Coleman's $\mathcal{L}$-invariant and Families of Modular forms

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§0. Statement of results.

Let p be a prime > 2 and let $\mathcal{X} := \mathbf{Z}/(p-1)\mathbf{Z} \times \mathbf{Z}_p$ with $\mathbf{Z}$ embedded in $\mathcal{X}$ diagonally. Let f be a classical newform of weight $k_0 + 2 \geq 2$ and assume that f is split multiplicative at p. Then Coleman has defined an $\mathcal{L}$-invariant $\mathcal{L}(f)$ which he conjectured should be equal to the Mazur-Tate-Teitelbaum $\mathcal{L}$-invariant. The purpose of this note is to outline a proof of Coleman's conjecture. More precisely we prove the following theorem.

Main Theorem. $L_p'(f, 1 + k_0/2) = \mathcal{L}(f) \cdot L_\infty(f, 1 + k_0/2)$.

This was proved by Ralph Greenberg and the author in the special case $k_0 = 0$ (weight 2) several years ago. Just as in the weight 2 case, the proof of the general case divides naturally into two steps (Theorems A and B below).

To state Theorems A and B, we first recall that Robert Coleman has constructed a p-adic analytic family f_k of overconvergent p-adic modular forms passing through our fixed newform f. This family is defined for k in an open set $B \subseteq \mathcal{X}$ containing k_0 and satisfies $f_{k_0+2} = f$. Coleman's family is an eigenfamily for the U-operator and we may therefore consider the eigenvalue $\alpha(k)$ of U acting on f_k. The function $\alpha(k)$ is a p-adic analytic function of $k \in B$ so we may consider the derivative of α at the special point $k_0 \in B$.

Theorem A. $L_p'(f, 1 + k_0/2) = -2 \cdot p^{-k_0/2} \cdot \alpha'(k_0) \cdot L_\infty(f, 1 + k_0/2)$.

Just as in the weight two case, the proof of Theorem A depends on the existence of a two variable p-adic L-function with certain properties. The existence of such a p-adic L-function was proven in the higher weight case about a year and a half ago. With the two-variable p-adic L-function in hand, the proof of theorem A proceeds exactly as in the weight two case. The details have been described elsewhere.

Theorem B. $\mathcal{L}(f) = -2 \cdot p^{-k_0/2} \cdot \alpha'(k_0)$.

This note is dedicated to proving Theorem B. The Main Theorem above is an immediate consequence of Theorems A and B.

§1. Coleman's $\mathcal{L}$-invariant.

We adopt Coleman's notations as in the BU monodromy proceedings volume with only one modification. Namely, we will add full level 2 structure to the moduli space. This rigidifies the setup and simplifies the calculation in (2) of Proposition 1 in section 2. (I have to admit that this point still confuses me. The calculation in (2) of proposition 1 really does seem to depend on this rigidification.) We fix a tame level N (the tame level of the newform f) and let X be the modular curve $X(Np, 2)$ with level Np structure (a cyclic

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subgroup of order Np) plus full level 2 structure. (If $2|N$ we assume that the additional level 2 structure extends the 2-part of the level N structure.) The rigid analytic space X^{an} underlying X is decomposed into the union of three disjoint parts, namely,

$$X^{an} = Z_\infty \cup W \cup Z_0$$

where Z_∞ and Z_0 are affinoids containing the ∞ and 0-cusps respectively, and W is the union of the supersingular annuli. Following Coleman, we write $W_\infty = Z_\infty \cup W$ and $W_0 = Z_0 \cup W$.

Let $Y = Y(N,p)$ denote X with the cusps deleted. Let E/Y be the universal elliptic curve with level structure over Y and let $\mathcal{H}$ be the relative de Rham cohomology sheaf over X with log singularities at the cusps. Then $\mathcal{H}$ is a coherent $\mathcal{O}$-module locally free of rank 2 over X. For any nonnegative integer k we let

$$\mathcal{H}_k := Symm^k(\mathcal{H}).$$

The Gauss-Manin connection $\nabla : \mathcal{H} \longrightarrow \mathcal{H} \otimes \Omega$ induces a connection

$$\nabla : \mathcal{H}_k \longrightarrow \mathcal{H}_k \otimes \Omega$$

for each integer $k \geq 0$, which we also call the Gauss-Manin connection.

The Deligne-Tate map preserves Z_∞ and extends to a wide open neighborhood of Z_∞ *properly* contained in W_∞. Accordingly, the Gauss-Manin connection is endowed with a natural frobenius structure over some sufficiently small wide open neighborhood of Z_∞. Katz spells out precisely how big this neighborhood can be, but this is a technical point that we will not need. It will be convenient to simplify the notation and write $Z_\infty^\dagger$ to denote such a sufficiently small wide open neighborhood of Z_∞ with the additional property that the intersection of $Z_\infty^\dagger$ with any supersingular annulus is a concentric subannulus.

We recall Coleman's definition of the $\mathcal{L}$-invariant $\mathcal{L}(f)$ of a split multiplicative p-newform f of weight $k+2 \geq 2$. Let $\mathcal{H}_k^*$ denote the complex of sheaves associated to $\mathcal{H}_k \xrightarrow{\nabla} \mathcal{H}_k \otimes \Omega$ and consider the hypercohomology $\mathbf{H}^1(X, \mathcal{H}_k^*)$ with respect to the covering $\{W_\infty, W_0\}$ of X. The Hecke operators act on this space and the systems of eigenvalues that occur in it are the same as those that occur in the space of classical modular forms of weight k and corresponding level. In particular, letting K be the field generated over $\mathbf{Q}_p$ by the eigenvalues of the Hecke operators acting on f, we obtain a $\mathbf{Q}_p$-subspace $H(f) \subseteq \mathbf{H}^1(X, \mathcal{H}_k^*)$ endowed with an action of the field K with the property that $H(f)$ is a 2-dimensional K-vector space on which the Hecke operators act as scalars according to the eigenvalues of f. Now what Coleman is able to do, using his theory of p-adic integration, is to endow $H(f)$ with a natural monodromy module structure in which the monodromy is *non-trivial.* Every two dimensional monodromy module with non-trivial monodromy has a well-definde $\mathcal{L}$-invariant. Thus Coleman's $\mathcal{L}$-invariant can be defined simply as the $\mathcal{L}$-invariant of Coleman's monodromy module.

We will use the more concrete definition that Coleman gives in his paper in the BU monodromy proceedings volume. For simplicity, we assume $k > 0$ so that there are no nonzero sections of $\mathcal{H}_k$ defined on all of W_∞ nor on all of W_0, i.e. $H^0(W_\infty, \mathcal{H}_k^*) =$

$H^0(W_0, \mathcal{H}_k^*) = 0$. On the other hand there are plenty of horizontal sections of $\mathcal{H}_k$ on $W = W_\infty \cap W_0$. Indeed, Coleman constructs two maps

$$\sigma, \rho : M_{k+2} \longrightarrow H^0(W, \mathcal{H}_k^*)$$

defined on the space M_{k+2} of classical modular forms of weight $k+2$ and appropriate level. The map σ is defined using Coleman's integration theory while the map ρ is defined in terms of residues.

For k an integer, let $M_{k+2}^\dagger$ denote the space of overconvergent p-adic modular forms of weight $k+2$ and appropriate level. If $k \geq 0$ we let

$$\kappa : M_{k+2}^\dagger \longrightarrow \mathcal{H}_k \otimes \Omega(Z_\infty^\dagger)$$

be the Kodoaira Spencer map. There is also a $\mathbf{Q}_p$-linear map

$$\nu : M_{-k}^\dagger \longrightarrow \mathcal{H}_k(Z_\infty^\dagger)$$

satisfying the equation

$$\nabla(\nu(g)) = \kappa(\theta^{k+1} g) \in \mathcal{H}_k \otimes \Omega(Z_\infty^\dagger)$$

for any $g \in M_{-k}$.

We now turn to the definitions of σ and ρ. Let $k \geq 0$ and $f \in M_{k+2}$ be a classical Hecke eigenform. Let α be the eigenvalue of the U-operator acting on f. We suppose $\alpha \neq 0$. The differential form $\omega_f := \kappa(f) \in \mathcal{H}_k \otimes \Omega(W_\infty)$ represents a cohomology class $[\omega_f] \in H^1(W_\infty, \mathcal{H}_k)$ and the Frobenius operator Φ acts on ω_f and also on $[\omega_f]$. Indeed, we have $\Phi([\omega_f]) = \frac{p^{k+1}}{\alpha} \cdot [\omega_f]$. Now Coleman's integration theory gives us a well-defined flabby antiderivative $I_\infty(f)$ defined on all of W_∞ which is analytic on the ordinary residue disks, is log-analytic on the supersingular annuli and satisfies the differential equation

$$\nabla(I_\infty(f)) = \omega_f \quad \text{on } W_\infty.$$

The additional property that characterizes $I_\infty(f)$ uniquely is that, though $I_\infty(f)$ need not be rigid analytic on W_∞ (or even on Z_∞), the section

$$I_\infty(f) - \frac{\alpha}{p^{k+1}} \Phi(I_\infty(f))$$

is rigid analytic on $Z_\infty^\dagger$ (i.e. not only on Z_∞, but also on some wide open neighborhood of Z_∞). Similar considerations give rise to a well-defined flabby solution $I_0(f)$ of the differential equation

$$\nabla(I_0(f)) = \omega_f \quad \text{on } W_0.$$

Now both $I_0(f)$ and $I_\infty(f)$ are defined on the overlap $W = W_\infty \cap W_0$. Coleman makes the following definition.

Definition 1. If $f \in M_{k+2}$ is a classical Hecke eigenform then we define $\sigma(f) \in H^0(W, \mathcal{H}_k^*)$ to be the horizontal section of $\mathcal{H}_k$ on W given by

$$\sigma(f) := I_\infty(f)|_W - I_0(f)|_W.$$

The residue map $\rho : M_{k+2} \longrightarrow H^0(W, \mathcal{H}_k^*)$ is easier to define. Indeed, ρ is defined on *all* overconvergent modular forms. Let

$$\text{Res} : \mathcal{H}_k \otimes \Omega(Z_\infty^\dagger) \longrightarrow H^0(W, \mathcal{H}_k^*)$$

be defined by $\text{Res}(\omega) :=$ the unique horizontal section of $\mathcal{H}_k$ on W whose restriction to $Z_\infty^\dagger \cap W$ is the residue of ω restricted to this disjoint union of oriented annuli. Given $f \in M_{k+2}^\dagger$ we let $\omega_f := \kappa(f) \in \mathcal{H}_k \times \Omega(Z_\infty^\dagger)$ and define $\rho(f)$ as follows.

Definition 2. Given $f \in M_{k+2}^\dagger$ we define

$$\rho(f) := \text{Res}(\omega_f).$$

Definition 3. Coleman's $\mathcal{L}$-invariant of a split multiplicative newform $f \in M_{k+2}$ is defined to be the unique element $\mathcal{L}(f) \in K$ for which

$$\sigma(f) = \mathcal{L}(f) \cdot \rho(f).$$

The existence and uniqueness of such an $\mathcal{L}$-invariant was, of course, proved by Coleman.

§2. Some families of modular forms.

First of all we have the Eisenstein family. For each integer k there is an overconvergent p-adic modular form E_k of weight k whose q-expansion is given by

$$E_k := 1 + 2\zeta_p(1-k)^{-1} \sum_{k \geq 1} \sigma_{k+1}^*(n) q^n.$$

Here $\zeta_p(s)$ is the Kubota–Leopoldt p-adic zeta function and when $k = 0$ the above equality is understood to mean $E_0 = 1$. (Recall $\zeta_p(s)$ has a simple pole at $s = 1$). For integral $k \geq 0$ we set

$$t_k := \frac{1}{2}\zeta_p(1+k) \cdot E_{-k}$$
$$G_k := \frac{1}{2}\zeta_p(-1-k) \cdot E_{k+2}$$

Then $t_k \in M_{-k}^\dagger$ is an overconvergent modular form of weight $-k$ and $G_k \in M_{k+2}$ is a classical modular form of weight $k+2$. The family t_k extends to a meromorphic family of Eisenstein series for $k \in \mathcal{X}$ with a simple pole at $k = 0$ and G_k defines a meromorphic family with a simple pole at $k=-2$. Moreover $G_k = t_{-2-k}$. The special point $k=0$ will play a crucial role in the proof of Theorem B.

Proposition 1.

1. *The family t_k, $k \in \mathcal{X}$, has a simple pole at $k=0$ with residue given by*

$$\lim_{k\to 0} kt_k = \frac{1}{2}\cdot\left(1-\frac{1}{p}\right).$$

2. *The residue of G_0 along any supersingular annulus is $1/2$:*

$$\rho(G_0) = \frac{1}{2}.$$

Proof. The first assertion is an immediate consequence of the well-known fact that the Kubota-Leopoldt p-adic zeta function $\zeta_p(s)$ has a simple pole at $s=1$ and that the residue at $s=1$ is given by

$$\lim_{s\to 1}(s-1)\zeta_p(s) = \left(1-\frac{1}{p}\right).$$

To prove the second assertion, we first consider the special case $N=1$. Then $\eta = \kappa(G_0)$ is a section of Ω over Y which extends to a meromorphic section over X with simple poles along the cusps. We want to compute

$$\mathrm{Res}(\eta) \in H^0(W).$$

We remark first of all that since the eigenvalues of the Hecke operators acting on η are known, they are also known on $\mathrm{Res}(\eta)$. Indeed, the eigenvalues are the same as those acting on constant functions on W. Hence $\mathrm{Res}(\eta)$ is a constant. To determine what the constant is we use the fact that the sum of all of the residues along the supersingular annuli and around the cusps contained in W_∞ is equal to zero. Now there are a total of three cusps in W_∞ corresponding to the three cusps of $X(2)$. The constant terms of G_0 are the same at all of these cusps since G_0 is modular of level p. Since the natural map $X \longrightarrow X_0(p)$ is ramified of order 2 at each of these cusps and since the constant term of G_0 at the infinity cusp is $(1-p)/24$ we conclude that the sum of the residues along the cusps is $(1-p)/4$. Hence the sum of the residues along the supersingular annuli is $(p-1)/4$. But a simple calculation shows that the number of supersingular annuli in X is $(p-1)/2$. Hence the residue along any supersingular annulus is $1/2$. This proves (2) when $N=1$.

The general case follows at once since for arbitrary N, the map $X(Np,2) \longrightarrow X(p,2)$ is unramified over the supersingular annuli. This completes the proof of the proposition.

We can remove Euler factors at p using the operator V on overconvergent modular forms defined on q-expansions by the formula $V(f)(q) = f(q^p)$. If F is an eigenform, then we let F^0 denote the eigenform obtained by removing the Euler factor at p. Thus, we have the families

$$\begin{aligned} t_k^0 &:= t_k - V(t_k) \\ G_k^0 &:= G_k - V(G_k) \\ f_k^0 &:= f_k - \alpha(k)V(f_k) \end{aligned}$$

For $k \geq 0$ we let $\eta_k := \kappa(G_k)$ and $\eta_k^0 := \kappa(G_k^0)$ where $\kappa : \mathcal{M}_{k+2} \longrightarrow \mathcal{H}_k \otimes \Omega$ is the Kodaira-Spencer map. We also set $g_k := \nu(t_k)$ and $g_k^0 := \nu(t_k^0)$. Then since $\theta^{k+1} t_k^0 = G_k^0$ it follows that

$$\nabla(g_k^0) = G_k^0.$$

Finally, for each integer $k \geq 0$ we may let $s_k := I_\infty(f_k)$ be the Coleman integral of f_k defined in section 1. Then s_k is a flabby section of $\mathcal{H}_k$ over W_∞. This section is characterized by the property that

$$s_k^0 := s_k - \frac{\alpha(k)}{p^{k+1}} \cdot \Phi(s_k)$$

is a rigid analytic section of $\mathcal{H}_k$ over $Z_\infty^\dagger$. Hence there is an overconvergent modular form $\phi_k^0 \in M_{-k}^\dagger$ such that

$$\nu(\phi_k^0) = s_k^0.$$

Hence $\theta^{k+1}(\phi_k^0) = f_k^0$. Finally, set

$$\begin{aligned} \omega_k &:= \kappa(f_k), \\ \omega_k^0 &:= \kappa(f_k^0). \end{aligned}$$

§3. Some Pairings.

As in the introduction, we fix an integer $k_0 \geq 0$. For each integer $k \geq 0$ cup product on the de Rham cohomology of the fibers of E/X induces a natural pairing

$$[\cdot,\cdot] : \mathcal{H}_k \times \mathcal{H}_{k+k_0} \longrightarrow \mathcal{H}_{k_0}.$$

This pairing induces natural pairings

$$\begin{aligned} [\cdot,\cdot] &: \mathcal{H}_k \times \mathcal{H}_{k+k_0} \otimes \Omega \longrightarrow \mathcal{H}_{k_0} \otimes \Omega; \\ [\cdot,\cdot] &: \mathcal{H}_k \otimes \Omega \times \mathcal{H}_{k+k_0} \longrightarrow \mathcal{H}_{k_0} \otimes \Omega. \end{aligned}$$

Proposition 2. *These pairings satisfy the following identity for all $x \in \mathcal{H}_k$, and $y \in \mathcal{H}_{k+k_0}$*

$$\nabla[x,y] = [x,\nabla y] + [\nabla x, y].$$

Proof. The proof follows from the product formula for differentiation.

We will attach a superscript † to denote over convergent section of a sheaf. For example, $\mathcal{H}_k^\dagger := \mathcal{H}_k(Z_\infty^\dagger)$. We may then define pairings

$$\langle\cdot,\cdot\rangle : \mathcal{H}_k^\dagger \times \mathcal{H}_{k+k_0}^\dagger \otimes \Omega^\dagger \longrightarrow H^0(W, \mathcal{H}_{k_0})$$
$$\langle\cdot,\cdot\rangle : \mathcal{H}_k^\dagger \otimes \Omega^\dagger \times \mathcal{H}_{k+k_0} \longrightarrow H^0(W, \mathcal{H}_{k_0}^\dagger)$$

by defining $\langle x, y\rangle := \mathrm{Res}([x,y])$ where $\mathrm{Res} : \mathcal{H}_{k_0}^\dagger \longrightarrow H^0(W, \mathcal{H}_{k_0})$ is the residue map.

We next record some basic properties of the Frobenius operator Φ, the involution W, and the operator U. Here we normalize W so that it is an involution on $\mathcal{H}_k(W)$: hence $W = p^{-k_0/2}w$ where w is the operator used by Coleman. We first remark that $\Phi = w$ on horizontal sections on the supersingular annuli. Hence $\Phi = p^{k_0/2}W$ on $H^0(W, \mathcal{H}_{k_0})$.

Proposition 3.

1. *For any $x \in \mathcal{H}_k^\dagger$ and $\omega \in \mathcal{H}_{k+k_0}^\dagger \otimes \Omega^\dagger$ we have* $\langle x, \Phi(\omega)\rangle = p^{k+\frac{k_0}{2}+1} \cdot W(\langle U(x), \omega\rangle)$;
2. *For any $\eta \in \mathcal{H}_k^\dagger \otimes \Omega^\dagger$ and $y \in \mathcal{H}_{k+k_0}^\dagger$ we have* $\langle \eta, \Phi(y)\rangle = p^{k+\frac{k_0}{2}} \cdot W(\langle U(\eta), y\rangle)$.

Proof. A simple calculation confirms the identities

$$U(\langle x, \Phi(\omega)\rangle) = p^{k+k_0+1} \cdot \langle U(x), \omega\rangle$$
$$U(\langle \eta, \Phi(y)\rangle) = p^{k+k_0} \cdot \langle U(\eta), y\rangle.$$

But $U \circ \Phi = p^{k_0}$ on $\mathcal{H}_{k_0}$, hence also on the finite dimensional space $H^0(W, \mathcal{H}_{k_0})$. Therefore $\Phi \circ U = p^{k_0}$ on $H^0(W, \mathcal{H}_{k_0})$. Hence applying Φ to the above identities gives us

$$\langle x, \Phi(\omega)\rangle = p^{k+1} \cdot \Phi(\langle U(x), \omega\rangle)$$
$$\langle \eta, \Phi(y)\rangle = p^{k} \cdot \Phi(\langle U(\eta), y\rangle).$$

But $\Phi = p^{k_0/2} \cdot W$ on $H^0(W, \mathcal{H}_{k_0})$ so proposition 3 follows.

§4. Some Lemmas.

The operator W is an involution on $H^0(W, \mathcal{H}_{k_0})$. We let superscript + denote projection to the +-component under the action of W. Consider the function $\psi : \mathcal{X} \longrightarrow H^0(W, \mathcal{H}_{k_0})^+$ defined by

$$\psi(k) := \rho(t_k^0 f_{k+k_0}^0)^+ \in H^0(W, \mathcal{H}_{k_0})^+.$$

Since $t_k^0 f_{k+k_0}^0$ is an analytic family of overconvergent modular forms of weight k_0 we see at once that $\psi(k)$ is an analytic function of k defined on a neighborhood of 0 in $\mathcal{X}$. For the proof of Theorem B we will calculate $\psi(0)$ in two ways. First, by direct calculation we express $\psi(0)$ in terms of $\rho(f)$. Then we apply the product rule (Proposition 2) to express $\psi(0)$ in terms of $\sigma(f)$. Comparing these two expressions, Theorem B follows.

Define $u(k) := p^{-k_0/2} \cdot \alpha(k)$, the "unit part" of $\alpha(k)$.

Lemma 1. *We have*

$$\psi(0) = -\frac{1}{2} \cdot \left(1 - \frac{1}{p}\right) \cdot u'(k_0) \cdot \rho(f).$$

Proof. For an arbitrary integer $k \geq 0$ we have

$$\psi(k) = \rho\left(t_k^0 f_{k+k_0}^0\right)^+ = \langle g_k^0, \omega_{k+k_0}^0 \rangle^+.$$

We also have

$$\begin{aligned}
\langle g_k^0, \omega_{k+k_0}^0 \rangle &= \langle g_k, \omega_{k+k_0}^0 \rangle \\
&= \left\langle g_k,\ \omega_{k+k_0} - \frac{\alpha(k+k_0)}{p^{k+k_0+1}} \Phi(\omega_{k+k_0}) \right\rangle \\
&= \langle g_k,\ \omega_{k+k_0} \rangle - \frac{\alpha(k+k_0)}{p^{k_0/2}} W(\langle U(g_k), \omega_{k+k_0} \rangle) \\
&= \langle g_k,\ \omega_{k+k_0} \rangle - u(k+k_0) \cdot W(\langle g_k, \omega_{k+k_0} \rangle).
\end{aligned}$$

The first equality above follows from three facts: (1) $g_k^0 - g_k$ is in the image of Φ; (2) $\omega_{k+k_0}^0$ is in the kernel of U; and (3) the image of Φ is perpendicular to the kernel of U by proposition 2. The last equality above follows from the fact that the Eisenstein series t_k is an eigenform for the U-operator with eigenvalue 1, hence $U(g_k) = g_k$.

Now project the above identity to the $+$-component for W to get

$$\begin{aligned}
\psi(k) &= (1 - u(k+k_0)) \cdot \langle g_k, \omega_{k+k_0} \rangle^+ \\
&= \frac{1 - u(k+k_0)}{k} \cdot \rho(k t_k f_{k+k_0})^+.
\end{aligned}$$

Setting $k = 0$, using (1) of propostion, and noting that $\rho(f)^+ = \rho(f)$ we obtain

$$\psi(0) = -\frac{1}{2} \cdot \left(1 - \frac{1}{p}\right) \cdot u'(k_0) \cdot \rho(f)$$

and the lemma is proved.

Let $C_\infty := Z_\infty^\dagger \setminus Z_\infty$. Then C_∞ is a union of concentric annuli in the supersingular annuli. Note that the pairings $\langle x, y \rangle$ are well-defined so long as x, y are rigid on C_∞. In particular we have a well-defined pairing

$$\langle \cdot, \cdot \rangle : \Omega^1(C_\infty) \times \mathcal{H}_{k_0}(C_\infty) \longrightarrow \mathcal{H}_{k_0}(W)^\nabla.$$

defined by $\langle \omega, h \rangle = \mathrm{Res}_W(h\omega)$, where this latter is defined to be the unique horizontal section on W extending $\mathrm{Res}_{C_\infty}(h\omega)$.

Lemma 2. *Let $e \in \mathcal{O}_{flog}(W_\infty)$ be any Coleman integral of η_0 (well-defined up to a constant). Restrict e to the supersingular annuli W and let $h = e - W(e) \in \mathcal{O}_{log}(W)$. Let*

$z = h \cdot \rho(f) \in \mathcal{H}_{k_0,log}(W)$, and let $z^0 := z - p^{-1-k_0/2}\Phi(z) \in \mathcal{H}_{k_0}(C_\infty)$. Then z, z_0 have the following properties.

(1) *z^0 is rigid on C_∞.*

(2) *$s_{k_0} - z$ is rigid on W.*

(3) *$\langle \eta_0, z^0 \rangle = 0$.*

(4) *$W(z) + z = 0$ on the supersingular annuli W.*

Proofs. (1) Since e is a Coleman integral of η_0, we have $e^0 := e - p^{-1}\Phi(e)$ is rigid on $Z_\infty^\dagger$. Since $W(\eta_0) = -\eta_0$ we have $W(e) + e$ is constant, and it follows that $h^0 := h - p^{-1}\Phi(h)$ is also rigid on C_∞. On the other hand, $\Phi(\rho(f)) = p^{k_0/2}\rho(f)$. Hence $z^0 = h^0 \cdot \rho(f)$, which is rigid on C_∞.

(2) By definition, $\nabla(s_{k_0}) = \kappa(f)$. Hence, $\mathrm{Res}_W(\nabla(s_{k_0})) = \rho(f)$. On the other hand, $\mathrm{Res}_W(\nabla(z)) = \mathrm{Res}_W(dh) \cdot \rho(f)$. But $dh = 2\eta_0$ and we have shown $\mathrm{Res}_W(\eta_0) = 1/2$, hence $\mathrm{Res}_W(\nabla(z)) = \rho(f)$. We therefore have $\mathrm{Res}_W(\nabla(s_{k_0} - z) = 0$ and it follows that $s_{k_0} - z$ is rigid on W, as claimed.

(3) We have $\langle \eta_0, z^0 \rangle = \langle \eta_0, h^0 \rangle \cdot \rho(f)$. Moreover, $\langle \eta_0, h^0 \rangle = \langle \eta_0^0, h^0 \rangle$ because the image of Φ is orthogonal to the kernel of U. But, $\langle \eta_0^0, h^0 \rangle = \mathrm{Res}_W(h^0\eta_0^0) = \frac{1}{2}\mathrm{Res}_W(h^0 dh^0) = 0$, since $h^0 dh^0$ is an exact differential on C_∞.

(4) Since $W(\rho(f)) = \rho(f)$, this follows immediately from the definition of z.

This completes the proof of Lemma 2.

Lemma 3.

$$\psi(0) = \frac{1}{4} \cdot \sigma(f).$$

Proof. As in the first line of the proof of lemma 1 we have

$$\psi(0) = \langle g_0^0, \omega_{k_0}^0 \rangle^+.$$

But $\omega_{k_0}^0$ is an exact differential, indeed $\nabla s_{k_0}^0 = \omega_{k_0}^0$. Moreover, $\nabla g_0^0 = \eta_0^0$. Hence, by lemma 1 we have

$$\nabla[g_0^0, s_{k_0}^0] = [\eta_0^0, s_{k_0}^0] + [g_0^0, \omega_{k_0}^0].$$

Taking residues of both sides of this equality along the supersingular annuli we obtain

$$0 = \langle \eta_0^0, s_{k_0}^0 \rangle + \langle g_0^0, \omega_{k_0}^0 \rangle.$$

Hence $\psi(0) = -\langle \eta_0^0, s_{k_0}^0 \rangle^+$. Now we just calculate as before, but in the second line we replace s_{k_0} by $s_{k_0} - z$. This gives us:

$$\begin{aligned}
\langle \eta_0^0, s_{k_0}^0 \rangle &= \langle \eta_0, s_{k_0}^0 \rangle \\
&= \langle \eta_0, s_{k_0}^0 - z^0 \rangle \\
&= \langle \eta_0, (s_{k_0} - z) - \frac{1}{p^{k_0/2+1}} \cdot \Phi(s_{k_0} - z) \rangle \\
&= \langle \eta_0, s_{k_0} - z \rangle - \frac{1}{p} W(\langle \eta_0, s_{k_0} - z \rangle) \\
&= \langle \eta_0, s_{k_0} - z \rangle - \frac{1}{p} W(\langle \eta_0, s_{k_0} - z \rangle)
\end{aligned}$$

Projecting to the +-component for W we obtain

$$\psi(0) = \left(1 - \frac{1}{p}\right) \cdot \langle \eta_0, s_{k_0} - z \rangle^+.$$

On the other hand, we have

$$\begin{aligned}
\langle \eta_0, s_{k_0} - z \rangle^+ &= \frac{1}{2} \cdot \Big(\langle \eta_0, s_{k_0} - z \rangle + W\left(\langle \eta_0, s_{k_0} - z \rangle\right) \Big) \\
&= \frac{1}{2} \cdot \Big(\langle \eta_0, s_{k_0} - z \rangle - \langle W(\eta_0), W(s_{k_0} - z) \rangle \Big) \\
&= \frac{1}{2} \cdot \Big(\langle \eta_0, s_{k_0} - z \rangle + \langle \eta_0, W(s_{k_0} - z) \rangle \Big) \\
&= \frac{1}{2} \cdot \langle \eta_0, (s_{k_0} - z) + W(s_{k_0} - z) \rangle \\
&= \frac{1}{2} \cdot \langle \eta_0, s_{k_0} + W(s_{k_0}) \rangle \\
&= \frac{1}{2} \cdot \langle \eta_0, \sigma(f) \rangle.
\end{aligned}$$

Finally, we use (2) of proposition 1 to conclude that $\langle \eta_0, \sigma(f) \rangle = \frac{1}{2} \cdot \sigma(f)$. Hence $\psi(0) = \frac{1}{4} \cdot \sigma(f)$ and lemma 3 is proved.

Proof of Theorem B. Combining lemma 2 and lemma 3 we obtain

$$-2 \cdot u'(k_0) \cdot \rho(f) = \sigma(f).$$

Hence $\mathcal{L}(f) = -2 \cdot u'(k_0) = -2 \cdot p^{-\frac{k_0}{2}} \cdot \alpha'(k_0)$ and Theorem B is proved.