Rigid Analytic Modular Symbols

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§0. Introduction.

Let p be a prime > 3 and consider the Tate algebra $\mathcal{A} := \mathbf{Q}_p\langle z\rangle$ defined to be the Banach algebra of formal power series over $\mathbf{Q}_p$ that converge on the closed unit disk $B[0,1] \subseteq \mathbf{C}_p$ with the supremum norm $||f|| := \sup_{z\in B[0,1]} |f(z)|$ for $f \in \mathcal{A}$. Equivalently, we have

$$\mathcal{A} = \left\{ f(z) = \sum_{k=0}^{\infty} a_k z^k \;\middle|\; a_k \in \mathbf{Q}_p \text{ and } \lim_{k\to\infty} a_k = 0 \right\} \tag{0.1}$$

and the norm is given by

$$||f|| = \sup_k |a_k|, \quad \text{for } f = \sum_{k=0}^{\infty} a_k z^k. \tag{0.2}$$

The pair $(\mathcal{A}, ||\ ||)$ defines a Banach algebra over $\mathbf{Q}_p$. We let $\mathbf{D}$ denote the Banach space of $\mathbf{Q}_p$-valued continuous linear functionals on $\mathcal{A}$. We will often regard the elements of $\mathcal{A}$ as analytic functions on $\mathbf{Z}_p$ and the elements of $\mathbf{D}$ as distributions on $\mathbf{Z}_p$. In this spirit we will often use measure-theoretic conventions and write

$$\int f(z)d\mu := \mu(f)$$

for the value of a linear functional $\mu \in \mathbf{D}$ on a power series $f \in \mathcal{A}$.

Let $\kappa : \mathbf{Z}_p^\times \longrightarrow \mathbf{Z}_p^\times$ be a locally analytic character and consider the semigroup

$$\Sigma_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2\times 2, \mathbf{Z}_p) \;\middle|\; a \in \mathbf{Z}_p^\times,\ c \in p\mathbf{Z}_p,\ ad - bc \neq 0 \right\}.$$

We define the weight κ action of $\Sigma_0(p)$ on $\mathcal{A}$ by the formulas

$$(\gamma_\kappa f)(z) := \kappa(a+cz)\cdot f\left(\frac{b+dz}{a+cz}\right), \tag{0.3}$$

for $f \in \mathcal{A}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma_0(p)$. A simple calculation shows that this defines a continuous action of $\Sigma_0(p)$ on the Tate algebra $\mathcal{A}$. Hence by duality, we also obtain a continuous action of $\Sigma_0(p)$ on $\mathbf{D}$, $\mu \mapsto \mu|_\kappa\gamma$ ($\mu \in \mathbf{D}$, $\gamma \in \Sigma_0(p)$), which we call the dual weight κ action.

(0.4) **Definition.** We will denote by $\mathcal{A}_\kappa$ the Banach space $\mathcal{A}$ equipped with the weight κ action of $\Sigma_0(p)$ described above. Similarly, we will denote by $\mathbf{D}_\kappa$ the dual space $\mathbf{D}$ equipped with the dual weight κ action.

In general, to any $\Gamma_0(p)$-space V there is associated a locally constant sheaf $\tilde{V}$ on $Y_0(p)$ and we may consider the space

$$H_c^1(\Gamma_0(p), V) := H_c^1(Y_0(p), \tilde{V}) \tag{0.5}$$

of one-dimensional compactly supported cohomology classes of $\tilde{V}$. As we will point out in more detail later, this space can be conveniently described in terms of modular symbols. In particular, if V is a Banach space with a unitary action of $\Gamma_0(p)$, then $H_c^1(\Gamma_0(p), V)$ has a natural Banach space structure. If, furthermore, V is equipped with a continuous right action of the semigroup

$$S_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2 \times 2, \mathbf{Z}) \;\middle|\; p \nmid a,\ c \in p\mathbf{Z},\ ad - bc \neq 0 \right\} \tag{0.6}$$

then $H_c^1(\Gamma_0(p), V)$ inherits a continuous action of the Hecke operators U and T_ℓ, $\ell \neq p$. In many examples, the U-operator is completely continuously on $H_c^1(\Gamma_0(p), V)$. This is trivially the case if V is finite dimensional over $\mathbf{Q}_p$. We will see that it is also true for $V = \mathbf{D}_\kappa$.

In general, if we have a completely continuous U-operator on a Banach space H and if $h \geq 0$ is a real number, then we define $H^{(h)} \subseteq H$ to be the subspace on which U acts with slope $\leq h$. The space $H^{(h)}$ is characterized as the largest U-invariant closed subspace of H satisfying the following properties:

(0.7)
(1) $U : H^{(h)} \longrightarrow H^{(h)}$ is an isomorphism.
(2) The sequence $p^{nh}U^{-n}$, $n \geq 0$ is a bounded sequence of operators on $H^{(h)}$.

The existence of $H^{(h)}$ follows from the theory of completely continuous operators. Moreover, $H^{(h)}$ is finite dimensional.

A locally analytic character $\kappa : \mathbf{Z}_p^\times \longrightarrow \mathbf{Z}_p^\times$ is said to be *arithmetic with signature* (k, ϵ) if $\kappa(t) = \epsilon(t)t^k$ $(t \in \mathbf{Z}_p^\times)$ where ϵ is a finite character of $\mathbf{Z}_p^\times$ and k is an integer ≥ 0. If κ is such a character, then we define the finite dimensional $\Sigma_0(p)$-module

$$L_\kappa := \left\{ F(Z) \in \mathbf{Q}_p[Z] \;\middle|\; \deg(F) \leq k \right\}$$

where the action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma_0(p)$ on a polynomial $F \in L_\kappa$ is given by

$$(F|\gamma)(Z) := \epsilon(a) \cdot (d - cZ)^k \cdot F\left(\frac{-b + aZ}{d - cZ}\right).$$

Now consider the function $\varphi_\kappa : \mathbf{Z}_p \longrightarrow L_\kappa$ defined by

$$\varphi_\kappa(z) := (Z - z)^k.$$

The coefficients of $\varphi(z)$ are polynomials in z, hence are elements of $\mathcal{A}_\kappa$. Integrating coefficient by coefficient, we obtain a map

$$\text{(0.8).} \qquad \begin{aligned} \phi_\kappa : \mathbf{D}_\kappa &\longrightarrow L_\kappa \\ \mu &\longmapsto \int \varphi_\kappa(z)\, d\mu \end{aligned}$$

and from the simple identity $(\gamma_\kappa \varphi_\kappa)(z) = \varphi_\kappa(z)|_\kappa \gamma$ it follows that this map intertwines the action of $\Sigma_0(p)$. It therefore induces a Hecke equivariant map

$$\phi_{\kappa,*} : H^1_c(\Gamma_0(p), \mathbf{D}_\kappa) \longrightarrow H^1_c(\Gamma_0(p), L_\kappa).$$

Since L_κ is a finite dimensional vector space over $\mathbf{Q}_p$ the cohomology group on the right is definitely finite dimensional. By the theory of Eichler and Shimura we know that classical cusp forms of weight $k + 2$ contribute elements to this group. In this spirit, we regard $H^1_c(\Gamma_0(p), L_\kappa)$ as the space of *classical modular symbols* of weight κ. On the other hand, the space $H^1_c(\Gamma_0(p), \mathbf{D}_\kappa)$ has all the appearances of being infinite dimensional. We will refer to its elements as *rigid analytic modular symbols* of weight κ. Since the U-operator acts completely continuously on $H^1_c(\Gamma_0(p), \mathbf{D}_\kappa)$, its subspaces of bounded slopes are finite dimensional. We will prove the following theorem.

(0.9) **Theorem.** (Comparison Theorem). *Suppose κ is an arithmetic character of signature (k, ϵ). Then for each real number h with $0 \le h < k + 1$, the map $\phi_\kappa : \mathbf{D}_\kappa \longrightarrow L_\kappa$ induces an isomorphism*

$$\phi^{(h)}_{\kappa,*} : H^1_c(\Gamma_0(p), \mathbf{D}_\kappa)^{(h)} \xrightarrow{\sim} H^1_c(\Gamma_0(p), L_\kappa)^{(h)}.$$

It is interesting to compare this theorem with the construction of p-adic distributions associated to classical modular symbols of slope less than $k + 1$ (by Mazur-Swinnerton-Dyer, Vishik, Amice-Velu, and Mazur-Tate-Teitelbaum). Indeed, the theorem should be viewed as an equivariant version of that construction. For more details, see section 8.

§1. Modular symbols.

Let V be a $\mathbf{Q}_p$-vector space with a right action of the semigroup $S_0(p)$ (see (0.6)). Then we may form the space $H^1_c(\Gamma_0(p), V)$ of one-dimensional compactly supported cohomology classes. This space is endowed with a natural action of the Hecke operators U and T_ℓ, $\ell \neq p$. As mentioned in the introduction, it is possible to give a fairly explicit description of this space in terms of modular symbols. We recall that description here.

Let $\mathcal{D}_0 = \mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q}))$ be the group of divisors of degree zero supported on $\mathbf{P}^1(\mathbf{Q})$ and note that $GL(2,\mathbf{Q})$ acts on $\mathcal{D}_0$ by fractional linear transformations. If $\Phi : \mathcal{D}_0 \longrightarrow V$ is an additive homomorphism and $\gamma \in S_0(p)$ we define $\Phi|\gamma : \mathcal{D}_0 \longrightarrow V$ by

$$(\Phi|\gamma)(D) := \Phi(\gamma D)|\gamma \tag{1.1}$$

for $D \in \mathcal{D}_0$. We say that Φ is a V-valued modular symbol over $\Gamma_0(p)$ if $\Phi|\gamma = \Phi$ for each $\gamma \in \Gamma_0(p)$ and denote the space of all V-valued modular symbols over $\Gamma_0(p)$ by $\mathrm{Symb}_{\Gamma_0(p)}(V)$. Hence, for an additive homomorphism $\Phi : \mathcal{D}_0 \longrightarrow V$, we have

$$\Phi \in \mathrm{Symb}_{\Gamma_0(p)}(V) \Longleftrightarrow \Phi|\gamma = \Phi \quad \text{for all } \gamma \in \Gamma_0(p). \tag{1.2}$$

The action of the Hecke operators is defined in the usual way in terms of double cosets. In particular, the U-operator is defined by

$$\Phi|U := \sum_{a=0}^{p-1} \Phi|\beta(a,p), \quad \text{where} \quad \beta(a,p) := \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}, \; a = 0, 1, \dots, p-1. \tag{1.3}$$

1.4. Proposition. *There is a canonical isomorphism $H^1_c(\Gamma_0(p), V) \cong \mathrm{Symb}_{\Gamma_0(p)}(V)$ and this isomorphism commutes with the action of the Hecke operators.*

Henceforth we will identify the spaces $H^1_c(\Gamma_0(p), V)$ and $\mathrm{Symb}_{\Gamma_0(p)}(V)$.

Now suppose V is complete with respect to a norm $||\ ||$. An operator L on V is called *unitary* if $||L(f)|| = ||f||$ for all $f \in V$. If $\Gamma_0(p)$ acts as on the right as a group of unitary operators on V, then $H^1_c(\Gamma_0(p), V)$ inherits a natural Banach space structure defined by the norm

$$||\Phi|| = \sup_{D \in \mathcal{D}_0} ||\Phi(D)|| \tag{1.5}$$

for $\Phi \in H^1_c(\Gamma_0(p), V)$. This supremum exists since $||\Phi(-)||$ is constant on $\Gamma_0(p)$-orbits in $\mathcal{D}_0$ and $\mathcal{D}_0$ is a finitely generated $\mathbf{Z}[\Gamma_0(p)]$-module.

In the applications, V will often be a Banach space endowed with a right action of $S_0(p)$ satisfying the following properties for $\gamma \in S_0(p)$.

(1.6)

(1) If $p \not| \det \gamma$ then γ induces a unitary operator on V.

(2) If $p | \det \gamma$ then γ induces a completely continuous operator on V with norm ≤ 1.

We have the following easily established proposition.

1.7. Proposition. *Suppose V is a Banach algebra with a continuous right action of $S_0(p)$ satisfying properties (1) and (2) above. Then (1.5) defines a Banach norm on $\mathrm{Symb}_{\Gamma_0(p)}(V)$. The Hecke operators U, T_ℓ define bounded linear operators of norm ≤ 1 and the U-operator is completely continuous.*

§2. Rigid analytic function spaces and distributions.

For each $r \in \mathbf{R}^+$ we define

$$B[\mathbf{Z}_p, r] := \{x \in \mathbf{C}_p \mid \exists a \in \mathbf{Z}_p \text{ such that } |x-a| \le r\}.$$

$\mathcal{A}[r]$:=the $\mathbf{Q}_p$-Banach algebra of rigid analytic functions on $B[\mathbf{Z}_p, r]$ whose Taylor expansions on $\mathbf{Z}_p$ have $\mathbf{Q}_p$-coefficients.

$\mathbf{D}[r]$:=the Banach dual of $\mathcal{A}[r]$.

The norm on $\mathcal{A}[r]$ is the usual supremum norm

(2.1)
$$||f||_r := \sup_{z \in B[\mathbf{Z}_p, r]} |f(z)|, \quad (f \in \mathcal{A}[r])$$

and the dual norm is defined by

(2.2)
$$||\mu||_r := \sup_{\substack{f \in \mathcal{A}[r] \\ f \neq 0}} \frac{|\mu(f)|}{||f||_r}, \quad (\mu \in \mathbf{D}[r]).$$

(2.3) **Proposition.** *Let $r, s \in \mathbf{R}$ with $r > s > 0$.*

a. *The restriction map $\rho_{r,s} : \mathcal{A}[r] \longrightarrow \mathcal{A}[s]$ is a completely continuous monomorphism. Moreover, the image is dense in $\mathcal{A}[s]$.*

b. *The dual map $\rho^*_{r,s} : \mathbf{D}[s] \longrightarrow \mathbf{D}[r]$ is also a completely continuous monomorphism.*

Proof. The first assertion is a well-known property of rigid analytic functions. The second assertion is then an immediate consequence of the first.

Define the limits

(2.4)
$$\mathcal{A}[0] := \varinjlim_{r>0} \mathcal{A}[r], \qquad \mathbf{D}[0] := \varprojlim_{r>0} \mathbf{D}[r]$$

with the respect to the connecting morphisms $rho_{r,s}$ and $\rho^*_{r,s}$ defined in proposition 2.3. We identify $\mathcal{A}[0]$ with the space of locally analytic $\mathbf{Q}_p$-valued functions on $\mathbf{Z}_p$ and endow it with the inductive topology. If $f \in \mathcal{A}[0]$ is a locally analytic function on $\mathbf{Z}_p$, we say that $r > 0$ is a *radius of definition* for f if $f \in \mathcal{A}[r]$.

Similarly we identify $\mathbf{D}[0]$ with the space of continuous linear functionals on $\mathcal{A}[0]$. The topology on $\mathbf{D}[0]$ is induced by the family of norms $|| \; ||_r$, $r > 0$, defined in (2.2). These norms increase as r approaches zero. More precisely,

(2.5)
$$||\mu||_r \le ||\mu||_s \quad \text{for} \quad r > s > 0.$$

§3. Representation spaces.

If $\kappa : \mathbf{Z}_p^\times \longrightarrow \mathbf{Z}_p^\times$ is a locally analytic character then for each real number r with $0 < r < p$, (0.3) defines an action of $\Sigma_0(p)$ on $\mathcal{A}[r]$ and hence by duality also on $\mathbf{D}[r]$. These actions commute with the connecting morphisms $\mathcal{A}[r] \longrightarrow \mathcal{A}[s]$ and $\mathbf{D}[s] \longrightarrow \mathbf{D}[r]$

for $1 \geq r > s > 0$, hence we also obtain a $\Sigma_0(p)$-action on the limits (2.4). We will denote the resulting $\Sigma_0(p)$-spaces, respectively, by

$$\mathcal{A}_\kappa[r] \qquad \text{and} \qquad \mathbf{D}_\kappa[r] \qquad \text{for } 0 \leq r < p. \tag{3.1}$$

In fact, a careful calculation reveals that if $\gamma \in \Sigma_0(p)$ and $\delta = |\det \gamma|$, then the action of γ on $\mathcal{A}_\kappa[0]$ increases the radius of definition by a factor of δ^{-1}. Indeed, we have the following proposition whose proof is straightforward.

(3.2) **Proposition.** *For each $r \in (0,p)$, and each $\gamma \in \Sigma_0(p)$ with $\delta = |\det \gamma|$, γ induces bounded linear operators of norm 1: $\gamma'_\kappa : \mathcal{A}_\kappa[r\delta] \longrightarrow \mathcal{A}_\kappa[r]$, $f \longmapsto \gamma'_\kappa f$, and $\gamma'_\kappa : \mathbf{D}_\kappa[r] \longrightarrow \mathbf{D}_\kappa[r\delta]$, $\mu \longmapsto \mu|\gamma'_\kappa$. Moreover, the following diagrams are commutative:*

$$\begin{array}{ccc} \mathcal{A}_\kappa[r] & \xrightarrow{\gamma_\kappa} & \mathcal{A}_\kappa[r] \\ {\scriptstyle \rho_{r,r\delta}}\downarrow & & \| \\ \mathcal{A}_\kappa[r\delta] & \xrightarrow{\gamma'_\kappa} & \mathcal{A}_\kappa[r] \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbf{D}_\kappa[r] & \xrightarrow{\gamma_\kappa} & \mathbf{D}_\kappa[r] \\ \| & & \uparrow{\scriptstyle \rho^*_{r,r\delta}} \\ \mathbf{D}_\kappa[r] & \xrightarrow{\gamma'_\kappa} & \mathbf{D}_\kappa[r\delta] \end{array}$$

The semigroup $S_0(p)$ (see (0.6)) is contained in $\Sigma_0(p)$ and therefore acts on the right on $\mathbf{D}_\kappa[r]$ for each $r > 0$. From proposition 3.2 it follows that this action satisfies the conditions (1.6). Hence, by proposition 1.7 we see that $H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[r])$ is naturally a Banach space on which the Hecke operators U, T_ℓ act as bounded operators of norm ≤ 1. Moreover U is completely continuous on this space.

Since the action of $\Sigma_0(p)$ intertwines the morphism $\rho^*_{r,s} : \mathbf{D}_\kappa[s] \longrightarrow \mathbf{D}_\kappa[r]$ for each r,s with $p > r > s > 0$, we can extend the $\Sigma_0(p)$-action to an action on the limit $\mathbf{D}_\kappa[0]$. Passing to cohomology, we obtain for each r,s with $p > r > s \geq 0$ a natural morphism

$$\rho^*_{r,s} : H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[s]) \longrightarrow H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[r]) \tag{3.3}$$

which commutes with the Hecke operators. Passing to the limit, we obtain an isomorphism of Hecke modules

$$H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[0]) \xrightarrow{\sim} \varprojlim_{r>0} H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[r]). \tag{3.4}$$

§4. Locally polynomial function spaces and distributions.

Fix a non-negative integer $k \geq 0$ and for each real number $r > 0$, let $\mathcal{P}_k[r] \subseteq \mathcal{A}[r]$ denote the finite dimensional subspace consisting of functions on $B[\mathbf{Z}_p, r]$ whose restriction to each closed ball of radius r is a polynomial function of degree $\leq k$. Let $\mathbf{L}_k[r]$ be the space of linear functionals on $\mathcal{P}_k[r]$ and endow $\mathbf{L}_k[r]$ with the norm $|| \; ||_{r,k}$ defined in the usual way by

$$||\mu||_{r,k} := \sup_{0 \neq f \in \mathcal{P}_k[r]} \frac{|\mu(f)|}{||f||_r} \tag{4.1}$$

for $\mu \in \mathbf{L}_k[r]$. For $r > s > 0$ we have the restriction map $\rho_{r,s} : \mathcal{P}_k[r] \longrightarrow \mathcal{P}_k[s]$ and its transpose $\rho^*_{r,s} : \mathbf{L}_k[s] \longrightarrow \mathbf{L}_k[r]$. We form the limits

$$\mathcal{P}_k[0] := \varinjlim_{r>0} \mathcal{P}_k[r] \quad \text{and} \quad \mathbf{L}_k[0] := \varprojlim_{r>0} \mathbf{L}_k[r] \tag{4.2}$$

with respect to the connecting morphisms $\rho_{r,s}$ and $\rho^*_{r,s}$. The space $\mathcal{P}_k[0]$ is canonically isomorphic to the space of functions on $\mathbf{Z}_p$ that are locally given by polynomials of degree $\leq k$ and $\mathbf{L}_k[0]$ is the space of linear functionals on $\mathcal{P}_k[0]$. We endow the space $\mathbf{L}_k[0]$ with the family of norms $|| \; ||_{r,k}$ inherited from the spaces $\mathbf{L}_k[r]$, $r > 0$.

Now suppose $\kappa : \mathbf{Z}_p^\times \longrightarrow \mathbf{Z}_p^\times$ is an arithmetic character of signature (k, ϵ). Then the subspace $\mathcal{P}_k[r] \subseteq \mathcal{A}_\kappa[r]$ is invariant under the weight κ action of $\Sigma_0(p)$. We let $\mathcal{P}_\kappa[r]$ denote $\mathcal{P}_k[r]$ with this action of $\Sigma_0(p)$ and let $\mathbf{L}_\kappa[r]$ denote the dual space $\mathbf{L}_k[r]$ with the dual right action. We write $\mathbf{L}_\kappa$ for $\mathbf{L}_\kappa[1]$ and note that there is a natural isomorphism between this and the space L_κ defined in the introduction.

As in the last section, the cohomology groups $H^1_c(\Gamma_0(p), \mathbf{L}_\kappa[r])$, $r \in [0, p)$, are endowed with natural actions of the Hecke operators U, T_ℓ and, analogous to (3.4), we have a natural isomorphism of Hecke modules

$$H^1_c(\Gamma_0(p), \mathbf{L}_\kappa[0]) \xrightarrow{\sim} \varprojlim_{r>0} H^1_c(\Gamma_0(p), \mathbf{L}_\kappa[r]). \tag{4.3}$$

§5. The operator U'.

From proposition 3.2 it easily follows that for each $r \in (0, p)$ there is a unique operator

$$U' : H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[r]) \longrightarrow H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[r/p]) \tag{5.1}$$

satisfying $U = \rho^*_{r,r/p} \circ U'$. Indeed, for $\Phi : \mathcal{D}_0 \longrightarrow \mathbf{D}_\kappa[r]$ an additive homomorphism and $\gamma \in S_0(p)$, define $\Phi|\gamma' : \mathcal{D}_0 \longrightarrow \mathbf{D}_\kappa[r/p]$ by $(\Phi|\gamma')(D) := \Phi(\gamma D)|\gamma'_\kappa$ for $D \in \mathcal{D}_0$. Then $\Phi|U'$ is given by $\Phi|U' := \sum_{a=0}^{p-1} \Phi|\beta(a,p)'$ where $\beta(a,p) \in S_0(p)$ are as in (1.3). A standard calculation shows that $\Phi|U' \in H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[r/p])$. From the definitions we have

$$U = \rho^*_{r,r/p} \circ U' = U' \circ \rho^*_{pr,r} \tag{5.2}$$

whenever $r < 1$.

(5.3) **Proposition.** *The operator U' is unitary. In other words, for each $r \in (0, p)$ and each $\Phi \in H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[r])$ we have*

$$||\Phi|U'||_{r/p} = ||\Phi||_r.$$

Proof. It follows immediately from the definitions that U' has norm ≤ 1. Hence, for any $\Phi \in H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[r])$ we have $||\Phi|U'||_{r/p} \leq ||\Phi||_r$.

For each $f \in \mathcal{A}[r]$ define the function $f_p \in \mathcal{A}[r/p]$ by

$$f_p(z) := \begin{cases} f(z/p) & \text{if } z/p \in B[\mathbf{Z}_p, r]; \\ 0 & \text{otherwise.} \end{cases}$$

A simple calculation confirms that $||f_p||_{r/p} = ||f||_r$ and moreover that $(\beta(0,p)'_\kappa f_p)(z) = f(z)$ and $(\beta(a,p)'_\kappa f_p)(z)$ vanishes identically for $a \in \mathbf{Z}_p^\times$. Hence

$$\begin{aligned} \int f_p(z)\, d(\Phi|U')(\beta(0,p)^{-1}D) &= \sum_{a=0}^{p-1} \int (\beta(a,p)_\kappa f_p)(z)\, d\Phi(\beta(a,p)\beta(0,p)^{-1}D) \\ &= \int f(z)\, d\Phi(D). \end{aligned}$$

We therefore have

$$\begin{aligned} ||\Phi||_r &= \sup_{\substack{0 \neq f \in \mathcal{A}[r] \\ D \in \mathcal{D}_0}} \frac{\left|\int f(z)\, d\Phi(D)\right|}{||f||_r} \\ &= \sup_{\substack{0 \neq f \in \mathcal{A}[r] \\ D \in \mathcal{D}_0}} \frac{\left|\int f_p(z)\, d(\Phi|U')(D)\right|}{||f_p||_{r/p}} \\ &\leq ||\Phi|U'||_{r/p} \end{aligned}$$

This completes the proof of proposition 5.3.

(5.4) **Corollary.** *For each $r \in [0,p)$ the U-operator acts injectively on $H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[r])$.*

Proof. If $r > 0$, then $U = \rho^*_{r,r/p} \circ U'$ is the composition of two injective maps, hence is injective. The statement for $r = 0$ follows by passing to the limit.

(5.5) **Definition.** If H is a vector space on which U operates, we define $H^\# := \cap_{n=1}^\infty H|U^n$. In other words, for $\Phi \in H$, we have

$$\Phi \in H^\# \iff \forall n > 0 \exists \Phi_n \in H \text{ such that } \Phi_n|U^n = \Phi.$$

Note that if H is a Banach space, then $H^\#$ is not necessarily a closed subspace, not even if U is completely continuous.

(5.6) **Proposition.** *For each $r > 0$ the natural maps*

$$H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[0])^\# \xrightarrow{\sim} H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[r])^\#, \qquad H^1_c(\Gamma_0(p), \mathbf{L}_\kappa[0])^\# \xrightarrow{\sim} H^1_c(\Gamma_0(p), \mathbf{L}_\kappa[r])^\#$$

are isomorphisms. Moreover, for each $r \in [0,p)$ the operator U acts invertibly on the spaces $H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[r])^\#$ and $H^1_c(\Gamma_0(p), \mathbf{L}_\kappa[r])^\#$.

Proof. By corollary 5.4, we have U acts injectively, hence invertibly on $H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[r])^\#$ for each $r \in [0,p)$. If $r > 0$, then $\mathbf{L}_\kappa[r]$ is finite dimensional, hence $H^1_c(\Gamma_0(p), \mathbf{L}_\kappa[r])$

is also finite dimensional. It follows that $H^1_c(\Gamma_0(p), \mathbf{L}_\kappa[r])^\#$ is the sum of the pseudo-eigensubspaces for U with non-zero pseudo-eigenvalue. In particular U acts invertibly on $H^1_c(\Gamma_0(p), \mathbf{L}_\kappa[r])^\#$. Passing to the limit we obtain isomorphisms

$$H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[0])^\# \xrightarrow{\sim} \varprojlim_{r>0} H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[r])^\#$$
$$H^1_c(\Gamma_0(p), \mathbf{L}_\kappa[0])^\# \xrightarrow{\sim} \varprojlim_{r>0} H^1_c(\Gamma_0(p), \mathbf{L}_\kappa[r])^\#$$

The proof will be complete if we can show that $\rho^*_{r,s}$ is an isomorphism whenever $p > r > s > 0$. For this, it suffices to prove that $\rho^*_{pr,r}$ is an isomorphism for every $r \in (0, 1)$.

We first show that $\rho^*_{pr,r}$ is injective. On $H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[r])^\#$ this follows from (4.3). So suppose $\Phi \in H^1_c(\Gamma_0(p), \mathbf{L}_\kappa[r])^\#$ and $\rho^*_{pr,r}(\Phi) = 0$. Then $\Phi|U = \rho^*_{pr,r}(\Phi)|U' = 0$. But we have already remarked that U is injective on $H^1_c(\Gamma_0(p), \mathbf{L}_\kappa[r])^\#$. Hence $\Phi = 0$.

Next we show that $\rho^*_{pr,r}$ is surjective. For $\Phi \in H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[pr])^\#$, consider the element $\Psi := (\Phi|U^{-1})|U'$ in $H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[r])^\#$. We have

$$\rho^*_{pr,r}(\Psi) = \rho^*_{pr,r}((\Phi|U^{-1})|U') = (\Phi|U^{-1})|U = \Phi.$$

This proves that $\rho^*_{pr,r}$ is surjective on $H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[pr])^\#$. A similar argument shows that $\rho^*_{pr,r}$ is surjective on $H^1_c(\Gamma_0(p), \mathbf{L}_\kappa[pr])^\#$. This completes the proof of the proposition.

§6. Weakly h-admissible distributions.

It is customary to classify distributions $\mu \in \mathbf{D}[0]$ (or $\mathbf{L}_k[0]$) according to the "rate of growth" of $||\mu||_r$ as $r \to 0^+$.

(6.1) **Definition.** (Weakly h-admissible distributions). For a real number $h \geq 0$ we say that a distribution $\mu \in \mathbf{D}_\kappa[0]$ is weakly h-admissible if $||\mu||_r = O(r^{-h})$. The space of all weakly h-admissible distributions will be denoted $\mathbf{D}^{(h)}$. Hence, for $\mu \in \mathbf{D}[0]$ we have

$$\mu \in \mathbf{D}^{(h)} \quad \Longleftrightarrow \quad ||\mu||_r = \mathrm{O}(r^{-h}) \quad \text{as} \quad r \to 0^+$$

Similarly, we say that a locally polynomial distribution $\mu \in \mathbf{L}_k[0]$ is weakly h-admissible if $||\mu||_{r,k} = O(r^{-h})$ as $r \to 0^+$ and denote the space of all weakly h-admissible elements by $\mathbf{L}_k^{(h)}$. Hence for $\mu \in \mathbf{L}_k[0]$ we have

$$\mu \in \mathbf{L}_k^{(h)} \quad \Longleftrightarrow \quad ||\mu||_{r,k} = \mathrm{O}(r^{-h}) \quad \text{as} \quad r \to 0^+$$

For an aritrary locally analytic character $\kappa : \mathbf{Z}_p^\times \longrightarrow \mathbf{Z}_p^\times$, the weight κ action of $\Sigma_0(p)$ on $\mathbf{D}[0]$ preserves the subspace $\mathbf{D}^{(h)}$. We let $\mathbf{D}_\kappa^{(h)}$ denote $\mathbf{D}^{(h)}$ with this action of $\Sigma_0(p)$. If κ is arithmetic of signature (k, ϵ), we define $\mathbf{L}_\kappa^{(h)}$ similarly.

(6.2) **Remarks.**

(a) In the literature it is customary to say that $\mu \in \mathbf{D}[0]$ is h-admissible if $||\mu||_r = o(r^{-h})$. We find the above notion of weak h-admissibility more useful for our purposes.

(b) The space of weakly 0-admissible distributions is just the space of bounded measures on $\mathbf{Z}_p$.

Every $\mu \in \mathbf{D}[r]$ restricts to an element $\mu \in \mathbf{L}_k[r]$ and we clearly have an inequality

$$||\mu||_{r,k} \leq ||\mu||_r.$$

From this we see that restriction gives us a natural map

$$\mathbf{D}^{(h)} \longrightarrow \mathbf{L}_k^{(h)}. \tag{6.3}$$

(6.4) **Theorem.** *Let k be a non-negative integer and h be a positive real number $< k+1$. Then the restriction morphism (6.3) is an isomorphism: $\mathbf{D}^{(h)} \xrightarrow{\sim} \mathbf{L}_k^{(h)}$.*

This is due to Vishik. A proof is outlined in the paper of Mazur, Tate, and Teitelbaum.

(6.5) **Corollary.** *If κ is arithmetic of signature (k, ϵ) and if $h < k+1$ then $\mathbf{D}_\kappa^{(h)} \xrightarrow{\sim} \mathbf{L}_\kappa^{(h)}$ is an isomorphism of $\Sigma_0(p)$-modules.*

(6.6) **Proposition.** *For each $r \in (0, p)$ the isomorphisms of proposition 5.6 restrict to isomorphisms*

$$H_c^1(\Gamma_0(p), \mathbf{D}_\kappa^{(h)})^\# \xrightarrow{\sim} H_c^1(\Gamma_0(p), \mathbf{D}_\kappa[r])^{(h)}, \qquad H_c^1(\Gamma_0(p), \mathbf{L}_\kappa^{(h)})^\# \xrightarrow{\sim} H_c^1(\Gamma_0(p), \mathbf{L}_\kappa[r])^{(h)}.$$

Proof. It suffices to show that $\rho^*_{r,0}(H^1_c(\Gamma_0(p), \mathbf{D}^{(h)}_\kappa)^\#) = H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[r])^{(h)}$ and correspondingly that $\rho^*_{r,0}(H^1_c(\Gamma_0(p), \mathbf{L}^{(h)}_\kappa)^\#) = H^1_c(\Gamma_0(p), \mathbf{L}_\kappa[r])^{(h)}$. For $\Phi \in H^1_c(\Gamma_0(p), \mathbf{D}^{(h)}_\kappa)^\#$ and $s > 0$ we let $\Phi_s := \rho^*_{s,0}(\Phi)$. With this notation we have

$$\begin{aligned} ||\Phi_r|U^{-n}||_r &= ||\Phi_r|U^{-n}|U'^n||_{r/p^n} \quad \text{(by proposition 5.3)} \\ &= ||\Phi_{r/p^n}||_{r/p^n} \end{aligned}$$

From this equality we therefore have

$$||\Phi_r|U^{-n}||_r = \mathrm{O}(p^{nh}) \text{ as } n \to \infty \iff ||\Phi_{r/p^n}||_{r/p^n} = \mathrm{O}(p^{nh}) \text{ as } n \to \infty.$$

Hence $\Phi_r \in H^1_c(\Gamma_0(p), \mathbf{D}_\kappa[r])^{(h)}$ if and only if $\Phi \in H^1_c(\Gamma_0(p), \mathbf{D}^{(h)}_\kappa)^\#$. This proves the first assertion. The second assertion follows similarly.

§7. The Comparison Theorem.

We are now prepared to prove the comparison theorem (0.9).

(7.1) **Theorem.** *Let κ be an arithmetic character of signature (k, ϵ) and suppose $0 \le h < k + 1$. Then the map*

$$\phi_\kappa : H^1_c(\Gamma_0(p), \mathbf{D}_\kappa)^{(h)} \longrightarrow H^1_c(\Gamma_0(p), \mathbf{L}_\kappa)^{(h)}$$

is an isomorphism of Hecke modules.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} H^1_c(\Gamma_0(p), \mathbf{D}_\kappa)^{(h)} & \longrightarrow & H^1_c(\Gamma_0(p), \mathbf{L}_\kappa)^{(h)} \\ \rho^*_{1,0}\Big\uparrow & & \Big\uparrow\rho^*_{1,0} \\ H^1_c(\Gamma_0(p), \mathbf{D}^{(h)}_\kappa)^\# & \longrightarrow & H^1_c(\Gamma_0(p), \mathbf{L}^{(h)}_\kappa)^\#. \end{array} \tag{7.4}$$

The vertical maps are isomorphisms by proposition 6.6 and the lower horizontal map is an isomorphism according to theorem 6.4 and its corollary 6.5. The theorem follows from the commutativity of the diagram.

§8. Final remarks. Since the U-operator is completely continuous on $H^1_c(\Gamma_0(p), \mathbf{D}_\kappa)$ we may form the Fredholm determinant

$$P_\kappa(t, U) := \det(1 - tU). \tag{8.1}$$

This is an entire function of t with coefficients in $\mathbf{Z}_p$. It is tempting to guess that $P_\kappa(t, U)$ is in fact identical to the characteristic power series of the U-operator acting on overconvergent modular forms of weight κ. Even though the evidence is rather scant at the moment we will call this a conjecture.

(8.2) **Conjecture.** *$P_\kappa(t,U)$ is equal to the characteristic power series of the U-operator acting on overconvergent modular forms of weight κ.*

The only evidence for this at the moment consists of the comparison theorem 7.1.

We view the comparison theorem as an equivariant version of the construction of p-adic L-functions associated to classical modular symbols. More precisely, if κ is arithmetic of signature (k,ϵ) and $h < k+1$, then the construction of Vishik, Amice-Velu, Mazur-Tate-Teitelbaum associates to each modular symbol $\varphi \in H^1_c(\Gamma_0(p), L_\kappa)^{(h)}$ a weakly h-admissible p-adic distribution $\mu_\varphi \in \mathbf{D}^{(h)}$. The following theorem is easily confirmed.

(8.3) **Theorem.** *Suppose $\Phi \in H^1_c(\Gamma_0(p), \mathbf{D}^{(h)}_\kappa)^\#$ corresponds to φ in diagram (7.4). Then*

$$\mu_\varphi = \Phi\left(\{\,0\,\} - \{\,i\infty\,\}\right).$$