Computations on $X_0(11)$

§0. Overview.

Let's start with a quick overview of the structure of the computation. Let $\Gamma = \Gamma_0(11)$, let p be a prime of good ordinary reduction for the elliptic curve $X_0(11)$, and let $\Gamma_0 = \Gamma_0(11p)$. Consider the p-adic topological vector spaces

$$\begin{aligned}
\mathcal{D} &:= \mathcal{D}(\mathbf{P}(\mathbf{Q}_p)),\\
\mathcal{D}^\dagger &:= \mathcal{D}^\dagger\big(\mathbf{P}(\mathbf{Q}_p), 1/p\big),\\
\mathcal{D}_0 &:= \mathcal{D}(\mathbf{Z}_p),\\
\mathcal{D}_0^\dagger &:= \mathcal{D}^\dagger(\mathbf{Z}_p, 1).
\end{aligned}$$

These are all described in the notes on distributions. The spaces $\mathcal{D}$, $\mathcal{D}^\dagger$ are Γ-modules, while $\mathcal{D}_0$, $\mathcal{D}_0^\dagger$ are Γ_0-modules. There is a natural commutative diagram

$$\begin{array}{ccccc}
\mathcal{D} & \hookrightarrow & \mathcal{D}^\dagger & \xrightarrow{\rho} & \mathrm{Ind}_{\Gamma_0}^{\Gamma}\mathbf{Q}_p\\
\downarrow & & \downarrow & & \downarrow\\
\mathcal{D}_0 & \hookrightarrow & \mathcal{D}_0^\dagger & \xrightarrow{\rho_0} & \mathbf{Q}_p
\end{array}$$

in which the maps in the first row are Γ-morphisms, those in the second row are Γ_0-morphisms, and the vertical arrows commute with the action of Γ_0. The vertical maps are defined by "restriction to $\mathbf{Z}_p$".

These maps induce a sequence of maps on modular symbols

$$\begin{array}{ccccc}
H^1_c(\Gamma, \mathcal{D}) & \hookrightarrow & H^1_c(\Gamma, \mathcal{D}^\dagger) & \xrightarrow{\rho_*} & H^1_c(\Gamma, \mathrm{Ind}_{\Gamma_0}^{\Gamma}\mathbf{Q}_p)\\
r\downarrow & & r\downarrow & & r\downarrow\\
H^1_c(\Gamma_0, \mathcal{D}_0) & \hookrightarrow & H^1_c(\Gamma_0, \mathcal{D}_0^\dagger) & \xrightarrow{\rho_{0*}} & H^1_c(\Gamma_0, \mathbf{Q}_p).
\end{array}$$

The Hecke operators T_n ($p \nmid n$) act on all of these spaces and all of the above arrows commute with these operators. We also have operators U_p, which commute with all arrows in the diagram. The operators T_n and U_p all commute with one another.

Important Fact: All maps in the above diagram, except ρ_* and ρ_{0*}, are isomorphisms when restricted to the "finite slope subspaces" for U_p. Moreover, the maps ρ_*, ρ_{0*} are isomorphisms when restricted to the "slope < 1" subspaces.

Our Goal: Let φ be a Hecke eigenvector in $H^1_c(\Gamma_0, \mathbf{Q}_p)$ of slope 1. Our goal is to compute a Hecke eigenvector $\Phi \in H^1_c(\Gamma_0, \mathcal{D}_0^\dagger)$ such that $\rho_{0*}(\Phi) = \varphi$.

More precisely, we let φ be a Hecke eigenvector in $H^1_c(\Gamma_0, \mathbf{Q}_p)$ associated to the elliptic curve $X_0(11)$ (i.e. an "old" cohomology class) and assume also that the eigenvalue β of U_p acting on φ is divisible by p, in fact $\mathrm{ord}_p(\beta) = 1$. Thus φ is one of the two (either even or odd) modular symbols to which the method of Vishik-Amice-Velu fails to attach a p-adic L-function. To attempt to construct such a p-adic L-function, we proceed in three steps:

Step 1: Compute an element $\Phi_0 \in H^1_c(\Gamma_0, \mathcal{D}_0^\dagger)$, which lifts φ, i.e. such that $\rho_{0*}(\Phi_0) = \varphi$.

Step 2: For each $n \geq 0$ compute $\Phi_n := \beta^{-n} \cdot \Phi_0 | U_p^n$ and compute the limit

$$\Phi := \lim_{n \to \infty} \Phi_n \in H_c^1(\Gamma_0, \mathcal{D}_0^\dagger).$$

I expect this limit exists, but I still have no complete proof. For now, let's assume the limit exists. In that case, we must have $\Phi | U_p = \beta \Phi$, so Φ has finite slope, and it follows that Φ lives in the image of the inclusion $H_c^1(\Gamma, \mathcal{D}_0) \hookrightarrow H_c^1(\Gamma, \mathcal{D}_0^\dagger)$. Thus we may regard Φ as an element of $H_c^1(\Gamma, \mathcal{D}_0)$. As a double-check, we should now confirm that this Φ is an eigenvector for all the Hecke operators with the same eigenvalues as φ.

Step 3: Finally, we calculate the p-adic L-function $L_p(\Phi, s)$. This is the p-adic L-function we are looking for.

Remarks:

1. Though the above outline does not explicitly mention any of the Γ-cohomology spaces, i.e. the first row of the above diagram, I think we will want to use them. I expect it will be easier to do some of our calculations at level 11 rather than level $11p$.
2. I believe the limit in (2) will exist, but am not certain. In that case, the p-adic L-function in (3) will interpolate the critical values of the complex L-function in the usual way. If I am wrong, and the limit does not exist, then we will have to find another way of constructing a Hecke eigenclass out of the lifting Φ_0. In any case, I don't think it will be hard to make such an eigenclass, but we'll have to see what comes out.

§1. The Steinberg module as Γ-module.

We need to have a very easy to use description of the Steinberg module as a Γ-module. This is given to us by the well-known Manin relations (see reference [3] in the notes on distributions), but it will be convenient to make these explicit in our special case. The theorem at the end of this section explains how, in this case, the Manin relations reduce to a single relation.

The Steinberg module is the $\mathbf{Z}[GL_2(\mathbf{Q})]$-module

$$\Delta_0 := \mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q})).$$

The action of $GL_2(\mathbf{Q})$ on Δ_0 is given by standard fractional linear transformations acting on $\mathbf{P}^1(\mathbf{Q})$ on the left. Our interest in this module is motivated by the fact that

$$H_c^1(\Gamma, M) \cong \mathrm{Hom}_\Gamma(\Delta_0, M)$$

for any Γ-module M. Before continuing, I should explain my conventions. I will always assume M is a *right* Γ-module. On the other hand, Δ_0 is a *left* Γ-module, so I'd better explain what I mean by a Γ-morphism $\Delta_0 \longrightarrow M$. My convention is that a right (left) Γ-module M may be considered a left (right) Γ-module by defining $\gamma m := m | \gamma^{-1}$ ($m | \gamma := \gamma^{-1} m$) for any $\gamma \in \Gamma$, $m \in M$. If M is a right Γ-module and $\varphi : \Delta_0 \longrightarrow M$ is any $\mathbf{Z}$-linear map, then for $\gamma \in \Gamma$ we define

$$\varphi | \gamma : \Delta_0 \longrightarrow M$$

by the rule $(\varphi|\gamma)(D) = \varphi(\gamma D)|\gamma$. Thus $\varphi : \Delta_0 \longrightarrow M$ is a Γ-morphism if and only if $\varphi|\gamma = \varphi$ for all $\gamma \in \Gamma$.

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbf{Q})$, I will use the notation $[\gamma] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to denote the singular 1-chain in the extended upper half-plane $\mathcal{H}^*$ represented by the geodesic path joining $\frac{a}{c}$ to $\frac{b}{d}$. I'll call such a 1-chain a "modular path" and any finite formal sum of such modular paths, a modular 1-chain. The $\mathbf{Z}$-module of all such modular chains will be denoted

$$Z_1 := Z_1(\mathcal{H}^*, \mathbf{P}^1(\mathbf{Q})),$$

which we regard as a module of 1-cycles in $\mathcal{H}^*$ relative to the boundary $\mathbf{P}^1(\mathbf{Q})$ of $\mathcal{H}^*$.

The group $PGL_2^+(\mathbf{Q})$ acts on Z_1 via standard fractional linear transformations on $\mathcal{H}^*$, hence Z_1 is naturally a $PGL_2(\mathbf{Q})$-module. If $\beta, \gamma \in GL_2^+(\mathbf{Q})$ then we have

$$\beta \cdot [\gamma] := [\beta\gamma].$$

The boundary map gives us a surjective $GL_2^+(\mathbf{Q})$-morphism

$$\partial : Z_1 \longrightarrow \Delta_0.$$

We say two modular chains c, c' are homologous if $\partial c = \partial c'$. Thus ∂ induces a $PGL_2^+(\mathbf{Q})$-isomorphism from the one-dimensional relative homology of the pair $(\mathcal{H}^*, \mathbf{P}^1(\mathbf{Q}))$ to the Steinberg module Δ_0:

$$\partial : H_1(\mathcal{H}^*, \mathbf{P}(\mathbf{Q}); \mathbf{Z}) \xrightarrow{\cong} \Delta_0.$$

Let $G = PSL_2(\mathbf{Z})$. A modular path of the form $[\gamma]$ with $\gamma \in G$ is called a "unimodular path" and any finite formal sum of such unimodular paths is called a unimodular 1-chain. Using continued fractions it is easy to see (and is a well-known result of Manin [3]) that every modular chain is homologous to a unimodular chain. Moreover, G acts transitively on the unimodular paths. Indeed, the map

$$\begin{aligned} G &\longrightarrow Z_1 \\ \gamma &\longmapsto [\gamma] \end{aligned}$$

is a bijection from G to the set of unimodular paths in Z_1. Extending by linearity, we obtain a G-morphism $\mathbf{Z}[G] \longrightarrow Z_1$, and composing with the boundary map ∂ we obtain a surjective G-morphism

$$e : \mathbf{Z}[G] \longrightarrow \Delta_0$$

is a surjective map of G-modules. We know from a result of Manin that the kernel of e is the right ideal

$$I := \mathbf{Z}[G](1 + \tau + \tau^2) + \mathbf{Z}[G](1 + \sigma)$$

where $\tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. These are the well-known Manin relations.

The Manin relations allow us to describe the structure of Δ_0 as a Γ-module in terms of generators and relations. The map $G \longrightarrow \mathbf{P}^1(\mathbf{F}_{11})$ defined by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{d}{c}$ is surjective and its fibers are the right Γ-cosets. We choose the section $g : \mathbf{P}^1(\mathbf{F}_{11}) \longrightarrow G$ of this map defined by

$$g_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_{10} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad g_0 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix};$$

$$g_1 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad g_9 = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix};$$

$$g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \quad g_4 = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}, \quad g_7 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix};$$

$$g_3 = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}, \quad g_6 = \begin{pmatrix} -1 & 1 \\ 3 & -4 \end{pmatrix}, \quad g_8 = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}.$$

Note that the elements of each of the four rows above is a left coset of the subgroup $\langle \tau \rangle \subseteq G$. Hence by the Manin relations, the sum of any row is an element of $\mathbf{Z}[G]$ which lies in the kernel of the map $e : \mathbf{Z}[G] \longrightarrow \Delta_0$. For each $i \in \mathbf{P}^1(\mathbf{F}_{11})$ define

$$D_i := e(g_i) \in \Delta_0.$$

Theorem. *Δ_0 is generated as a Γ-module by D_∞, D_7, and D_9. The only $\mathbf{Z}[\Gamma]$-relation between these elements is*

$$\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} - 1\right) D_\infty + \left(1 - \begin{pmatrix} -5 & -1 \\ 11 & 2 \end{pmatrix}\right) D_7 + \left(1 - \begin{pmatrix} 4 & 3 \\ -11 & -8 \end{pmatrix}\right) D_9 = 0.$$

Proof: A simple calculation shows that the above identity is valid. Indeed, $D_\infty = \{0\} - \{\infty\}$, so $\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} - 1\right) D_\infty = \{-1\} - \{0\}$. Similarly, $D_7 = \{0\} - \{-1/3\}$, hence $\left(1 - \begin{pmatrix} -5 & -1 \\ 11 & 2 \end{pmatrix}\right) D_7 = \{0\} - \{-\frac{1}{3}\} - \{-\frac{1}{2}\} + \{-\frac{2}{5}\}$; and $D_9 = \{-\frac{1}{2}\} - \{-1\}$, hence $\left(1 - \begin{pmatrix} 4 & 3 \\ -11 & -8 \end{pmatrix}\right) D_9 = \{-\frac{1}{2}\} - \{-1\} - \{-\frac{2}{5}\} + \{-\frac{1}{3}\}$. These sum to zero.

It is not hard to see that this relation is equivalent to the Manin relations. I'll leave the details for later.

Corollary: *Let M be an arbitrary right Γ-module and choose three elements $m_7, m_9, m_\infty \in M$ such that*

$$m_\infty|\Delta = m_7 \left| \left(\begin{pmatrix} 2 & 1 \\ -11 & -5 \end{pmatrix} - 1\right) + m_9 \right| \left(\begin{pmatrix} -8 & -3 \\ 11 & 4 \end{pmatrix} - 1\right)$$

where Δ is the difference operator $\Delta := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1$. *Now define*

$$\begin{aligned}
m_\infty &:= m_\infty \\
m_0 &:= -m_\infty \,\Big|\, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\
m_1 &:= m_\infty \,\Big|\, \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1 \right) \\
m_2 &:= m_9 + m_\infty \,\Big|\, \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1 \right) \\
m_3 &:= -m_7 \\
m_4 &:= -m_7 - m_9 - m_\infty \,\Big|\, \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1 \right) \\
m_5 &:= -m_9 - m_\infty \,\Big|\, \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1 \right) \\
m_6 &:= -m_9 \,\Big|\, \begin{pmatrix} -7 & -2 \\ 11 & 3 \end{pmatrix} \\
m_7 &:= m_7 \\
m_8 &:= m_7 + m_9 \,\Big|\, \begin{pmatrix} -7 & -2 \\ 11 & 3 \end{pmatrix} \\
m_9 &:= m_9 \\
m_{10} &:= m_\infty \,\Big|\, \left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} - 1 \right).
\end{aligned}$$

Then there is a unique modular symbol $\varphi \in H^1_c(\Gamma, M)$ *such that*

$$\varphi(D_i) = m_i, \quad \text{for all } i \in \mathbf{P}^1(\mathbf{F}_{11}).$$

Moreover, once the m_i *are known, the continued fraction algorithm of Manin will efficiently compute the value of* φ *on any* $D \in \Delta_0$. *Conversely, every modular symbol* $\varphi \in H^1_c(\Gamma, M)$ *arises in this manner.*

Proof: This follows immediately from the Manin relations.

§2. The main diagram.

Our main diagram is

$$\begin{array}{ccccc}
H^1_c(\Gamma, \mathcal{D}) & \hookrightarrow & H^1_c(\Gamma, \mathcal{D}^\dagger) & \xrightarrow{\rho_*} & H^1_c(\Gamma, \mathrm{Ind}^\Gamma_{\Gamma_0} \mathbf{Q}_p) \\
r\big\downarrow & & r\big\downarrow & & r\big\downarrow \\
H^1_c(\Gamma_0, \mathcal{D}_0) & \hookrightarrow & H^1_c(\Gamma_0, \mathcal{D}^\dagger_0) & \xrightarrow{\rho_{0*}} & H^1_c(\Gamma_0, \mathbf{Q}_p),
\end{array}$$

which was briefly described in §**0**. In this section I will recall what all the terms in the diagram are and make a few remarks about how we might calculate in them.

I will use the words "distribution" and "log-differential" almost interchangeably. The dictionary between the two is described in the notes on distributions. With these identifications we have

$$\begin{aligned}\mathcal{D} &:= \mathcal{D}(\mathbf{P}(\mathbf{Q}_p)) = \Omega_{\log}(\mathcal{H}_p), \text{ and} \\ \mathcal{D}^\dagger &:= \mathcal{D}^\dagger(\mathbf{P}(\mathbf{Q}_p), 1/p) = \bigoplus_{\mathbf{x}\in\mathbf{P}(\mathbf{F}_p)} \mathcal{D}^\dagger(\mathbf{x}, 1/p) = \bigoplus_{\mathbf{x}\in\mathbf{P}(\mathbf{F}_p)} \Omega_{\log}(W(\mathbf{x}, 1/p)).\end{aligned}$$

We also have

$$\begin{aligned}\mathcal{D}_0 &:= \mathcal{D}_0(\mathbf{Z}_p) = \Omega_{\log}(\mathcal{H}_p(\mathbf{Z}_p)), \text{ and} \\ \mathcal{D}_0^\dagger &:= \mathcal{D}_0^\dagger(\mathbf{Z}_p, 1) = \Omega_{\log}(W(\mathbf{Z}_p, 1)).\end{aligned}$$

We will need programs that work efficiently with elements of $\mathcal{D}^\dagger$ and $\mathcal{D}_0^\dagger$. Then $\mathcal{D}$ and $\mathcal{D}_0$ will take care of themselves.

Let's start by describing the elements of $\mathcal{D}^\dagger$. Let j run over the set $\{\infty, 0, 1, \ldots, p-1\} \subseteq \mathbf{P}(\mathbf{C}_p)$ and for each such j, let $\mathbf{x}_j \in \mathbf{P}(\mathbf{F}_p)$ be the congruence class to which j belongs. For each j we also choose the following uniformizer at j:

$$w_j := \begin{cases} z - j & \text{if } j \neq \infty; \\ 1/z & \text{if } j = \infty. \end{cases}$$

An element $\mu \in \mathcal{D}^\dagger$ has a unique representation as a sum of log-differentials

$$\mu = \bigoplus_j \mu_j$$

where each $\mu_j \in \Omega_{\log}(W(\mathbf{x}_j, 1/p))$ and therefore has a unique expansion in the form

$$\mu_j = a_0(j)\delta_j + \sum_{n=1}^{\infty} a_n(j) w_j^{-n} \cdot \frac{dw_j}{w_j}$$

where δ_j is the Dirac distribution concentrated at j, or equivalently,

$$\delta_j := \frac{d(Y - jX)}{Y - jX} \in \Omega_{\log}(W(\mathbf{x}_j, 1/p)).$$

There are growth conditions on the coefficients $a_n(j)$ (see the proposition in section 3 of the notes on distributions).

The elements of $\mathcal{D}_0^\dagger$ are simpler to represent. Each $\mu \in \mathcal{D}_0^\dagger$ is a log-differential $\mu \in \Omega_{\log}(W(\mathbf{Z}_p, 1))$ and can therefore be expressed uniquely in the form

$$\mu = a_0\delta_0 + \sum_{n=1}^{\infty} a_n z^{-n} \cdot \frac{dz}{z}.$$

There are also growth conditions on the coefficients a_n.

The map $r : \mathcal{D}^\dagger \longrightarrow \mathcal{D}_0^\dagger$ is given by

$$\bigoplus_j \mu_j \longmapsto \sum_j \mu_j|_{W(\mathbf{Z}_p,1)}.$$

Explicitly, if $\mu_j = a_0(j)\delta_j + \sum_{n=1}^\infty a_n(j) w_j^{-n} \cdot \frac{dw_j}{w_j}$ then to calculate the series of $\mu_j|_{W(\mathbf{Z}_p,1)}$ we use the identity

$$\delta_j = \delta_0 + (\delta_j - \delta_0) = \delta_0 + \frac{d(1-j/z)}{1-j/z} = \delta_0 + j \cdot w_j^{-1} \frac{dz}{z}$$

and then replace w_j^{-1} by its taylor series in z^{-1}

$$w_j^{-1} = \frac{1}{z-j} = \frac{1/z}{1-j/z} = \sum_{n=1}^\infty j^{n-1} z^{-n}.$$

The induced module $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathbf{Q}_p)$ is defined by

$$\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathbf{Q}_p) := \mathbf{Q}_p[\mathbf{P}^1(\mathbf{F}_p)]$$

where we write the elements of $\mathbf{P}^1(\mathbf{F}_p)$ as homogeneous row vectors and let Γ act on these by matrix multiplication on the right. We identify $\mathbf{P}(\mathbf{F}_p)$ with $\mathbf{F}_p \cup \{\infty\}$ by the correspondence $[c,d] \leftrightarrow \frac{d}{c}$.

The "residue" map ρ is defined by

$$\begin{aligned} \rho : \mathcal{D}^\dagger &\longrightarrow \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathbf{Q}_p) \\ \mu &\longmapsto \sum_{\mathbf{x} \in \mathbf{P}(\mathbf{F}_p)} \left(\int_{\mathbf{x}} 1 \cdot d\mu \right) \cdot \mathbf{x} = \sum_j a_0(j) \cdot \mathbf{x}_j. \end{aligned}$$

The "restriction map" $r : \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathbf{Q}_p) \longrightarrow \mathbf{Q}_p$ is defined by

$$\begin{aligned} \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathbf{Q}_p) &\longrightarrow \mathbf{Q}_p \\ \sum_{\mathbf{x}} m_{\mathbf{x}} \cdot \mathbf{x} &\longmapsto \sum_{\mathbf{x} \neq \mathbf{x}_\infty} m_{\mathbf{x}}. \end{aligned}$$

We also have the Shapiro map

$$\mathcal{S} : \sum_{\mathbf{x}} m_{\mathbf{x}} \cdot \mathbf{x} \mapsto m_{\mathbf{x}_\infty}.$$

Both r and $\mathcal{S}$ commute with the action of Γ_0.

Lemma. *The restriction map r induces an isomorphism on modular symbols:*

$$r_* : H^1_c(\Gamma, \mathrm{Ind}_{\Gamma_0}^{\Gamma} \mathbf{Q}_p) \xrightarrow{\sim} H^1_c(\Gamma_0, \mathbf{Q}_p).$$

Proof. Define the star operator St on $\mathrm{Ind}_{\Gamma_0}^{\Gamma} \mathbf{Q}_p$ by

$$\begin{array}{c} \mathrm{Ind}_{\Gamma_0}^{\Gamma} \mathbf{Q}_p \xrightarrow{St} \mathrm{Ind}_{\Gamma_0}^{\Gamma} \mathbf{Q}_p \\ \sum_{\mathbf{x}} a_{\mathbf{x}} \cdot \mathbf{x} \longmapsto \sum_{\mathbf{x}} \left(\sum_{\mathbf{y} \neq \mathbf{x}} a_{\mathbf{y}} \right) \cdot \mathbf{x} \end{array}$$

This map is easily seen to be a Γ-isomorphism, hence induces an isomorphism on cohomology. Moreover, the following diagram is commutive

$$\begin{array}{ccc} H^1_c(\Gamma, \mathrm{Ind}_{\Gamma_0}^{\Gamma} \mathbf{Q}_p) & \xrightarrow{St_*} & H^1_c(\Gamma, \mathrm{Ind}_{\Gamma_0}^{\Gamma} \mathbf{Q}_p) \\ r_* \downarrow & & \mathcal{S}_* \downarrow \\ H^1_c(\Gamma_0, \mathbf{Q}_p) & \xrightarrow{=} & H^1_c(\Gamma_0, \mathbf{Q}_p). \end{array}$$

where $\mathcal{S}_*$ is the Shapiro isomorphism.

Corollary. *The operator $U_p : H^1_c(\Gamma_0, \mathbf{Q}_p) \longrightarrow H^1_c(\Gamma_0, \mathbf{Q}_p)$ is invertible.*

Proof: The following diagram is commutative.

$$\begin{array}{ccc} H^1_c(\Gamma, \mathrm{Ind}_{\Gamma_0}^{\Gamma} \mathbf{Q}_p) & \xrightarrow{=} & H^1_c(\Gamma, \mathrm{Ind}_{\Gamma_0}^{\Gamma} \mathbf{Q}_p) \\ \mathcal{S}_* \downarrow & & r_* \downarrow \\ H^1_c(\Gamma_0, \mathbf{Q}_p) & \xrightarrow{w_p U_p} & H^1_c(\Gamma_0, \mathbf{Q}_p). \end{array}$$

The result now follows from the lemma and the fact that w_p is invertible on $H^1_c(\Gamma_0, \mathbf{Q}_p)$.

Theorem. *The map $\rho_* : H^1_c(\Gamma, \mathcal{D}^\dagger) \longrightarrow H^1_c(\Gamma, \mathrm{Ind}_{\Gamma_0}^{\Gamma} \mathbf{Q}_p)$ is surjective. More precisely, for arbitrary $\psi \in H^1_c(\Gamma, \mathrm{Ind}_{\Gamma_0}^{\Gamma} \mathbf{Q}_p)$ we may construct $\Psi \in H^1_c(\Gamma, \mathcal{D}^\dagger)$ such that $\rho_*(\Psi) = \psi$ as follows. For each $D \in \Delta_0$ write*

$$\psi(D) =: \sum_{\mathbf{x} \in \mathbf{P}(\mathbf{F}_p)} m_{\mathbf{x}}(D) \cdot \mathbf{x}$$

where the $m_{\mathbf{x}}(D) \in \mathbf{Q}_p$ are uniquely determined by these identities. Choose $\mu_7, \mu_9 \in \mathcal{D}^\dagger$ to be arbitrary overconvergent distributions for which

$$\int_{\mathbf{x}} 1 \cdot d\mu_i = m_{\mathbf{x}}(D_i)$$

for $i = 7$ and $i = 9$ and for all $\mathbf{x} \in \mathbf{P}^1(\mathbf{F}_p)$. If we set

$$\nu := \mu_7 \Big| \left(\begin{pmatrix} 2 & 1 \\ -11 & -5 \end{pmatrix} - 1 \right) + \mu_9 \Big| \left(\begin{pmatrix} -8 & -3 \\ 11 & 4 \end{pmatrix} - 1 \right)$$

then there is a unique solution $\mu_\infty \in \mathcal{D}^\dagger$ *of the difference equation*

$$\mu_\infty | \Delta = \nu$$

with the property $\rho(\mu_\infty) = \psi(D_\infty)$. *Then there is a unique modular symbol* $\Psi \in H^1_c(\Gamma, \mathcal{D}^\dagger)$ *such that* $\Psi(D_i) = \mu_i$ *for* $i = \infty, 7, 9$. *Furthermore, we have* $\rho_*(\Psi) = \psi$.

Proof. By the corollary at the end of the last section, we just need to show that the difference equation $\mu_\infty|\Delta = \nu$ has a solution $\mu_\infty \in \mathcal{D}^\dagger$.

By the results of the last section

$$\psi(D_\infty)|\Delta = \psi(D_7) \left| \left(\begin{pmatrix} 2 & 1 \\ -11 & -5 \end{pmatrix} - 1 \right) + \psi(D_9) \right| \left(\begin{pmatrix} -8 & -3 \\ 11 & 4 \end{pmatrix} - 1 \right).$$

But the right hand side of this identity is just $\rho(\nu)$. Thus $\rho(\nu)$ is in the image of the difference operator Δ. From this it follows at once that

$$\int_{\mathbf{X}_\infty} 1 \cdot d\nu = \int_{\mathbf{Z}_p} 1 \cdot d\nu = 0.$$

Thus, by our results on distributions, there is a unique μ_∞ satisfying the difference equation. This completes the proof of the theorem.

§3. Step One:

Let $\varphi \in H^1_c(\Gamma_0, \mathbf{Q}_p)$. We wish to compute $\Phi_0 \in H^1_c(\Gamma_0, \mathcal{D}_0^\dagger)$ such that $\rho_{0*}(\Phi_0) = \varphi$. We do this by taking the following steps.

Step 1a. *Compute* $\psi \in H^1_c(\Gamma, \mathrm{Ind}_{\Gamma_0}^{\Gamma} \mathbf{Q}_p)$ *such that* $r_*(\psi) = \varphi$.

First define a section of the map $\Gamma \longrightarrow \mathbf{P}(\mathbf{F}_p)$, $\gamma \mapsto \mathbf{x}_\infty \gamma$. In other words, for each $\mathbf{x} \in \mathbf{P}(\mathbf{F}_p)$ choose an element $\gamma_\mathbf{x} \in \Gamma$ such that $\mathbf{x}_\infty \gamma_\mathbf{x} = \mathbf{x}$.

Now regard φ as a Γ_0-morphism $\varphi : \Delta_0 \longrightarrow \mathbf{Q}_p$ and define $\psi : \Delta_0 \longrightarrow \mathrm{Ind}_{\Gamma_0}^{\Gamma} \mathbf{Q}_p$ by

$$\psi(D) := \sum_{\mathbf{x} \in \mathbf{P}(\mathbf{F}_p)} m_\mathbf{x}(D) \cdot \mathbf{x} \in \mathrm{Ind}_{\Gamma_0}^{\Gamma} \mathbf{Q}_p$$

where

$$m_\mathbf{x}(D) := -\varphi(\gamma_\mathbf{x} D) + \frac{1}{p} \sum_{\mathbf{y}} \varphi(\gamma_\mathbf{y} D).$$

A simple calculation shows $\psi|\gamma = \psi$ for all $\gamma \in \Gamma$, hence $\psi \in H^1_c(\Gamma, \mathrm{Ind}_{\Gamma_0}^{\Gamma} \mathbf{Q}_p)$. Furthermore, we have

$$r_*(\psi) = \varphi$$

as desired.

Step 1b. *Compute an element* $\Psi \in H^1_c(\Gamma, \mathcal{D}^\dagger)$ *such that* $\rho_*(\Psi) = \psi$.

For this, we just follow the steps described in the statement of the theorem at the end of the last section. The key step is to solve the difference equation

$$\mu_\infty | \Delta = \nu$$

for $\mu_\infty \in \mathcal{D}^\dagger$. An explicit formula for such a μ is given in the notes on distributions.

Finally, define μ_i for $i = \infty, 0, 1, \ldots, 10$ as in the statement of the corollary at the end of section 1. We then define $\Psi \in H^1_c(\Gamma, \mathcal{D}^\dagger)$ to be the unique modular symbol for which

$$\Psi(D_i) = \mu_i$$

for each $i = \infty, 0, 1, \ldots, 10$. This modular symbol can be calculated quickly on any $D \in \Delta_0$ using this data and Manin's continued fraction algorithm.

Step 1c. *Compute* $\Phi_0 := r_*(\Psi) \in H^1_c(\Gamma_0, \mathcal{D}^\dagger_0)$.

First compute a section of the map $G \longrightarrow \mathbf{P}^1(\mathbf{Z}/11p\mathbf{Z})$. In other words, for each $\mathbf{x} \in \mathbf{P}^1(\mathbf{Z}/11p\mathbf{Z})$, choose an element $g_\mathbf{x} \in G$ whose bottom row represents $\mathbf{x}$ in the usual way. Then for each $\mathbf{x} \in \mathbf{P}^1(\mathbf{Z}/11p\mathbf{Z})$ let $D_\mathbf{x} := e(g_\mathbf{x})$. and set

$$\mu_\mathbf{x} := r(\Psi(D_\mathbf{x})) \in \mathbf{D}^\dagger_0.$$

We define $\Phi_0 \in H^1_c(\Gamma_0, \mathcal{D}^\dagger_0)$ to be the unique modular symbol for which

$$\Phi_0(D_\mathbf{x}) = \mu_\mathbf{x}$$

for all $\mathbf{x} \in \mathbf{P}^1(\mathbf{Z}/11p\mathbf{Z})$. This modular symbol can be calculated on any $D \in \Delta_0$ using Manin's continued fraction algorithm.

§4 ... Steps two and three.

These deserve to be the easy part. To be continued ...