j) = \int_{\mathbf{Z}_p} x^j \, d\mu_\omega = \rho_{\partial W}(x^j \omega) = a_j$$

since the residue function returns the coefficient of dz/z. □

With this equivalence between distributions and log-differentials in hand, we will tacitly identify these two spaces.

4 Lifting modular symbols

Let $D_k := \mathcal{D}_k(\mathbf{Z}_p), \mathbf{D}_k[\mathbf{Z}_p, 1]$ or $\mathcal{D}_k^\dagger(\mathbf{Z}_p, 1)$; we will refer to $\mathrm{Symb}_\Gamma(D_k)$ as the space of overconvergent modular symbols (endowed with the weight k action).

The space of overconvergent modular symbols naturally maps to the space of classical modular symbols. Indeed, there is a natural $\Sigma_0(p)$-equivariant map

$$\begin{aligned} \rho_k : D_k &\to L_k \\ \mu &\mapsto \int (Z - x)^k d\mu(x) \end{aligned}$$

where the integration takes place coefficient by coefficient. This induces a map

$$\rho_k^* : \mathrm{Symb}_\Gamma(D_k) \to \mathrm{Symb}_\Gamma(L_k)$$

which we will refer to as the *specialization map.* Note that ρ_k^* also intertwines the action of $\Sigma_0(p)$ and is thus Hecke-equivariant. It is with respect to this map that we will be a lifting classical modular symbols.

The main result of this section is that ρ_k^* is a surjective map. A cohomological proof of this theorem is given in [4] by analyzing $H_c^2(\Gamma, D_k)$. In this section, we give a proof that is completely explicit. Namely, by Corollary 2.11, in order to write down a V-valued modular symbol for Γ, we have to be able to solve the equations

$$v\big|(1+\sigma) = 0,$$

$$v\big|(1+\tau+\tau^2) = 0,$$

and

$$v\big|\Delta = w$$

for $v, w \in V$. Recall that $\Delta = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) - 1$ while σ and τ have orders two and three respectively.

The first two equations are easy to explicitly solve in any Γ-module V (where 2 and 3 act invertibly) while the last equation (the difference equation) is more subtle and its solution will depend heavily upon the module V. For $V = \mathcal{D}_k^\dagger(\mathbf{Z}_p, 1)$, we give an explicit solution to this equation in section 4.2.

4.1 Torsion equations

To solve the equation $v\big|(1+\sigma) = 0$, note that

$$V \cong V^{\sigma=1} \oplus V^{\sigma=-1}$$

given by

$$v \mapsto \left(\frac{v\big|(1+\sigma)}{2}, \frac{v\big|(1-\sigma)}{2} \right)$$

since σ acts as an involution on V. Thus, the set of all $v \in V$ that are killed by $1+\sigma$ is equal to $V\big|(1-\sigma)$.

Similarly, for the equation $v\big|(1+\tau+\tau^2) = 0$, we have that

$$V \cong V^{\tau=1} \oplus V^{1+\tau+\tau^2=0}$$

given by

$$v \mapsto \left(\frac{v\big|(1+\tau+\tau^2)}{3}, \frac{v\big|(2-\tau-\tau^2)}{3} \right).$$

Thus, the set of all $v \in V$ that are killed by $1+\tau+\tau^2$ is equal to $V\big|(2-\tau-\tau^2)$.

4.2 The difference equation

We begin by discussing the case where $V = L_k$.

Proposition 4.1. *Consider $\Delta : L_k \to L_k$. We have that*

1. $\ker(\Delta)$ *equals the constant functions in L_k.*
2. $\mathrm{im}(\Delta) = L_{k-1} \subseteq L_k$.

Thus, for each non-zero $g \in L_k$ with $\deg(g) < k$ there exists $f \in L_k$ such that

$$f\big|\Delta = g$$

Moreover, f is unique up to the addition of a constant.

Proof. First note that if $h \in L_k$ and $h\big|\Delta = 0$, then $h(x-1) = h(x)$; thus, h is constant. Therefore, we have

$$0 \to \mathbf{Q}_p \to L_k \xrightarrow{\Delta} L_k \to \mathrm{coker}(\Delta) \to 0$$

where $\mathrm{coker}(\Delta)$ is one dimensional over $\mathbf{Q}_p$. Moreover, directly from the definition of acting by Δ, one sees that $\mathrm{im}(\Delta) \subseteq L_{k-1}$. Since L_{k-1} is of codimension 1 in L_k, we must have that $\mathrm{im}(\Delta) = L_{k-1}$. $\square$

We next consider the case of $V = \mathcal{D}_k^\dagger(\mathbf{Z}_p, 1) \cong \Omega_{\log}(W(\mathbf{Z}_p, 1))$.

Lemma 4.2. *For* $\Delta : \mathcal{D}_k^\dagger(\mathbf{Z}_p, 1) \to \mathcal{D}_k^\dagger(\mathbf{Z}_p, 1)$*, we have that* $\ker(\Delta) = 0$.

Proof. Let $\mu \in \ker(\Delta)$ be a non-zero distribution and let n be the smallest non-negative integer such that $\mu(x^n) \neq 0$. Then since $\mu\big|_k\Delta = 0$, we have

$$\mu((x-1)^{n+1}) = \mu(x^{n+1}).$$

(Note that the weight k action of Δ on $\mathcal{D}_k^\dagger(\mathbf{Z}_p, 1)$ is the same as its weight 0 action). We then have that

$$\mu(x^{n+1}) = \mu(x^{n+1}) + (-1)^{n+1}(n+1)\mu(x^n)$$

since $\mu(x^j) = 0$ if $j < n$. Thus, $\mu(x^n) = 0$ and this contradiction implies that μ is identically zero. □

Remark 4.3. In what follows, we write $\mu\big|\Delta$ for $\mu\big|_k\Delta$ since this action is independent of the weight k.

Note that

$$\left(\mu\big|\Delta\right)(\mathbf{Z}_p) = \left(\mu\big|\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)\right)(\mathbf{Z}_p) - \mu(\mathbf{Z}_p) = 0$$

and, thus, $\mathrm{im}(\Delta)$ is contained in the set of distributions with total measure zero. We will see that this inclusion is in fact an equality. To this end, we will work with log-differentials rather than distributions and we begin by solving the difference equation

$$\mu\big|\Delta = \frac{1}{z^{j+1}}\frac{dz}{z}$$

for $j \geq 0$. (Recall that under the dictionary between log-differentials and distributions, $z^{-j}dz/z$ corresponds to the distribution that takes the value 1 on x^j and vanishes on all other monomials.) In the following lemma, we write down explicit log-differentials that solve this equation whose coefficients are given in terms of Bernoulli numbers.

Lemma 4.4. *Let*

$$\eta_j = \begin{cases} \displaystyle\sum_{r=j}^{\infty} \binom{r}{j} b_{r-j} z^{-r}\frac{dz}{z} & j \neq 0 \\ \displaystyle\delta_0 + \sum_{r=1}^{\infty} b_r z^{-r}\frac{dz}{z} & j = 0 \end{cases}$$

where b_r *is the* r*-th Bernoulli number. Then* $\eta_j \in \Omega_{\log}(W(\mathbf{Z}_p, 1))$ *and*

$$\eta_j\big|\Delta = \frac{j+1}{z^{j+1}}\frac{dz}{z}.$$

Proof. By the von Staudt-Clausen theorem, $pb_n \in \mathbf{Z}_p$ for each n. Thus, η_j is in $\Omega_{\log}(W(\mathbf{Z}_p, 1))$ by Proposition 3.2. Now, for $j > 0$ we compute:

$$\begin{aligned}
\eta_j \Big| \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &= \left(\sum_{r=j}^{\infty} \binom{r}{j} b_{r-j} z^{-r} \frac{dz}{z} \right) \Big| \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
&= \sum_{r=j}^{\infty} \binom{r}{j} b_{r-j} (z-1)^{-r} \frac{dz}{z-1} \\
&= \sum_{r=j}^{\infty} \binom{r}{j} b_{r-j} z^{-r} (1 - z^{-1})^{-r-1} \frac{dz}{z} \\
&= \sum_{r=j}^{\infty} \binom{r}{j} b_{r-j} z^{-r} \left(\sum_{m=0}^{\infty} \binom{-r-1}{m} (-1)^m z^{-m} \right) \frac{dz}{z} \\
&= \sum_{r=j}^{\infty} \sum_{m=0}^{\infty} \binom{r}{j} \binom{r+m}{m} b_{r-j} z^{-r-m} \frac{dz}{z} \\
&= \sum_{s=j}^{\infty} \left(\sum_{r=j}^{s} \binom{r}{j} \binom{s}{r} b_{r-j} \right) z^{-s} \frac{dz}{z}.
\end{aligned}$$

But we have the following identity of Bernoulli numbers:

$$\sum_{r=j}^{n} \binom{n}{r} \binom{r}{j} b_{r-j} = \begin{cases} \binom{n}{j} b_{n-j} & n \neq j+1 \\ \binom{n}{j} b_{n-j} + (j+1) & n = j+1 \end{cases}.$$

Thus

$$\eta_j \big| \Delta = \frac{j+1}{z^{j+1}} \frac{dz}{z}$$

as claimed. The case $j = 0$ is done similarly. □

Theorem 4.5. *For any $\nu \in \Omega_{\log}(W(\mathbf{Z}_p, 1))$ of total measure zero, there exists a unique $\mu \in \Omega_{\log}(W(\mathbf{Z}_p, 1))$ such that*

$$\mu \big| \Delta = \nu.$$

Proof. Let

$$\nu = \sum_{j=1}^{\infty} a_j z^{-j} \frac{dz}{z}.$$

Then consider

$$\mu = \sum_{j=1}^{\infty} \frac{a_j}{j} \eta_{j-1}.$$

Since $\eta_j \in \Omega_{\log}(W(\mathbf{Z}_p, 1))$, by Proposition 3.2, the coefficients of μ grow slowly enough so that $\mu \in \Omega_{\log}(W(\mathbf{Z}_p, 1))$. Now from Lemma 4.4, it is clear that $\mu \big| \Delta = \nu$. Finally, the uniqueness of μ follows as $\ker(\Delta) = 0$. □

Remark 4.6. Note that the proof of Theorem 4.5 is completely explicit since the elements η_j are completely explicit.

4.3 Explicit lifts of classical symbols

Theorem 4.7. *We have that*

$$\rho_k^* : \mathrm{Symb}_\Gamma(\mathcal{D}^\dagger(\mathbf{Z}_p, 1)) \to \mathrm{Symb}_\Gamma(L_k)$$

is surjective.

Proof. For each $\varphi \in \mathrm{Symb}_\Gamma(L_k)$, we wish to find $\Phi \in \mathrm{Symb}_\Gamma(\mathcal{D}^\dagger(\mathbf{Z}_p, 1))$ such that $\rho_k^*(\Phi) = \varphi$. By Corollary 2.11, to define Φ it suffices to give the value of Φ on $D_1, \cdots, D_t, D_\infty, D_1', \cdots, D_s', D_1'', \cdots, D_r''$ subject to certain relations.

Defining $\Phi(D_i)$ is easy for $i < \infty$; simply pick any $\mu_i \in \mathcal{D}^\dagger(\mathbf{Z}_p, 1)$ such that $\rho_k^*(\mu_i) = \varphi(D_i)$. To define $\Phi(D_i')$, we need to find μ_i' such that

$$\rho_k^*(\mu_i') = \varphi(D_i') \text{ and } \mu_i'\big|(1+\sigma) = 0.$$

To do this, pick any $\hat{\mu}_i' \in \mathcal{D}^\dagger(\mathbf{Z}_p, 1)$ such that $\rho_k^*(\hat{\mu}_i') = \varphi(D_i')$ and set $\mu_i' = \frac{1}{2}\hat{\mu}_i'\big|(1-\sigma)$. Since μ_i' is in the image of $1-\sigma$ it is automatically killed by $1+\sigma$. Moreover, since $\varphi(D_i')\big|\sigma = -\varphi(D_i')$, we have that

$$\rho_k^*(\mu_i') = \frac{1}{2}\rho_k^*(\hat{\mu}_i'\big|(1-\sigma)) = \frac{1}{2}(\varphi(D_i') - \varphi(D_i')\big|\sigma) = \varphi(D_i').$$

A similar argument can be used to define $\Phi(D_i'')$.

Set

$$\nu = \sum_{i=1}^{t} \mu_i\big|(\gamma_i - 1) + \sum_{i=1}^{s} \mu_i' + \sum_{i=1}^{r} \mu_i''. \tag{2}$$

All that is left now is to find $\mu_\infty \in \mathcal{D}^\dagger(\mathbf{Z}_p, 1)$ such that μ_∞ lifts $\varphi(D_\infty)$ and $\mu_\infty\big|\Delta = \nu$. To this end, note that $\mu_i\big|(\gamma_i - 1)$ has total measure zero. Also, since μ_i' is the image of $1-\sigma$ and μ_i'' is in the image of $2-\tau-\tau^2$ all of these distributions also have total measure zero. Thus, ν has total measure zero and, by Theorem 4.5, there exists a unique $\mu_\infty \in \mathcal{D}^\dagger(\mathbf{Z}_p, 1)$ satisfying the difference equation.

However, μ_∞ need not lift $\varphi(D_\infty)$. Indeed, from our construction, we only know that $\varphi(D_\infty) - \rho_k^*(\mu_\infty)$ is killed by Δ. But since Δ has a kernel on L_k, we can only conclude that $\varphi(D_\infty) - \rho_k^*(\mu_\infty)$ is a constant (see Proposition 4.1).

From the definition of specialization, we see that the k-th moment of μ_∞ is the only one that is possibly taking on the wrong value. To remedy this problem, we can make use of the fact that we have great flexibility in our choice of each μ_i. By carefully modifying one of these distributions, we can change the value of $\mu_\infty(x^k)$ to any value in $\mathbf{Q}_p$.

Choose some j such that γ_j has all non-zero entries. **(Not always possible in small level...need to use torsion elements.)** Let ν_r be the distribution defined by

$$\nu_r(x^i) = \begin{cases} 1 & \text{if } i = r \\ 0 & \text{otherwise.} \end{cases}$$

For any $r > k$, the distribution $\mu_j + \lambda \cdot \nu_r$ lifts $\varphi(D_j)$ for any choice of $\lambda \in \mathbf{Q}_p$. Define a new distribution ν' by a modified version of the right-hand side of equation (2); namely, replace μ_j by $\mu_j + \lambda \cdot \nu_r$. Then ν' has total measure zero and, thus, there is a unique distribution μ'_∞ such that $\mu'_\infty\big|\Delta = \nu'$.

In general, if $\mu\big|\Delta = \nu$, then

$$\mu(x^k) = \sum_{i=0}^{k} \frac{b_{k-i}}{i+1}\binom{k}{i} \cdot \nu(x^{i+1}). \tag{3}$$

This formula follows from our explicit description of the unique solution to the difference equation given in Theorem 4.5. In particular, the k-th moment of μ only depends upon the first $k+1$ moments of ν.

Note that since $r > k$, we have $\rho_k(\nu_r) = 0$ and, thus, $\rho_k(\nu_r\big|_k(\gamma_j - 1)) = 0$. Therefore, ν and ν' have the same first k moments. Then, from (3),

$$\mu'_\infty(x^k) = \mu_\infty(x^k) + \alpha \cdot \lambda \cdot (\nu_r\big|_k(\gamma_j - 1))(x^{k+1})$$

for α some non-zero constant. If $r > k+1$, then

$$\mu'_\infty(x^k) = \mu_\infty(x^k) + \alpha \cdot \lambda \cdot (\nu_r\big|_k\gamma_j)(x^{k+1}).$$

It thus suffices to show that $(\nu_r\big|_k\gamma_j)(x^{k+1})$ is non-zero, since then, by varying λ, we can force $\mu'_\infty(x^k)$ to take on any value in $\mathbf{Q}_p$. In particular, we can choose λ so that μ' lifts $\varphi(D_\infty)$.

Computing, we have that

$$\begin{aligned}
(\nu_r\big|\gamma_i)(x^{k+1}) &= (\nu_r\big|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))(x^{k+1}) \\
&= \nu_r\left((a+cx)^{-1} \cdot (b+dx)^{k+1}\right) \\
&= \nu_r\left(a^{-1}\left(1+ca^{-1}x\right)^{-1} \cdot (b+dx)^{k+1}\right) \\
&= \nu_r\left(a^{-1}\left(\sum_{s=0}^{\infty}(-c)^s a^{-s}x^s\right)\left(\sum_{s=0}^{k}\binom{k+1}{s}b^{k+1-s}d^s x^s\right)\right) \\
&= a^{-1}\sum_{s=0}^{k+1}\binom{k+1}{s}b^{k+1-s}d^s(-c)^{r-s}a^{s-r}
\end{aligned}$$

Up to a non-zero constant, this last expression equals

$$\begin{aligned}\sum_{s=0}^{k+1}\binom{k+1}{s}b^{-s}d^s(-c)^{-s}a^s &= \sum_{s=0}^{k+1}\binom{k+1}{s}\left(\frac{-ad}{bc}\right)^s \\ &= \left(1-\frac{ad}{bc}\right)^{k+1} \\ &= \left(\frac{bc-ad}{bc}\right)^{k+1} \\ &= \left(\frac{-1}{bc}\right)^{k+1}\end{aligned}$$

which is non-zero. Thus, there is some value of λ such that μ'_∞ lifts $\varphi(D_\infty)$.

Now, by Corollary 2.11, there is a unique modular symbol Φ defined by

$$\Phi(D_\infty) = \mu'_\infty,\ \Phi(D_i) = \mu_i,\ \Phi(D'_i) = \mu'_i \text{ and } \Phi(D''_i) = \mu''_i$$

which by construction now lifts φ. □

Remark 4.8. In practice, if one is lifting a non-Eisenstein eigensymbol, one does not need to go through the extra process of modifying the distribution that lifts $\varphi(D_j)$ to force the value at D_∞ to work out correctly. Instead, note that the symbol φ_{eis} defined by $\varphi_{eis}(D_\infty) = 1$ and $\varphi_{eis}(D_i) = 0$ for all i is an Eisenstein symbol. Thus, in the notation of the above proof, one first forms a symbol Φ such that $\Phi(D_\infty) = \mu_\infty$ and $\Phi(D_i) = \mu_i$. Then the symbol $\Phi|\left(T_\ell - (\ell^{k+1}+1)\right)$ is a lift of $(a_\ell - (\ell^{k+1}+1))\varphi$. As long as φ is not an Eisenstein symbol (at ℓ) then one can rescale to get an overconvergent lift of φ.

The following lemma will allow us to show that Φ can be constructed to take values in $\mathbf{D}[\mathbf{Z}_p, 1]$.

Lemma 4.9. *If $\mu \in \mathcal{D}^\dagger(\mathbf{Z}_p, 1)$ then $\mu\big|\beta(a,p) \in \mathbf{D}[\mathbf{Z}_p, 1]$ where $\beta(a,p) = \left(\begin{smallmatrix}1 & a\\ 0 & p\end{smallmatrix}\right)$. In particular, if $\Phi \in \mathrm{Symb}_\Gamma(\mathcal{D}^\dagger(\mathbf{Z}_p, 1))$ then $\Phi\big|U_p \in \mathrm{Symb}_\Gamma(\mathbf{D}[\mathbf{Z}_p, 1])$.*

Proof. We have

$$(\mu\big|\beta(a,p))(x^j) = \mu\left((a+px)^j\right) = \sum_{r=0}^{j}\binom{j}{r}a^{j-r}p^r\mu(x^r).$$

But $\{p^r\mu(x^r)\}$ is bounded by Proposition 3.2 and thus, again by Proposition 3.2, $\mu\big|\beta(a,p) \in \mathbf{D}[\mathbf{Z}_p, 1]$.

For the second part of the lemma,

$$(\Phi\big|U_p)(D) = \sum_{a=0}^{p-1}\Phi\left(\beta(a,p)D\right)\big|\beta(a,p)$$

which lies in $\mathbf{D}[\mathbf{Z}_p, 1]$ by the first part of the lemma. □

Corollary 4.10. *We have that*

$$\rho_k^* : \mathrm{Symb}_\Gamma(\mathbf{D}_k[\mathbf{Z}_p, 1]) \to \mathrm{Symb}_\Gamma(L_k)$$

is surjective.

Proof. Let $\varphi \in \mathrm{Symb}_\Gamma(L_k)$. Without loss of generality, we may assume that φ is an eigensymbol for U_p with eigenvalue λ. Then, by Theorem 4.7, there is some $\Psi \in \mathrm{Symb}_\Gamma(\mathcal{D}^\dagger(\mathbf{Z}_p, 1))$ lifting φ. Set $\Phi := \frac{1}{\lambda}\Psi|U_p$ which is in $\mathrm{Symb}_\Gamma(\mathbf{D}_k[\mathbf{Z}_p, 1])$ by Lemma 4.9. Moreover,

$$\rho_k^*(\Phi) = \frac{1}{\lambda}\rho_k^*\left(\Psi\middle|U_p\right) = \frac{1}{\lambda}\varphi|U_p = \varphi$$

and, thus, Φ lifts φ. □

5 Overconvergent eigensymbols

In this section, we prove the existence of overconvergent modular symbols that are *Hecke-eigensymbols* with the same eigenvalues as classical modular symbols.

5.1 The ordinary subspace of an operator

We begin by discussing generalities about the slope zero subspace of a operator. In latter sections, we will apply this to the operator U_p/p^h to recover the slope h subspace of modular symbols.

Lemma 5.1. *Let X be an abelian group and $u : X \to X$ a homomorphism with finite image. Then there exists a unique decomposition*

$$X = X^{\mathrm{ord}} \oplus X^{\mathrm{nil}}$$

such that u is invertible on X^{ord} and nilpotent on X^{nil}. Moreover, X^{ord} is a finite set.

Proof. For any element $x \in X$, consider the sequence, $\{x, ux, u^2x, \dots\}$. Since u has finite image, this sequence will eventually become periodic; denote by $N(x)$ the length of this period. Set X^{ord} be to the subgroup of $x \in X$ such that this sequence is *purely* periodic; that is,

$$X^{\mathrm{ord}} := \{x \in X \ : \ u^j x = x \text{ for some } j > 0\}.$$

Clearly, u acts invertibly on X^{ord} and since the image of u is finite, X^{ord} is finite. Set X^{nil} equal to the subgroup of X consisting of all elements killed by some power of u.

For any $x \in X$, the sequence $\{u^{N(x)j}\}$ eventually stabilizes; denote by x^{ord} this stable value. Then if $x^{\mathrm{nil}} = x - x^{\mathrm{ord}}$, we have that $u^{N(x)j}$ kills x^{nil} for j large enough and thus $x^{\mathrm{nil}} \in X^{\mathrm{nil}}$. Therefore, since $X^{\mathrm{ord}} \cap X^{\mathrm{nil}} = \{0\}$, we have $X = X^{\mathrm{ord}} \oplus X^{\mathrm{nil}}$. □

Proposition 5.2. *Let V be a Banach space over $\mathbf{Q}_p$ and let $u : V \to V$ be a completely continuous operator that preserves some bounded $\mathbf{Z}_p$-submodule L of V such that $L \otimes \mathbf{Q}_p = V$. Then there is a unique decomposition of Banach spaces*

$$V = V^{\mathrm{ord}} \oplus V^{\mathrm{nil}}$$

where u is topologically nilpotent on V^{nil}, invertible on V^{ord} and such that $\{u^{-n}\}$ forms a bounded sequence of operators on V^{ord}. Moreover, the space V^{ord} has finite dimension over $\mathbf{Q}_p$ and the projection $V \to V^{\mathrm{ord}}$ is given by $x \mapsto \lim_{n\to\infty} u^{n!}x$.

Proof. Since u is a completely continuous operator, u has finite image in L/p^nL for any n. In particular, by the previous lemma, we have a decomposition

$$L/p^nL = (L/p^nL)^{\mathrm{ord}} \oplus (L/p^nL)^{\mathrm{nil}}$$

for each n. Moreover, the natural projection maps respect this decomposition on every level. Thus, if we define

$$L^{\mathrm{ord}} := \varprojlim_n (L/p^nL)^{\mathrm{ord}} \text{ and } L^{\mathrm{nil}} := \varprojlim_n (L/p^nL)^{\mathrm{nil}},$$

we get the decomposition

$$L = L^{\mathrm{ord}} \oplus L^{\mathrm{nil}}.$$

Since $(L/pL)^{\mathrm{ord}}$ is finite, by Nakayama's lemma, L^{ord} has finite rank over $\mathbf{Z}_p$. Finally, tensoring by $\mathbf{Q}_p$, yields the result of the proposition. □

5.2 Forming overconvergent eigensymbols

Let $\varphi \in \mathrm{Symb}_\Gamma(L_k) \otimes K$ be an eigensymbol for the full Hecke-algebra of slope h where K is some finite extension of $\mathbf{Q}_p$ containing the system of eigenvalues attached to φ. Let us denote by λ the eigenvalue of U_p acting on φ; note that $\mathrm{ord}_p(\lambda) = h$ where h lies between 0 and $k+1$. Set

$$X := (\rho_k^*)^{-1}(K \cdot \varphi) \subseteq \mathrm{Symb}_\Gamma(\mathbf{D}_k[\mathbf{Z}_p, 1]) \otimes K.$$

(In particular, X contains the kernel of specialization.)

Proposition 5.3. *There is a unique decomposition*

$$X \cong X^{(=h)} \oplus X^{(>h)}$$

where $\lambda^{-1}U_p$ is topologically nilpotent on $X^{(>h)}$, invertible on $X^{(=h)}$ and such that $\{\lambda^n U_p^{-n}\}$ forms a bounded sequence of operators on $X^{(=h)}$. Moreover, $X^{(=h)}$ is a finite dimensional space over $\mathbf{Q}_p$ (which we will refer to as the slope h subspace).

Proof. Let $u := \lambda^{-1}U_p$. By Proposition 5.2, it suffices to show that the $\mathbf{Z}_p$-submodule:

$$L = \left\{\Phi \in X \ : \ ||\Phi|| \le 1 \text{ and } ||\Phi(D)(x^j)|| \le p^{-(k+1-j)} \text{ for all } D \in \Delta_0\right\}$$

is preserved by u. In the course of the proof, we will see that the unit ball of X is *not* necessarily preserved by u which is the reason we are imposing the second condition in the definition of L. Also, this second condition is explained well by a natural filtration that is defined on $\mathbf{D}_k[\mathbf{Z}_p, 1]$ in [4] and recalled in section 7.1.

Let Φ be any symbol in L and we will check that $\Phi|u \in L$. Note first that if a symbol Ψ_1 satisfies the second condition defining L and $\rho_k^*(\Psi_1) = \rho_k^*(\Psi_2)$, then Ψ_2 also satisfies this condition. Since

$$\rho_k^*(\Phi|u) = \lambda^{-1}\rho_k^*(\Phi)|U_p = \rho_k^*(\Phi),$$

we see that $\Phi|u$ satisfies the second condition defining L.

For the first condition, let D be any divisor in Δ_0 and, then, by definition

$$(\Phi|u)(D) = \lambda^{-1}\sum_{a=0}^{p-1}\Phi(\beta(a,p)D)\big|\beta(a,p)$$

where $\beta(a,p) = \left(\begin{smallmatrix}1 & a\\ 0 & p\end{smallmatrix}\right)$. Thus, it suffices to show that $||\Phi(D')\big|\beta(a,p)|| \le p^{-h}$ for an arbitrary divisor $D' \in \Delta_0$ since $||\lambda^{-1}|| = p^h$.

Write

$$\Phi(D) = \sum_{j=0}^{\infty} a_j z^{-j}\frac{dz}{z} = \sum_{j=0}^{k} a_j z^{-j}\frac{dz}{z} + \sum_{j=k+1}^{\infty} a_j z^{-j}\frac{dz}{z} = \mu_1 + \mu_2.$$

The μ_2 term is handled by Lemma 5.5 (below) which tells us directly that

$$||\mu_2|\beta(a,p)|| \le p^{-(k+1)}||\mu_2|| \le p^{-(k+1)} \le p^{-h}.$$

The last inequality follows since h lies between 0 and $k+1$.

For the first term, by our assumption on Φ, we have that $||a_j|| \le p^{-(k+1-j)}$ for j between 0 and k. Since $||\left(z^{-j}dz/z\right)\big|\beta(a,p)|| \le p^{-j}$ (again by Lemma 5.5), we have $||\mu_1|\beta(a,p)|| \le p^{-(k+1)} \le p^{-h}$. (Note that without the addition assumption on the first $k+1$ moments of $\Phi(D)$, this above argument could not be made.)

Thus L is preserved by u and our proposition follows from Proposition 5.2. □

Remark 5.4. By carefully following the above proof, we will see in section 7.4 that one can apply $u := \lambda^{-1}U_p$ to our representation of an element of X without losing any accuracy even in the case when λ is not a unit. Maintaining a constant accuracy will be essential in our applications since in order to project to the slope h subspace one must iterate the operator u.

Lemma 5.5. *If $\mu_j = z^{-j} dz/z$ then*

$$||\mu_j | \beta(a,p)|| = p^{-j}.$$

where $\beta(a,p) = \left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right)$.

Proof. We have

$$\begin{aligned} \mu_j \Big| \beta(a,p) &= \left(\frac{z-a}{p} \right)^{-j-1} \cdot \frac{1}{p} \cdot dz \\ &= \frac{p^j}{z^j} \cdot \frac{1}{(1-az^{-1})^{j+1}} \cdot \frac{dz}{z} \\ &= p^j \cdot \sum_{r=j}^{\infty} a^{r-j} z^{-r} \frac{dz}{z} \end{aligned}$$

which has norm p^{-j} since $a \in \mathbf{Z}_p^\times$. □

Theorem 5.6. *If $\varphi \in \mathrm{Symb}_\Gamma(L_k) \otimes K$ is a Hecke eigensymbol then there exists a non-zero Hecke eigensymbol $\Phi \in \mathrm{Symb}_\Gamma(\mathbf{D}_k[\mathbf{Z}_p, 1]) \otimes K$ with the same system of eigenvalues as φ.*

Proof. Let h be the slope of φ and set $X := (\rho_k^*)^{-1} (K \cdot \varphi)$. Then, by Corollary 4.10 and Proposition 5.3, we have a surjective map

$$X^{(=h)} \oplus X^{(>h)} \to K \cdot \varphi.$$

Since φ has slope h, ρ_k^* kills $X^{(>h)}$ and thus

$$X^{(=h)} \to K \cdot \varphi$$

is surjective. The source and (of course) the target of this map are finite dimensional over K. The following lemma from linear algebra then establishes our theorem. □

Lemma 5.7. *Let V and W be finite dimensional vector spaces over a field K. Let $\{A_i\}$ be a countable family of commuting operators on these spaces and let $T : V \to W$ be a surjective linear map equivariant for each A_i. If there is some $w \in W$ such that $w|A_i = \lambda_i w$ for each i with $\lambda_i \in K$ then there is some $v \in V$ such that $v|A_i = \lambda_i v$ for each i.*

Proof. Replacing W with $K \cdot w$ and V with $T^{-1}(K \cdot w)$, we may assume that W is one dimensional. Then, write $V = \oplus_j V_j$ with each V_j a simultaneous eigenspace for all the A_i. Since T is non-zero and W is itself a simultaneous eigenspace for the family $\{A_i\}$, one of the V_j must be a λ_i-eigenspace for each A_i. Since all of the λ_i are in K, V_j has some bonafide eigenvector v which proves the lemma. □

Remark 5.8. One cannot in general find an $\{A_i\}$-eigenvector that maps to w. The set of all such eigenvectors with the same eigenvalues as w might lie entirely within the kernel of T. It is for this reason that in Theorem 5.6 we cannot conclude solely from the linear algebra of the situation that there is an overconvergent eigensymbol that *lifts* φ.

Remark 5.9. Note that the proof of Theorem 5.6 is essentially constructive. First one lifts φ to any overconvergent symbol in $\mathrm{Symb}_\Gamma(\mathcal{D}^\dagger(\mathbf{Z}_p, 1))$. This lifting is explicitly described in Theorem 4.7. To get a lifting in $\mathrm{Symb}_\Gamma(\mathbf{D}_k[\mathbf{Z}_p, 1])$ (and thus X) one then applies $u := \lambda^{-1}U_p$ as described in Corollary 4.10. Set Φ equal to this lifting and let h be the slope of φ. Then the projection of Φ to $X^{(=h)}$ is given by $\Phi^0 := \lim_{n\to\infty} \Phi|u^{n!}$.

In the case when $h < k+1$, Φ^0 is the overconvergent symbol we are seeking (see Theorem 5.10 below). When the slope is exactly $k+1$, by Proposition 5.3, we know that the Hecke span of Φ^0 is finite dimensional. Thus, extracting a vector in this space with the same Hecke-eigenvalues as φ is just an exercise in linear algebra.

We now give an improvement of Theorem 5.6 in the case when the slope is non-critical (i.e. strictly less than $k+1$).

(Is it okay to include this theorem? My paper with Henri relies upon it. We currently refer to a preprint of yours where you prove it using Visik's theorem. If we do include it, comments are in order to explain that this was known by you for sometime already...)

Theorem 5.10. *We have that*

$$\mathrm{Symb}_\Gamma(\mathbf{D}_k[\mathbf{Z}_p, 1])^{(<k+1)} \to \mathrm{Symb}_\Gamma(L_k)^{(<k+1)}$$

is an isomorphism. That is, the specialization map restricted to the subspace where U_p acts with slope strictly less than $k+1$ is an isomorphism.

Proof. We first check that the kernel of specialization is contained in the subspace where U_p acts with slope greater than or equal to $k+1$. Let Φ be any symbol in the kernel of specialization. By definition,

$$(\Phi|U_p)(D) = \sum_{a=0}^{p-1} \Phi\left(\beta(a,p)D\right) \big|\beta(a,p).$$

Since Φ is in the kernel of specialization,

$$\Phi(D') = \sum_{j=k+1}^{\infty} a_j z^{-j} \frac{dz}{z}$$

for any divisor D'. By Lemma 5.5, for $j \geq k+1$,

$$\left|\left| \left(z^{-j}\frac{dz}{z}\right) \big|\beta(a,p)\right|\right| \leq p^{-(k+1)}$$

Thus,

$$||\Phi|U_p|| \leq p^{-(k+1)}||\Phi||$$

which proves that U_p acts with slope at least $k+1$ on Φ. In particular, the map considered in this theorem is injective.

The surjectivity of our map follows from Theorem 5.6. Indeed, let $\varphi \in \mathrm{Symb}_\Gamma(L_k)^\pm \otimes \overline{\mathbf{Q}}_p$ be a Hecke-eigensymbol. By the above arguments, the surjective map in the proof of Theorem 5.6,

$$X^{(=h)} \to \overline{\mathbf{Q}}_p \cdot \varphi,$$

is also injective. Thus, the overconvergent symbol Φ of Theorem 5.6 actually specializes to a non-zero multiple of φ. Hence, the specialization map is an isomorphism after extending scalars to $\overline{\mathbf{Q}}_p$ and thus an isomorphism before extending scalars. □

In [4], the following theorem on the slope $k+1$ subspace is proven.

Theorem 5.11. *Let $f = \sum_n a_n q^n$ be a normalized eigenform in $S_k(\Gamma_0(Np), \overline{\mathbf{Q}}_p)$ of slope $k+1$. If*

1. *there does not exist a system of eigenvalues occurring in*

$$S_2(\Gamma_1(Np), \omega^{k+2}, \overline{\mathbf{Q}}_p)$$

that is congruent to the system of eigenvalues $\{\ell a_\ell\}$,

2. *there is some $\ell \nmid Np$ such that $a_\ell - (\ell^k + 1)$ is a unit,*

3. *there is some $\ell \nmid Np$ such that $a_\ell - (\ell^{k+2} + \ell^{-1})$ is a unit,*

then

$$\mathrm{Symb}_\Gamma(\mathbf{D}_k[\mathbf{Z}_p, 1])_{(f)} \to \mathrm{Symb}_\Gamma(L_k)_{(f)}$$

is an isomorphism. (Here the subscript of (f) denotes the pseudo-eigenspace with the same eigenvalues as f.)

Remark 5.12. If f is the evil twin of a CM modular form that is ordinary at p, then f fails the hypothesis of Theorem 5.11. In section 8.1, numerical data will be presented that suggests that the above theorem is not true for such a form. In fact, it appears that there is an overconvergent Hecke-eigensymbol with the same system of eigenvalues of f lying in the kernel of specialization.

6 The connection to p-adic L-functions

Let f be an eigenform of weight two modular form and level N where $p||N$. (We are working in weight two only for convenience; the results of this section apply to higher weight modular forms with the appropriate modifications.) Attached to f, we have the classical modular symbol $\varphi_f^\pm \in \mathrm{Symb}_\Gamma(K_f)^\pm$ where K_f is a number field containing the system of eigenvalues of f. This symbol is defined by

$$\varphi_f^{\pm}(\{r\}-\{s\}) := \left(\int_r^s f(z)dz \pm \int_{-r}^{-s} f(z)dz\right)\Omega_f^{-1} \in K_f \tag{4}$$

where $r, s \in \mathbf{P}^1(\mathbf{Q})$ and Ω_f is the canonical period of f. The symbol $\varphi_f^{\pm}$ is a Hecke-eigensymbol with the same eigenvalues as f.

Let $\mathfrak{p}$ be some prime of K_f sitting over p and set K to be the completion of K_f at $\mathfrak{p}$. We then view $\varphi_f^{\pm}$ as a p-adic object by viewing it in $\mathrm{Symb}_\Gamma(K)^{\pm}$. Note that we are not assuming that f is new at p. Indeed, many of our applications will come from the case when f is one of the two p-stabilizations of a newform with good reduction at p.

Set $\lambda := a_p(f)$ the p-th Fourier coefficient of f viewed as an element of K. In the case when $\mathrm{ord}_p(\lambda) < 1$, the p-adic L-function of f can be defined from the data of the symbol φ_f^+. Indeed, the p-adic L-function $L_p(f)$ is the distribution on $\mathbf{Z}_p^\times$ given by:

$$L_p(f)(\mathbf{1}_{a+p^n\mathbf{Z}_p}) = \lambda^{-n}\varphi_f^+(\{a/p^n\}-\{\infty\})$$

(the sign should be checked here) where $\mathbf{1}_X$ is the characteristic function of X. (Note that the above data defines the value of $L_p(f)$ on locally constant functions. Since $\mathrm{ord}_p(\lambda) < 1$, this uniquely defines a distribution on $\mathbf{Z}_p^\times$.)

In the above definition, $L_p(f)$ is determined by the symbol φ_f^+ evaluated at infinitely many divisors. We will see that the p-adic L-function of f can be recovered by evaluating an overconvergent version of φ_f^+ at one divisor. We begin with a few lemmas.

Lemma 6.1. *For any $h < \infty$, the natural maps*

$$\mathrm{Symb}_\Gamma(\mathcal{D}(\mathbf{Z}_p))^{(<h)} \to \mathrm{Symb}_\Gamma(\mathbf{D}[\mathbf{Z}_p,1])^{(<h)} \to \mathrm{Symb}_\Gamma(\mathcal{D}^\dagger(\mathbf{Z}_p,1))^{(<h)}$$

are isomorphisms.

Proof. As the maps are clearly injective, we just need to show that any $\Phi \in \mathrm{Symb}_\Gamma(\mathcal{D}^\dagger(\mathbf{Z}_p,1))^{(<h)}$ actually takes values in $\mathcal{D}(\mathbf{Z}_p)$. Since $h < \infty$, U_p acts injectively on these spaces; since the spaces are finite dimensional, we thus have that every symbol is in the image of U_p. In particular, by Lemma 4.9, Φ takes values in $\mathbf{D}[\mathbf{Z}_p,1]$.

Moreover, Φ is in the image of U_p^n for every n and, thus, for some symbol

Ψ, we have

$$\begin{aligned}\Phi(D)(g) &= (\Psi|U_p^n)(D)(g) \\ &= \sum_{a=0}^{p^n-1} \left(\Psi\middle|\beta(a,p^n)\right)(D)(g) \\ &= \sum_{a=0}^{p^n-1} \left(\Psi\left(\beta(a,p^n)D\right)\middle|\beta(a,p^n)\right)(g) \\ &= \sum_{a=0}^{p^n-1} \Psi\left(\beta(a,p^n)D\right)\left(\beta(a,p^n)g\right)\end{aligned}$$

where $\beta(a,p^n) = \left(\begin{smallmatrix} 1 & a \\ 0 & p^n \end{smallmatrix}\right)$. If $g \in A[\mathbf{Z}_p, p^{-n}]$, then $\beta(a,p^n)g \in A[\mathbf{Z}_p, 1]$. Thus, the above computation shows that $\Phi(D)$ extends to $\mathcal{D}[\mathbf{Z}_p, p^{-n}]$ for all n and thus to $\mathcal{D}(\mathbf{Z}_p)$. □

The space of distributions $\mathcal{D}[\mathbf{Z}_p, r]$ is naturally a Banach space with its norm being given by $||\cdot||_r$ as defined in section 3.2. Since $\mathcal{D}(\mathbf{Z}_p)$ lies inside of $\mathcal{D}[\mathbf{Z}_p, r]$ for every $r > 0$, it inherits a family of norms $\{||\cdot||_r\}$ satisfying $||\mu||_{r_1} \geq ||\mu||_{r_2}$ for $r_1 \leq r_2$. It is natural to classify distributions $\mu \in \mathcal{D}(\mathbf{Z}_p)$ by the growth of $||\mu||_r$ as $r \to 0^+$.

Definition 6.2. For $\mu \in \mathcal{D}(\mathbf{Z}_p)$, we say that μ is *h-admissible* if $||\mu||_r$ is $\mathrm{O}(r^{-h})$ as $r \to 0^+$.

Lemma 6.3. *Let $\Phi \in \mathrm{Symb}_\Gamma(\mathcal{D}(\mathbf{Z}_p))$ be an eigensymbol of slope h. For any divisor $D \in \Delta_0$, we have that $\Phi(D)$ is an h-admissible distribution.*

Proof. Let the U_p-eigenvalue of Φ be λ (which has valuation h). Let r be any real number greater than 0 and n any positive integer. Then for each $D \in \Delta_0$, we have

$$\begin{aligned}||\Phi(D)||_{\frac{r}{p^n}} &= |\lambda|^{-n} ||(\Phi\middle|U_p^n)(D)||_{\frac{r}{p^n}} \\ &\leq |\lambda|^{-n} \max_{0\leq a\leq p^n-1} ||\Phi(\beta(a,p^n)D)\middle|\beta(a,p^n)||_{\frac{r}{p^n}} \\ &\leq |\lambda|^{-n} \max_{0\leq a\leq p^n-1} ||\Phi(\beta(a,p^n)D)||_r \\ &\leq |\lambda|^{-n} ||\Phi||_r.\end{aligned}$$

The second to last inequality uses the fact that for any $\mu \in \mathcal{D}[\mathbf{Z}_p, r]$ and any $\gamma \in \Sigma_0(p)$ with $\det(\gamma) = p^n$, we have

$$||\mu|\gamma||_{\frac{r}{p^n}} \leq ||\mu||_r.$$

Thus, by definition, $\Phi(D)$ is h-admissible. □

Proposition 6.4. *Assume that the symbol φ_f^+ defined in (4) has slope strictly less than 1. Let Φ_f be the unique overconvergent eigensymbol that specializes to φ_f^+ as guaranteed by Theorem 5.6. Then*

$$L_p(f) = \Phi_f\left(\{0\} - \{\infty\}\right).$$

Proof. First note that by Lemma 6.1, Φ_f actually takes values in $\mathcal{D}(\mathbf{Z}_p)$. Furthermore, by Lemma 6.3, $\Phi_f(\{0\} - \{\infty\})$ is a 1-admissible distribution. Thus, by a theorem of Visik ([6]), to check the equality of this proposition, it suffices to check that $\Phi_f(\{0\} - \{\infty\})$ and $L_p(f)$ agree on locally constant functions.

If the U_p-eigenvalue of Φ is λ, we have

$$\begin{aligned}\Phi_f(\{0\} - \{\infty\}) &= \frac{1}{\lambda^n}(\Phi_f|U_p^n)(\{0\} - \{\infty\}) \\ &= \frac{1}{\lambda^n}\sum_{a=0}^{p^n-1}\Phi_f\left(\{a/p^n\} - \{\infty\}\right)\big|\beta(a,p^n).\end{aligned}$$

For any distribution $\mu \in \mathcal{D}(\mathbf{Z}_p)$, the support of $\mu\big|\beta(a,p^n)$ is contained in $a + p^n\mathbf{Z}_p$. Thus,

$$\begin{aligned}\Phi(\{0\} - \{\infty\})(\mathbf{1}_{a+p^n\mathbf{Z}_p}) &= \lambda^{-n}\left(\Phi_f\left(\{a/p^n\} - \{\infty\}\right)\big|\beta(a,p^n)\right)(\mathbf{1}_{a+p^n\mathbf{Z}_p}) \\ &= \lambda^{-n}\Phi_f\left(\{a/p^n\} - \{\infty\}\right)(\mathbf{1}_{\mathbf{Z}_p}) \\ &= \lambda^{-n}\rho_k^*(\Phi_f)\left(\{a/p^n\} - \{\infty\}\right) \\ &= \lambda^{-n}\varphi_f^+\left(\{a/p^n\} - \{\infty\}\right) \\ &= L_p(f)(\mathbf{1}_{a+p^n\mathbf{Z}_p})\end{aligned}$$

which proves the theorem. □

In light of the previous proposition, the following is a natural definition of the p-adic L-function of a modular form of critical slope.

Definition 6.5. Let f be an eigenform in $S_k(\Gamma_0(Np), \overline{\mathbf{Q}}_p)$ of slope $k+1$ that satisfies the hypotheses of Theorem 5.11. Let Φ_f be the unique overconvergent eigensymbol (of Theorem 5.11) that specializes to φ_f^+. Define the p-adic L-function of f to be

$$L_p(f) := \Phi_f(\{0\} - \{\infty\}) \in \mathcal{D}(\mathbf{Z}_p).$$

Note that this definition applies to the evil twin of an eigenform with good ordinary reduction at p (as long as it satisfies the hypotheses of Theorem 5.11). Thus, as in the supersingular case, one can naturally attach *two* p-adic L-functions to an eigenform that has good ordinary reduction at p.

The following proposition describes the interpolation property of this critical p-adic L-function. It is a formal consequence of the fact that Φ_f is a U_p-eigensymbol lifting ϕ_f^+.

Proposition 6.6. *Let f be an eigenform of slope $k+1$ as in Definition 6.5. Let χ be a character of $\mathbf{Z}_p^\times$ of order $n > 1$ that factors through $1 + p\mathbf{Z}_p$. Then for $0 \leq j \leq k$, we have*

$$\int_{\mathbf{Z}_p^\times} x^j \chi(x) dL_p(f)(x) = \frac{1}{\lambda^{n+1}} \cdot \frac{p^{(n+1)(j+1)}}{(-2\pi i)^j} \cdot \frac{j!}{\tau(\overline{\chi})} \cdot \frac{L(f, \overline{\chi}, j+1)}{\Omega_f}$$

where $\tau(\chi)$ is a Gauss sum and Ω_f is the canonical period of f.

Note that since $L_p(f)$ is a $(k+1)$-admissible distribution, it is *not* uniquely determined by the above interpolation property. To uniquely determine this distribution by interpolation, one would also need to specify its values at the characters of the form $x^{k+1}\chi$. We point out here that our method of producing $L_p(f)$ from overconvergent modular symbols does not directly give a way of understanding its values at such characters.

Examples of these critical p-adic L-functions (especially their zeroes) will be given in section 8.

7 Finite approximation modules

In this section, we will discuss how one can actually compute approximations to the eigensymbol of Theorem 5.6.

7.1 Approximating distributions

Since we want to be able to perform explicit computations in spaces of overconvergent modular symbols, we need some way of storing a distribution on a computer; that is, we need a systematic way of approximating a distribution by a finite amount of data.

One first guess on how to form such an approximation to an element of $\mathcal{D}^\dagger(\mathbf{Z}_p, 1)$ is to view it as a log-differential in $\Omega_{\log}(W(\mathbf{Z}_p, 1))$. Then for two integers M and N, one can store its first M coefficients mod p^N. Unfortunately, these approximations are not stable under the action of $\Sigma_0(p)$. Indeed, for any fixed $r \geq 1$, the subspace

$$\left\{ \sum_{j=0}^{\infty} a_j z^{-j} \frac{dz}{z} \in \Omega_{\log}(W(\mathbf{Z}_p, 1)) \;\middle|\; a_0 = a_1 = \cdots = a_r = 0; a_j \in \mathbf{Z}_p \right\}$$

is not $\Sigma_0(p)$-stable.

Let $\Omega_0 \subseteq \Omega_{\log}(W(\mathbf{Z}_p, 1))$ be the subspace of log-differentials all of whose coefficients are integral. In [4], the following $\Sigma_0(p)$-stable filtration on Ω_0 is introduced:

$$\mathrm{Fil}^r(\Omega_0) = \left\{ \sum_{j=0}^{\infty} a_j z^{-j} \frac{dz}{z} \in \Omega_0 \;\middle|\; a_j \in p^{r-j}\mathbf{Z}_p \right\}.$$

This filtration yields a systematic method of approximating distributions that is compatible with the $\Sigma_0(p)$-action. Namely,

Definition 7.1. For $M > 0$, define the M-th *finite approximation module* of Ω_0 to be

$$\mathcal{F}(M) := \Omega_0 / \operatorname{Fil}^M(\Omega_0).$$

Proposition 7.2. *We have that $\mathcal{F}(M)$ is a $\Sigma_0(p)$-module and that the map*

$$\begin{aligned} \mathcal{F}(M) &\longrightarrow (\mathbf{Z}/p^M\mathbf{Z}) \times (\mathbf{Z}/p^{M-1}\mathbf{Z}) \times \cdots \times (\mathbf{Z}/p\mathbf{Z}) \\ \overline{\omega} &\mapsto (a_0 + p^M\mathbf{Z}_p, a_1 + p^{M-1}\mathbf{Z}_p, \ldots, a_{M-1} + p\mathbf{Z}_p) \end{aligned}$$

is an isomorphism. (Here $\omega = \sum_{j=0}^{\infty} a_j z^{-j} \frac{dz}{z}$.)

Proof. Since $\operatorname{Fil}^M(\Omega_0)$ is a $\Sigma_0(p)$-module, $\mathcal{F}(M)$ is also a $\Sigma_0(p)$-module. The fact that this map is an isomorphism follows directly from the definition of the filtration. $\square$

Proposition 7.2 tells us that $\mathcal{F}(M)$ is a finite set which is thus easily represented on a computer. For a given element $\omega \in \Omega_0$, we can store its image in $\mathcal{F}(M)$ as a sequence of integers modulo various powers of p.

One downside to this description of $\mathcal{F}(M)$ is that it only allows us to approximate log-differentials with integral coefficients. However, as we saw in solving the difference equation, it will be important to be able to work with log-differentials whose coefficients are not integral (not even bounded!). To remedy this problem, consider the space:

$$\mathcal{K}_0 = \left\{ \sum_{j=0}^{\infty} a_j z^{-j} \frac{dz}{z} \;\middle|\; p^j a_j \in \mathbf{Z}_p \right\} \subseteq \Omega_{\log}(W(\mathbf{Z}_p, p)).$$

Note that by definition, $\Omega_0 \cap p^M\mathcal{K}_0 = \operatorname{Fil}^M(\Omega_0)$. This gives an alternate description of $\mathcal{F}(M)$:

$$\mathcal{F}(M) \cong \Omega_0/(\Omega_0 \cap p^M\mathcal{K}_0) \cong (\Omega_0 + p^M\mathcal{K}_0)/\mathcal{K}_0.$$

Using this description, we can approximate any log-differential $\omega \in \Omega_{\log}(W(\mathbf{Z}_p, 1))$. Namely, fix some r such that the first M coefficients of $p^r\omega$ are integral. Then $p^r\omega \in \Omega_0 + p^M\mathcal{K}_0$ and one can consider its image in $\mathcal{F}(M)$. Such an approximation will be useful as long as r is small relative to M.

7.2 Solving the difference equation in $\mathcal{F}(M)$

Theorem 4.5 gives an explicit solution to the difference equation $\mu|\Delta = \nu$ in $\mathcal{D}^\dagger(\mathbf{Z}_p, 1)$ which required division by powers of p. Following the same approach in $\mathcal{F}(M)$ leads to mild difficulties since p does not act invertibly on $\mathcal{F}(M)$. To remedy this problem, we will need to assume that ν is divisible by a power of p that is small relative to M. The following lemma keeps track of the denominators that appear from solving the difference equation in $\mathcal{D}^\dagger(\mathbf{Z}_p, 1)$; from this lemma, it will be easy to describe how to solve the difference equation in $\mathcal{F}(M)$.

Lemma 7.3. *Let $\mu \in \mathcal{D}^\dagger(\mathbf{Z}_p, 1)$ such that $\mu\big|\Delta = \nu \in \Omega_0$. If $m, M \geq 0$ are integers for which $p^m > M + 1$ then*

$$p^m \mu \in \Omega_0 + p^M \mathcal{K}_0.$$

Proof. From the explicit construction of Theorem 4.5, we know that μ equals a sum of terms of the form $\eta_j/(j+1)$ scaled by integral coefficients. Thus, to prove the lemma, it suffices to prove that if $p^m > M + 1$ then for all $j \geq 0$,

$$p^m \cdot \frac{1}{j+1} \cdot \eta_j \in \Omega^0 + p^M \mathcal{K}_0.$$

For this, it suffices to show that for all integers n, j with $n \geq j \geq 0$,

$$\frac{p^m}{j+1}\binom{n}{j} b_{n-j} \in \mathbf{Z}_p \quad \text{if} \quad n < M,$$
$$p^n \cdot \frac{p^m}{j+1}\binom{n}{j} b_{n-j} \in p^M \mathbf{Z}_p \quad \text{if} \quad n \geq M.$$

If $n < M$ then $j < M$ and hence, $p^m > j + 1$. It then follows that $\mathrm{ord}_p(p^m/(j+1)) \geq 1$ and so the first assertion follows from the Clausen-von Staudt theorem.

On the other hand, if $n \geq M$ then $n = M + r$ with $r \geq 0$ and so $p^{m+r} > M + 1 + r = n + 1$. Thus $\mathrm{ord}_p(p^{m+r}/(j+1)) \geq 1$ for every $j \geq 0$ with $j \leq n$. Again, from the Clausen-von Staudt theorem it follows that

$$\frac{p^{m+r}}{j+1}\binom{n}{j} b_{n-j} \in \mathbf{Z}_p$$

and consequently that

$$p^n \cdot \frac{p^m}{j+1}\binom{n}{j} b_{n-j} = p^M \cdot \frac{p^{m+r}}{j+1}\binom{n}{j} b_{n-j} \in p^M \mathbf{Z}_p.$$

□

Corollary 7.4. *If $\overline{\nu} \in p^m \mathcal{F}(M)$ has total measure zero with $p^m > M + 1$ then there exists $\overline{\mu} \in \mathcal{F}(M)$ such that $\overline{\mu}\big|\Delta = \overline{\nu}$.*

Proof. First lift $p^{-m}\overline{\nu}$ to some element ν of Ω_0. Then, solving the difference equation, yields some μ such that $\mu\big|\Delta = \nu$. By Lemma 7.3, we have that $p^m \mu \in \Omega_0 + p^M \mathcal{K}_0$. Since $\mathcal{F}(M) \cong (\Omega_0 + p^M \mathcal{K}_0)/\mathcal{K}_0$, we can set $\overline{\mu}$ to be the image of $p^m \mu$ in $\mathcal{F}(M)$. Then $\overline{\mu}\big|\Delta = \overline{\nu}$ as desired. □

7.3 Lifting symbols

We begin by describing approximations to elements of L_k that are compatible with our approximations of distributions (relative to the specialization map). For $M > 0$ define,

$$L_k(M) = \left\{ \sum_{j=0}^{k} a_j Z^k \;\middle|\; a_j \in \mathbf{Z}/p^{M-j+e_j}\mathbf{Z} \right\}$$

where $e_j = \mathrm{ord}_p \binom{k}{j}$. (The e_j terms are present to account for the powers of p that appear in the specialization map ρ_k^*.)

The map ρ_k defined in section 4.3 reduces to give a map $\mathcal{F}(M) \xrightarrow{\overline{\rho}_k} L_k(M)$ which in turn induces

$$\overline{\rho}_k^* : \mathrm{Symb}_\Gamma(\mathcal{F}(M)) \to \mathrm{Symb}_\Gamma(L_k(M)).$$

It is with respect to this map that we will be lifting our classical modular symbols.

In the following theorem, let $e = \max_{0 \le j \le k} \mathrm{ord}_p \begin{pmatrix} k \\ j \end{pmatrix}$.

Theorem 7.5. *Let $\overline{\varphi} \in p^{m+e}\,\mathrm{Symb}_\Gamma(L_k(M))$ for some integer $m \geq 0$. If $p^m > M+1$ then there exists a modular symbol $\overline{\Phi} \in \mathrm{Symb}_\Gamma(\mathcal{F}(M))$ such that $\rho_k^*(\overline{\Phi}) = \overline{\varphi}$.*

Proof. We can proceed as in Theorem 4.7 with minor modifications. Note that in defining $\overline{\Phi}(D_i')$ (resp. $\overline{\Phi}(D_i''))$ we need to divide by 2 (resp. 3). Since $p^m > M+1$, m is necessarily positive and this division is possible.

To define $\overline{\Phi}(D_\infty)$, we begin by forming some lift $\overline{\nu}$ of $\overline{\varphi}(D_\infty)$ that is divisible by p^m which is possible since $\overline{\varphi}(D_\infty)$ is divisible by p^{m+e}. (The extra e in the exponent of p is needed because in the lifting one needs to divide by certain binomial coefficients.) Then by Corollary 7.4 we can solve the difference equation with $\overline{\nu}$. The remainder of the proof of Theorem 4.7 then carries through in this setting. □

7.4 Iterating $\lambda^{-1}U_p$

Let $\Phi \in \mathrm{Symb}_\Gamma(\mathcal{F}(M))$ be a symbol lifting a classical U_p-eigensymbol φ with eigenvalue λ. The first step in producing a Hecke-eigensymbol lifting φ is to project Φ to the slope h subspace where h is the valuation of λ. To do this projection, we must iterate the operator $\lambda^{-1}U_p$ on the symbol Φ as in Proposition 5.3.

In this section, we point out that the proof of Proposition 5.3 can be made completely explicit. Namely, for the the class of symbols $\Phi \in \mathrm{Symb}_\Gamma(\mathcal{F}_M)$ satisfying

$$||\Phi(D)(x^j)|| \leq p^{-(k+1-j)} \text{ for all } D \in \Delta_0,$$

the symbol $\lambda^{-1}(\Phi|U_p)$ naturally lives in $\mathrm{Symb}_\Gamma(\mathcal{F}_M)$. (If λ is a unit, there is nothing to show here; however, if λ has positive valuation, it's *a priori* possible that $\lambda^{-1}(\Phi|U_p)$ would be known to less accuracy than Φ.) We have

$$
\begin{aligned}
(\Phi|U_p)(D) &= \sum_{a=0}^{p-1} \Phi(\beta(a,p)D)\big|\beta(a,p) \\
&= \sum_{a=0}^{p-1} \left(\sum_{j=0}^{M-1} c_j^{(a)} z^{-j} \frac{dz}{z} \right) \Big| \beta(a,p) \\
&= \sum_{j=0}^{M-1} \sum_{a=0}^{p-1} c_j^{(a)} \left(z^{-j} \frac{dz}{z} \right) \Big| \beta(a,p).
\end{aligned}
$$

Recall that by definition $c_j^{(a)}$ is only defined modulo p^{M-j}.

We wish to make sense of how to divide this last expression by λ. Since

$$\overline{\rho}_k^*(\lambda^{-1}\Phi\big|U_p) = \overline{\rho}_k^*(\Phi),$$

we know the first $k+1$ moments of $(\Phi\big|U_p)(D)$ to the correct accuracy. In Lemma 5.5, $\left(z^{-j}\frac{dz}{z}\right)\big|\beta(a,p)$ is expressed as p^j times another log-differential with integral coefficients. Thus, one can perform this division by λ on these terms for $j \geq k+1 \geq \mathrm{ord}_p(\lambda)$.

Thus, all that is left to examine is the terms

$$\lambda^{-1} c_j^{(a)} \left(z^{-j} \frac{dz}{z} \Big| \beta(a,p) \right)$$

for $j < k+1$. At this point, we need to use our hypothesis on the symbol Φ, namely, that $c_j^{(a)}$ is divisible by p^{k+1-j}. Combining this power of p with the other factor of p^j arising from $\left(z^{-j}\frac{dz}{z}\right)\big|\beta(a,p)$, gives us an extra factor of p^{k+1}. Since the λ has valuation less than or equal to $k+1$, we can now perform the needed to division. Lastly, note that $c_j^{(a)}$ is only known modulo p^{M-j} and thus its quotient by p^{k+1-j} is only known modulo $p^{M-(k+1)}$. Thus, the above argument only defines the $(k+1)$-st and higher coefficients to sufficient accuracy. Fortunately, as noted above, we know the first $k+1$ coefficients to the correct accuracy by (7.4).

7.5 Forming Hecke-eigensymbols

8 Examples, p-adic L-functions and Data

8.1 Examples

Using the algorithms described in the previous section, we numerically explored the slope one subspace of $\mathrm{Symb}_\Gamma(\mathbf{D}_k[\mathbf{Z}_p, 1])$ with $k = 0$ and $\Gamma = \Gamma_0(Np)$ for various values of N and p.

Example 8.1 **($N = 11$ and $p = 3$).** For $\Gamma = \Gamma_0(33)$, the slope one subspace of $\mathrm{Symb}_\Gamma(\mathbf{Q}_3)$ is three dimensional. Indeed, let $f = \sum_n a_n q^n$ be the modular form corresponding to the elliptic curve $X_0(11)$. If α and β are the two roots of $x^2 - a_3 x + 3$, ordered so that

$$0 = \mathrm{ord}_p(\alpha) < \mathrm{ord}_p(\beta) = 1,$$

then there are two modular forms, f_α and f_β, of level 33 with the same eigenvalues as f away from 3. At 3, we have

$$f_\alpha | U_3 = \alpha f_\alpha \quad \text{and} \quad f_\beta | U_3 = \beta f_\beta.$$

So the eigenform f_β is of slope one and thus gives rise to modular symbols $\varphi_\beta^\pm \in \mathrm{Symb}_\Gamma(\mathbf{Q}_3)^\pm$ (for each choice of sign) such that

$$\begin{aligned} \varphi_\beta^\pm | T_\ell &= a_\ell \varphi_\beta^\pm \quad (\text{for } \ell \nmid 33), \\ \varphi_\beta^\pm | U_p &= \beta \varphi_\beta^\pm. \end{aligned}$$

The third dimension of $\mathrm{Symb}_\Gamma(\mathbf{Q}_3)^{(=1)}$ comes from an Eisenstein symbol. That is, there is a symbol $\varphi_{eis} \in \mathrm{Symb}_\Gamma(\mathbf{Q}_3)^+$ such that

$$\begin{aligned} \varphi_{eis} \big| T_\ell &= (\ell + 1) \cdot \varphi_{eis}, \\ \varphi_{eis} \big| U_3 &= 3 \cdot \varphi_{eis} \text{ and} \\ \varphi_{eis} \big| U_{11} &= \varphi_{eis}. \end{aligned}$$

The hypotheses of Theorem 5.11 are met for the modular form f_β. To see this, one needs to determine if the sequence of Hecke-eigenvalues $\{\ell a_\ell\}$ modulo 3 occurs in

$$S_2(\Gamma_1(33), \omega^2, \mathbf{C}) = S_2(\Gamma_0(33), \mathbf{C}).$$

This space of cuspforms is 2 dimensional with one dimension coming from f_β and the other dimension comes from an elliptic curve of conductor 33 that is congruent to f mod 3. Thus, the only sequence of Hecke-eigenvalues that occurs is $\{a_\ell\}$.

We are therefore guaranteed that there exists an overconvergent Hecke-eigensymbol $\Phi_\beta^\pm$ lifting $\varphi_\beta^\pm$. Numerically, we were able to find this symbol (for both choices of sign) in $\mathrm{Symb}_\Gamma(\mathcal{F}_M)$ for M around 100.

Since Theorem 5.11 does not apply to Eisenstein symbols, we are not a priori ensured that φ_{eis} lifts to an overconvergent eigensymbol. Nonetheless, numerically, we did find a Hecke-eigensymbol Φ_{eis}^+ in $\mathrm{Symb}_\Gamma(\mathcal{F}_M)^+$ that specializes to φ_{eis}. (**This computation and the one below were actually carried out by Glenn and Vincent.**)

These three lifts do not account for all of the symbols in the slope one subspace of $\mathrm{Symb}_\Gamma(\mathcal{F}_M)$. We also found a three dimensional subspace of eigensymbols in the kernel of specialization. Namely, there was a symbol Φ_{eis}^- in

$\mathrm{Symb}_\Gamma(\mathbf{D}[\mathbf{Z}_p, 1])^-$ such that

$$\begin{aligned}\Phi^-_{eis}\big|T_\ell &= (\ell+1)\cdot\Phi^-_{eis}\\ \Phi^-_{eis}\big|U_3 &= 3\cdot\Phi^-_{eis},\\ \Phi^-_{eis}\big|U_{11} &= \Phi^-_{eis}\end{aligned}$$

and such that $\rho_k^*(\Phi) = 0$. (**I think I recall that there is an easy explanation for the presence of this symbol.**) There were also two slope one Hecke-eigensymbols in the kernel of specialization whose eigenvalues appeared to lie in a degree two extension of $\mathbf{Q}_3$. For example, the characteristic polynomial of U_3 acting on the span of these two symbols was congruent to

$$x^2 + 6501x + 4248 \pmod{3^8}$$

which has its roots defined over $\mathbf{Q}_3(\sqrt{-3})$.

The presense of these last two symbols can be explained by Hida theory. First note that the Hida algebra of tame level 11 with $p = 3$ has rank 2 over Λ. This is true because there is an elliptic curve with conductor 33 that is congruent mod 3 to $X_0(11)$. Then, specializing the Hida algebra to weight 0 (i.e. $k = -2$) yields a two dimensional space of overconvergent modular forms of weight 0 and level 33. The image under the theta operator of this two dimensional space should correspond to the two dimensional space found in the kernel of specialization.

Example 8.2 ($N = 11$ and $p = 5$). In this case, the slope one subspace of $\mathrm{Symb}_\Gamma(\mathbf{Q}_p)$ is again three dimensional with one dimension coming from an Eisenstein symbol and the other two dimensions coming from the slope one 5-stabilization of the modular form attached to $X_0(11)$. We point out this example in particular because $X_0(11)$ has rational 5 torsion and thus its associated modular form is congruent to an Eisenstein series mod 5. In particular, the hypotheses of Theorem 5.11 are not satisfied for this cuspform. In particular, we are not guaranteed that there exist overconvergent lifts of any of these symbols.

Numerically though, we did found Hecke-eigensymbols lifting all three of these symbols in $\mathrm{Symb}_\Gamma(\mathcal{F}_M)$. Note that the standard 5-adic L-function of $X_0(11)$ has a positive μ-invariant. We will see that this seems to have a profound affect on the zeroes of the 5-adic L-function of the corresponding slope one form.

Example 8.3 ($N = 32$ and $p = 5$). Again the slope one subspace of $\mathrm{Symb}_\Gamma(\mathbf{Q}_p)$ is three dimensional with one dimension coming from an Eisenstein symbol and the other two dimensions coming from the 5-stabilization of the modular form attached to an elliptic curve of conductor 32. Again, in this example, this cuspform does not satisfy the hypotheses of Theorem 5.11. However, the reason it fails these hypotheses is not because its mod p representation is reducible; instead, it fails these hypotheses because it arises from a CM elliptic curve.

To see this, let f be any modular form corresponding to a CM elliptic curve with ordinary reduction at p. Since the mod p Galois representation of a CM elliptic curve is split on inertia, by Gross' tameness result [2], there exists a mod p companion form g of level N and weight $p-1$ such that θf is congruent to

$\theta^2 g$ mod p. (Here $\theta = q\frac{d}{dq}$ and increases the weight of a mod p modular form by $p+1$.) Thus, the system of eigenvalues associated to θf occurs in

$$S_{3p+1}(\Gamma_0(N), \overline{\mathbf{F}}_p).$$

By [1, Lemma 3.3], for $j \geq 2$, any system of eigenvalues in

$$S_j(\Gamma_0(N), \overline{\mathbf{F}}_p)$$

also occurs in

$$S_2(\Gamma_1(Np), \omega^{j-2}, \overline{\mathbf{F}}_p).$$

Hence, the system of eigenvalues of θf occurs in

$$S_2(\Gamma_1(Np), \omega^{3p-1}, \overline{\mathbf{F}}_p) = S_2(\Gamma_1(Np), \omega^2, \overline{\mathbf{F}}_p).$$

Finally, by [1, ?], this system of eigenvalues lifts to characteristic zero, which exactly violates the hypotheses of Theorem 5.11.

Unlike in the previous examples, we were not able to find a Hecke-eigensymbol in $\mathrm{Symb}_\Gamma(\mathcal{F}_M)$ lifting the classical symbol arising from this CM curve. However, numerically, we found a Hecke-eigensymbol in $\mathrm{Symb}_\Gamma(\mathcal{F}_M)^\pm$ with the same eigenvalues as the classical symbol but whose specialization was zero. Furthermore, it appeared that up to scaling and choice of sign, there was a unique Hecke-eigensymbol in $\mathrm{Symb}_\Gamma(\mathcal{F}_M)^\pm$ with these eigenvalues. If this observation is true then one could define the p-adic L-function of this slope one cuspform to be the value of this symbol at $\{\infty\} - \{0\}$.

8.2 Power series representations of p-adic L-functions

We wish to study the p-adic L-functions of the critical slope symbols discussed in the previous section, especially their zeroes. To do this, we introduce in this section a power series representation of these distributions which are readily computable.

Let $\mu \in \mathcal{D}(\mathbf{Z}_p^\times)$ be any locally analytic distribution on $\mathbf{Z}_p^\times$. Then, by integrating, we can view μ as a function on $\mathrm{Hom}_{cont}(\mathbf{Z}_p^\times, \mathbf{C}_p^\times)$. Since $\mathbf{Z}_p^\times \cong (\mathbf{Z}/p\mathbf{Z})^\times \times (1+p\mathbf{Z}_p)$, by choosing a topological generator γ of $1+p\mathbf{Z}_p$, we can identify $\mathrm{Hom}_{cont}(\mathbf{Z}_p^\times, \mathbf{C}_p^\times)$ with $p-1$ copies of the open unit disc around 0 each indexed by a character of $(\mathbf{Z}/p\mathbf{Z})^\times$. Concretely, for ψ a character of $(\mathbf{Z}/p\mathbf{Z})^\times$ and z in the open unit disc around 0, we have a character $\chi_{\psi,z}$ on $\mathbf{Z}_p^\times$ defined by

$$\chi_{\psi,z}(x) = \psi(x) \cdot (z+1)^{\log_\gamma \langle x \rangle}$$

where $\langle \cdot \rangle$ is the projection of $\mathbf{Z}_p^\times$ onto $1+p\mathbf{Z}_p$ and $\log_\gamma(x) = \frac{\log_p(x)}{\log_p(\gamma)}$ for x a 1-unit.

Thus, for each fixed character ψ, μ gives rise to a function on the open unit disc. If we write this function as $L_p(\mu,\psi,z)$ then

$$\begin{aligned} L_p(\mu,\psi,z) &= \int_{\mathbf{Z}_p^\times} \chi_{\psi,z}(x)\ d\mu(x) \\ &= \int_{\mathbf{Z}_p^\times} \psi(x)\cdot(z+1)^{\log_\gamma\langle x\rangle}\ d\mu(x) \\ &= \int_{\mathbf{Z}_p^\times} \psi(x)\left(\sum_{n\geq 0}\binom{\log_\gamma\langle x\rangle}{n} z^n\right)\ d\mu(x) \\ &= \sum_{n\geq 0}\left(\int_{\mathbf{Z}_p^\times}\psi(x)\binom{\log_\gamma\langle x\rangle}{n}\ d\mu(x)\right) z^n \end{aligned}$$

By construction, $L_p(\mu,\psi,z)$ is in fact a rigid analytic function on the open unit disc and we will refer to this power series in z as the power series representation of $\mu \in \mathcal{D}(\mathbf{Z}_p^\times)$.

If μ arises as the p-adic L-function of some modular form f of level Np with $f\big|U_p = \lambda f$, we then write $L_p(f,\lambda,\psi,z)$ for $L(\mu,\psi,z)$. If ψ is the trivial character, we simply write $L_p(f,\lambda,z)$.

8.3 Computing p-adic L-functions

Let $\Phi_{f,\lambda}$ be an overconvergent eigensymbol with the same Hecke-eigenvalues as a modular form f of level Np such that $f\big|U_p = \lambda f$. We will now discuss how to compute the coefficients of $L_p(f,\lambda,\psi,z)$, the power series expansion of the distribution $\mu := \Phi_{f,\lambda}(\{0\}-\{\infty\})$, in terms of the modular symbol $\Phi_{f,\lambda}$.

From the proceeding section, we have that the n-th coefficient of $L_p(f,\lambda,\psi,z)$ is

$$\int_{\mathbf{Z}_p^\times}\psi(x)\binom{\log_\gamma\langle x\rangle}{n}\ d\mu(x) = \sum_{a=1}^{p-1}\psi(a)\int_{a+p\mathbf{Z}_p}\binom{\log_\gamma\left(x\cdot\{a\}^{-1}\right)}{n}\ d\mu.$$

Since

$$\log_\gamma\left(x\cdot\{a\}^{-1}\right) = \frac{1}{\log_p(\gamma)}\cdot\sum_{j\geq 1}\frac{(-1)^{j+1}}{j}\left(\frac{x}{\{a\}}-1\right)^j,$$

we have that

$$\binom{\log_\gamma\left(x\cdot\{a\}^{-1}\right)}{n} = \sum_{j\geq 1} c_j^{(n)}\left(\frac{x}{\{a\}}-1\right)^j$$

for some elements $c_j^{(n)} \in \mathbf{Q}_p$. In particular, the n-th coefficient of $L_p(f,\lambda,\psi,z)$ equals

$$\sum_{j\geq 1} c_j^{(n)}\left(\sum_{a=1}^{p-1}\psi(a)\{a\}^{-j}\int_{a+p\mathbf{Z}_p}(x-\{a\})^j\ d\mu\right). \tag{5}$$

To compute these integrals on $a+p\mathbf{Z}_p$, note that since $\Phi_{f,\lambda}$ is a U_p-eigensymbol, we have that

$$\Phi_{f,\lambda} = \frac{1}{\lambda}\Phi_{f,\lambda}\big|U_p = \frac{1}{\lambda}\sum_{b=0}^{p-1}\Phi_{f,\lambda}\big|\left(\begin{smallmatrix}1 & b\\ 0 & p\end{smallmatrix}\right).$$

For any distribution $\nu \in \mathcal{D}(\mathbf{Z}_p)$, $\nu\big|\left(\begin{smallmatrix}1 & b\\ 0 & p\end{smallmatrix}\right)$ has support inside of $b+p\mathbf{Z}_p$. Thus,

$$\begin{aligned}\int_{a+p\mathbf{Z}_p}(x-\{a\})^j\ d\mu &= \Phi_{f,\lambda}(D_\infty)\left(\mathbf{1}_{a+p\mathbf{Z}_p}(x)\cdot(x-\{a\})^j\right)\\ &= \frac{1}{\lambda}\left(\left(\Phi_{f,\lambda}\big|\left(\begin{smallmatrix}1 & a\\ 0 & p\end{smallmatrix}\right)\right)(D_\infty)\right)\left(\mathbf{1}_{a+p\mathbf{Z}_p}(x)\cdot(x-\{a\})^j\right)\\ &= \frac{1}{\lambda}\Phi_{f,\lambda}(D_{a,p})\left((a-\{a\})+px)^j\right)\\ &= \frac{1}{\lambda}\sum_{r=0}^{j}\binom{j}{r}(a-\{a\})^{j-r}p^r\cdot\Phi_{f,\lambda}(D_{a,p})(x^r)\end{aligned}$$

where $D_{a,p} = \{\frac{a}{p}\}-\{\infty\}$. This last expression is simply a linear combination of the moments of $\Phi_{f,\lambda}(D_{a,p})$. Thus, we have succeeded in expressing the power series $L_p(f,\lambda,\psi,z)$ in terms of the symbol $\Phi_{f,\lambda}$.

In practice, note that the values $\Phi_{f,\lambda}(D_{a,p})(x^r)$ are easy to compute once the symbol $\Phi_{f,\lambda}$ is known. Indeed, one uses Manin's continued fraction algorithm to express $\Phi_{f,\lambda}(D_{a,p})$ is terms of the values of $\Phi_{f,\lambda}$ on our generating set of divisors. Moreover, these distributions are stored precisely by their sequence of moments.

Another practical point to consider is how many terms in (5) one needs to sum to achieve a desired level of accuracy. To answer this, note that

$$\int_{a+p\mathbf{Z}_p}(x-\{a\})^j\ d\mu$$

is divisible by $p^{j-\mathrm{ord}_p(\lambda)}$ (as is seen from the above computation since a and $\{a\}$ are congruent mod p). The constants $c_j^{(n)}$ are independent of the form f and their p-adic valuations are easily computed. Thus, one can bound the size of the tail-end of the series in (5) and make an appropriate error estimate.

For the computations carried out in this paper (discussed in section 8.5), we added up enough terms to guarantee that we knew the exact value of the p-adic valuation of the coefficients of $L_p(f,\lambda,\psi,z)$. By knowing these p-adic valuations, we could then compute the newton polygon of these p-adic L-functions.

8.4 Twists

Let Φ be an overconvergent modular symbol for the group $\Gamma_0(Np)$ and let χ be some quadratic character of conductor m with $p \nmid m$. Define a new symbol by

$$\Phi_\chi = \sum_{b \pmod{m}}\chi(b)\cdot\Phi\ \Big|\left(\begin{smallmatrix}1 & b/m\\ 0 & 1\end{smallmatrix}\right)$$

which is invariant under the group $\Gamma_0(Npm^2)$ (and possibly invariant under a larger group if $(N,m) \neq 1$).

If Φ is a U_p-eigensymbol with eigenvalue λ, one has that Φ_χ is a U_p-eigensymbol with eigenvalue $\chi(p)\lambda$. In particular, the p-adic L-function of f twisted by χ is

$$\Phi_\chi(D_\infty) = \sum_{b \pmod m} \chi(b) \cdot \Phi\left(\{b/m\} - \{\infty\}\right) \Big| \left(\begin{smallmatrix} 1 & b/m \\ 0 & 1 \end{smallmatrix}\right).$$

Thus, the power series $L_p(f_\chi, \chi(p)\lambda, z)$ is computable using the formulas of the previous section.

8.5 Data of critical slope p-adic L-functions

We computed the newton polygons of the critical slope p-adic L-functions attached to $X_0(11)$ for $p = 3, 5$, $X_0(14)$ for $p = 3$ and the CM elliptic curve of conductor 32 for $p = 3$. Moreover, we did the same computation for all quadratic twists of conductor in size less than 200.

We begin by analyzing the data for $X_0(11)$ and $p = 3$. The following table lists the slopes of the zeroes of the critical p-adic L-function twisted by quadratic characters of conductor D.

D	zeros and their slopes
1	6 of slope $\frac{1}{3}$, 12 of slope $\frac{1}{12}$, 36 of slope $\frac{1}{36}$
5	2 of slope $\frac{1}{2}$, 6 of slope $\frac{1}{6}$, 14 of slope $\frac{1}{14}$, 36 of slope $\frac{1}{36}$
8	1 of slope ∞, 2 of slope $\frac{1}{2}$, 4 of slope $\frac{1}{4}$, 12 of slope $\frac{1}{12}$, 36 of slope $\frac{1}{36}$
13	1 of slope ∞, 2 of slope $\frac{1}{2}$, 4 of slope $\frac{1}{4}$, 12 of slope $\frac{1}{12}$, 36 of slope $\frac{1}{36}$
17	1 of slope ∞, 6 of slope $\frac{1}{3}$, 12 of slope $\frac{1}{12}$, 36 of slope $\frac{1}{36}$
28	1 of slope ∞, 2 of slope $\frac{1}{2}$, 6 of slope $\frac{1}{3}$, 16 of slope $\frac{1}{16}$, 36 of slope $\frac{1}{36}$

Note that in each example the zeroes of larger slope behave erratically while the ones of smaller slope begin to settle down. Indeed, all of these p-adic L-functions have 36 zeroes of slope $\frac{1}{36}$ and, in fact, each also had 108 zeroes of slope $\frac{1}{108}$.

Since the slopes of the zeroes close enough to the boundary appear to be stabilizing, it makes sense to ask how many "extra" zeroes there are of large slope. The following table gives a running total of the number of zeroes of each p-adic L-function as we take larger and larger discs about the origin inside of the unit disc. The last column is the difference between the total number of zeroes and smallest total number of zeroes appearing in the table.

D	number of zeroes in expanding discs	"extra" zeroes
1	6, 18, 54, 162	0
5	2, 8, 22, 58, 166	4
8	1, 3, 7, 19, 55, 163	1
13	1, 3, 7, 19, 55, 163	1
17	1, 7, 19, 55, 163	1
28	1, 3, 9, 25, 61, 169	7

It is naturally to guess that these extra zeroes are somehow related to the ordinary p-adic L-function attached to these curves. Indeed, in each of these examples, the number of extra zeroes is exactly equal to the λ-invariant (i.e. the number of zeroes) of the p-adic L-function of the ordinary p-stabilization of this curve.

Moreover, we computed all twists with $-200 < D < 200$ and D prime to 3. We found that if $r_n = (p^n(p-1))^{-1}$, the number of zeros in the open disc of radius p^{-r_n} equaled

$$p^{n-1}(p-1) + \lambda_D \tag{6}$$

for n large enough (i.e. $n \geq 3$) where λ_D is the λ-invariant of the ordinary p-adic L-function of $X_0(11)$ twisted by the quadratic character of conductor D.

We repeated this experiment for $X_0(14)$ with $p = 3$ for discriminants between -200 and 200. We found the exact same results with the number of zeroes in increasing discs being given by (6).

In repeating the experiment for $X_0(11)$ with $p = 5$, we observed something new. When twisting by characters with negative discriminants we observed the same behavior as described by (6). However, when twisting by characters with positive discriminants, the behavior was remarkably different. The following table illustrates this.

D	zeros and their slopes
1	4 of slope $\frac{1}{4}$, 20 of slope $\frac{1}{20}$, 100 of slope $\frac{1}{100}$
8	1 of slope ∞, 4 of slope $\frac{1}{4}$, 20 of slope $\frac{1}{20}$, 100 of slope $\frac{1}{100}$
12	2 of slope 1, 4 of slope $\frac{1}{4}$, 20 of slope $\frac{1}{20}$, 100 of slope $\frac{1}{100}$
13	1 of slope ∞, 4 of slope $\frac{1}{4}$, 20 of slope $\frac{1}{20}$, 100 of slope $\frac{1}{100}$
17	1 of slope ∞, 4 of slope $\frac{1}{4}$, 20 of slope $\frac{1}{20}$, 100 of slope $\frac{1}{100}$

In this case, the pattern of 4 zeroes of slope $\frac{1}{4}$, 20 zeroes of slope $\frac{1}{20}$ and 100 zeroes of slope $\frac{1}{100}$ appear in every example. Note that the zeroes of the p-adic logarithm have the same slopes that are appearing in this table. However, it is not possible that the p-adic logarithm divides these p-adic L-functions because of their interpolation property. (If this divisibility occurred this would translate into the vanishing at 1 of the associated complex L-series twisted by any character of p-power order and conductor which violates a theorem of Rohrlich [5])

From the data collected for all real quadratic twists of conductor less than 200, it appears that the number of zeroes of these twisted L-functions in the open disc of radius p^{-r_n} equals

$$p^n - 1 + \epsilon_D$$

where ϵ_D is some non-negative integer. At present, we have no interpretation of the constants ϵ_D.

However, we have a guess as to why the behavior in this example is different than the previous ones. The curve $X_0(11)$ and $p = 5$ has a non-zero μ-invariant. This is also true of any real quadratic twist, but not true of an imaginary quadratic twist. **(I should check if this is known or just conjectured.)** Moreover, the previous examples with $p = 3$ all had zero μ-invariant. It thus appears that the non-vanishing of μ of an ordinary p-adic L-function has an affect on the zeroes of the associated critical slope p-adic L-function.

Our last examples comes from the CM elliptic curve of conductor 32 with $p = 5$. This example is fundamentally different from the proceeding ones since the overconvergent Hecke-eigensymbol attached to the slope one p-stabilization of this form appears to be in the kernel of specialization. Thus, the associated power series of its p-adic L-function vanishes at $\zeta_n - 1$ for all n, where ζ_n is a primitive p^n-th root of unity. In particular, these p-adic L-functions are divisible by the p-adic log. We must have then that the number of zeroes of twists of this p-adic L-functions in the open disc of radius p^{-r_n} equals

$$p^n - 1 + \epsilon_D$$

as in the previous example.

Since these p-adic L-functions are divisible by $\log_p$ and are $\mathrm{O}(\log_p)$, it is natural to consider their quotient which is an Iwasawa function (up to a power of p). Our first guess as to what this Iwasawa function could be was the p-adic L-function attached to the ordinary p-stabilization of this curve. Unfortunately, this is not the case since $\epsilon_D \neq \lambda_D$ for many values of D. At present, we again have no interpretation of the constants ϵ_D.

References

[1] Ash and Stevens

[2] Gross

[3] Manin

[4] R. Pollack and G. Stevens, The "missing" p-adic L-function.

[5] Rohrlich

[6] Visik