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Efficient Estimation of Spectral Functionals for Continuous-Time Stationary Models

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Abstract The paper considers a problem of construction of asymptotically efficient estimators for functionals defined on a class of spectral densities, and bounding the minimax mean square risks. We define the concepts of H- and IK-efficiency of estimators, based on the variants of Hájek-Ibragimov-Khas'minskii convolution theorem and Hájek-Le Cam local asymptotic minimax theorem, respectively, and show that the simple "plug-in" statistic $\Phi(I_T)$, where $I_T = I_T(\lambda)$ is the periodogram of the underlying stationary Gaussian process $X(t)$ with an unknown spectral density $\theta(\lambda)$, $\lambda \in \mathbb{R}$, is H- and IK-asymptotically efficient estimator for a linear functional $\Phi(\theta)$, while for a nonlinear smooth functional $\Phi(\theta)$ an H- and IK-asymptotically efficient estimator is the statistic $\Phi(\widehat{\theta_T})$, where $\widehat{\theta_T}$ is a suitable sequence of the so-called "undersmoothed" kernel estimators of the unknown spectral density $\theta(\lambda)$. Exact asymptotic bounds for minimax mean square risks of estimators of linear functionals are also obtained.

Keywords Efficient nonparametric estimation · Spectral functionals · Continuous-time stationary process · Spectral density · Singularities · Local asymptotic normality (LAN) · Asymptotic bounds

Mathematics Subject Classification (2000) 62G05 · 62G20 · 62M15 · 60G10

1 Introduction

1.1 The Problem and Objectives

The problem of efficient nonparametric estimation of different kind of functionals for various statistical models has been extensively discussed in the literature (see, for instance,

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Ibragimov and Khas'minskii [21], Pfanzagl [34], Taniguchi and Kakizawa [39], Kutoyants [26], and references therein).

This paper is concerned with the problem of efficient nonparametric estimation of spectral functionals for continuous-time stationary Gaussian models.

The problem. Suppose we observe a realization $\mathbf{X}_T = \{X(t),\ 0 \le t \le T\}$ of a zero mean real-valued mean square continuous stationary Gaussian process $X(t)$ with an *unknown* spectral density function $\theta(\lambda)$, $\lambda \in \mathbb{R}$. We assume that $\theta(\lambda)$ belongs to a given class $\boldsymbol{\Theta} \subset L^p = L^p(\mathbb{R})$ $(p \ge 1)$ of spectral densities possessing some smoothness properties. Let $\Phi(\cdot)$ be some *known* functional, the domain of definition of which contains $\boldsymbol{\Theta}$. The distribution of the process $X(t)$ is completely determined by the spectral density, and we consider $\theta(\lambda)$ as an infinite-dimensional "parameter" on which the distribution of $X(t)$ depends. The problem is to estimate the value $\Phi(\theta)$ of the functional $\Phi(\cdot)$ at an unknown point $\theta \in \boldsymbol{\Theta}$ on the basis of an observation $\mathbf{X}_T$, and investigate the asymptotic (as $T \to \infty$) properties of the suggested estimators. The main objective is construction of asymptotically efficient estimators for $\Phi(\theta)$.

This problem for discrete-time stationary Gaussian processes has been considered in a number of articles. We cite merely the papers Millar [31], Ibragimov and Khas'minskii [19, 23], Ginovyan [8, 14], and Dahlhaus and Wefelmeyer [5].

Notice that in Millar [31] and Dahlhaus and Wefelmeyer [5] were considered efficiency concept based on a nonparametric version of Hájek convolution theorem, while in Ibragimov and Khas'minskii [19, 23] and Ginovyan [8] the efficiency is based on a nonparametric version of Hájek-Le Cam local asymptotic minimax theorem. In Ginovyan [14] were considered both efficiency concepts in the class of spectral densities possessing singularities.

For continuous-time processes the problem was partially studied in Ginovyan [9–11], where efficient nonparametric estimators for linear functionals were constructed and asymptotic upper bounds for minimax mean square risks of these estimators were obtained.

The objective of the present paper is construction of asymptotically efficient nonparametric estimators for linear and some nonlinear smooth spectral functionals and bounding the minimax mean square risks of suggested estimators in the case where the underlying model is a continuous-time stationary processes with possibly unbounded or vanishing spectral density function. For construction of asymptotically efficient estimators we use a general powerful method developed by Ibragimov and Khas'minskii [19, 23] (see, also, Goldstein and Khas'minskii [15]). Our plan will be as follows:

- We define the concept of local asymptotic normality (LAN) in the spirit of Ibragimov and Khas'minskii [23], and derive conditions under which the underlying family of Gaussian distributions is LAN at a point $\theta \in \boldsymbol{\Theta}$.
- Using LAN we state variants of Hájek-Le Cam local asymptotic minimax theorem and Hájek-Ibragimov-Khas'minskii convolution theorem.
- We define the concepts of H- and IK-efficiency of estimators, based on the variants of Hájek-Ibragimov-Khas'minskii convolution theorem and Hájek-Le Cam local asymptotic minimax theorem, respectively, and prove that the simple "plug-in" statistic $\Phi(I_T)$, where $I_T = I_T(\lambda)$ is the periodogram of the underlying stationary Gaussian process $X(t)$ with an unknown spectral density $\theta(\lambda)$, $\lambda \in \mathbb{R}$, is H- and IK-asymptotically efficient estimator for a linear functional $\Phi(\theta)$, while for a nonlinear smooth functional $\Phi(\theta)$ an H- and IK-asymptotically efficient estimator is the statistic $\Phi(\widehat{\theta}_T)$, where $\widehat{\theta}_T$ is a suitable sequence of the so-called "undersmoothed" kernel estimators of the unknown spectral density $\theta(\lambda)$.
- We obtain exact asymptotic bounds for minimax mean square risks of estimators of linear functionals.

1.2 The Model

Statistical analysis of Gaussian stationary processes usually requires two type of conditions imposed on the spectral density $\theta(\lambda)$. The first type of these conditions controls the singularities (zeros and poles) of function $\theta(\lambda)$, and describes the dependence structure of the underlying process $X(t)$, while the second type conditions requires smoothness of spectral density $\theta(\lambda)$. Much of statistical inferences (parametric and non-parametric) is concerned with the so-called *short-memory* stationary models, in which case the spectral density $\theta(\lambda)$ of the model $X(t)$ is assumed to be separated from zero and infinity, that is, $0 < C_1 \le \theta(\lambda) \le C_2 < \infty$ with some constants C_1 and C_2. However, the data in many fields of science (e.g. in economics, engineering, finance, hydrology, etc.) occur in the form of a realization of a stationary process $X(t)$ with possibly unbounded (*long-memory* model) or vanishing (*anti-persistent* model) spectral density (see, for instance, Beran [1]). So, it is important to consider a model that will include all these cases.

To specify our model we need the following definition (see, e.g., Hunt, Muckenhoupt and Wheeden [20], Böttcher and Karlovich [3], Sect. 2.1, Stein [37], Sect. 5.1).

Definition 1.1 (Muckenhoupt condition (A_2)) We say that a nonnegative locally integrable function $f(\lambda)$ $(\lambda \in \mathbb{R})$ satisfies the *Muckenhoupt condition* $(\mathcal{A}_2)$ (or has Muckenhoupt type singularities), if

$$\sup \frac{1}{|J|^2} \int_J f(\lambda)\, d\lambda \int_J \frac{1}{f(\lambda)}\, d\lambda < \infty, \qquad (A_2)$$

where the supremum is over all intervals J, and $|J|$ stands for the length of an interval J.

The class of functions $f(\lambda)$ satisfying condition (A_2) we denote by $\mathcal{A}_2$.

Remark 1.1 It is clear that the spectral densities of short-memory processes belong to $\mathcal{A}_2$. The class $\mathcal{A}_2$ also contains spectral densities possessing singularities. In particular, it is known (see, e.g., Böttcher and Karlovich [3], Sect. 2.1) that if $\lambda_k, \alpha_k \in \mathbb{R}$, $k = \overline{1,n}$, then functions of the form

$$f(\lambda) = \prod_{k=1}^{n} |\lambda - \lambda_k|^{\alpha_k}$$

belong to $\mathcal{A}_2$ if and only if $-1 < \alpha_k < 1$ for all $k = \overline{1,n}$.

Remark 1.2 Condition (A_2) controls the singularities of the spectral density $\theta(\lambda)$, and describe the dependence structure of the underlying process $X(t)$ (see Ibragimov and Rozanov [24], Chap. 6).

Hölder classes Given numbers $0 < \alpha \le 1$, $p \ge 1$, and $r \in \mathbb{N}_0$, where $\mathbb{N}_0$ stands for the set of nonnegative integers. We put $\beta = r + \alpha$, and denote by $\mathbf{H}_p(\beta)$ the Hölder class of functions, that is, the class of functions $\psi(\lambda) \in L^p := L^p(\mathbb{R})$, which have r-th derivatives in L^p and satisfy

$$\|\psi^{(r)}(\cdot + u) - \psi^{(r)}(\cdot)\|_p \le C|u|^{\alpha},$$

where $\|h\|_p$ denotes the L^p-norm of a function h, and C is a positive constant. Also, by $\mathbf{\Sigma}_p(\beta)$ we denote the set of all spectral densities which belong to the class $\mathbf{H}_p(\beta)$.

The assumption on the observed process $X(t)$ is the following.

Assumption 1.1 $X(t)$ $(t \in \mathbb{R})$ is a zero mean real-valued mean square continuous stationary Gaussian process with a spectral density $\theta(\lambda)$ satisfying Muckenhoupt condition $(\mathcal{A}_2)$ and belonging to a Hölder class $\mathbf{H}_p(\beta)$. Thus, $\theta(\lambda) \in \mathbf{\Theta} \subseteq \mathcal{A}_2 \cap \mathbf{\Sigma}_p(\beta)$, where $0 < \alpha \leq 1$, $p \geq 1$, $\beta = r + \alpha$ and $r \in \mathbb{N}_0$.

The rest of the paper is organized as follows. In Sect. 2 we state some preliminary results: LAN, variants of Hájek-Ibragimov-Khas'minskii convolution theorem and Hájek-Le Cam local asymptotic minimax theorem, and define the concepts of H- and IK-efficiency of estimators. In Sect. 3 we state the main results of the paper: construction of asymptotically efficient estimators for linear and nonlinear smooth spectral functionals, and bounding the minimax mean square risks. Section 4 is devoted to the proofs of results stated in Sect. 3.

Throughout the paper the letters C, C_k, $C(\cdot)$, c and c_k are used to denote positive constants.

2 Preliminary Results

In this section we establish local asymptotic normality of families of distributions generated by a continuous-time stationary Gaussian process, then state variants of Hájek-Ibragimov-Khas'minskii convolution theorem and Hájek-Le Cam local asymptotic minimax theorem, and define the concepts of H- and IK-efficiency of estimators.

2.1 Local Asymptotic Normality

The notion of local asymptotic normality (LAN) of families of distributions, introduced by Le Cam in 1960 (see Le Cam [27]), plays an important role in asymptotic estimation theory. Le Cam, Hájek, Ibragimov and Khas'minskii and others have shown (see, for instance, Hájek [17, 18], Ibragimov and Khas'minskii [21, 23], Le Cam [27, 28], Kutoyants [26], and references therein) that many important properties of statistical estimators (characterization of limiting distributions, lower bounds on the accuracy, asymptotic efficiency, etc.) follow in fact from LAN condition.

The importance of LAN concept for nonparametric estimation problems has been emphasized by Levit [29, 30], Ibragimov and Khas'minskii [21, 23], Millar [31], Kutoyants [26], and others. The LAN for families of distributions generated by discrete-time stationary Gaussian processes has been studied by Davies [6], Dzhaparidze [7], Ginovyan [12], for continuous-time processes sufficient conditions for LAN were obtained in Solev and Zerbet [36].

Following Ibragimov and Khas'minskii [23], where a definition of LAN concept for the case where the underlying parametric set is a subset of a normed space or a smooth infinite-dimensional manifold was suggested, we define LAN for our model.

Definition 2.1 Let $\mathbb{P}_{T,\theta}$ be the probability distribution of the observation $\mathbf{X}_T = \{X(t), 0 \leq t \leq T\}$ with spectral density $\theta(\lambda)$. A family of distributions $\{\mathbb{P}_{T,\theta}, \theta \in \mathbf{\Theta}\}$ is called *locally asymptotically normal* (*LAN*) at a point $\theta_0 \in \mathbf{\Theta}$ in the direction $L^2 := L^2(\mathbb{R})$ with norming factors $A_T := A_T(\theta_0)$, if there exist a linear manifold $H := H(\theta_0) \subset L^2$ with closure $\overline{H} = L^2$ and a family $\{A_T\}$ of linear operators $A_T : L^2 \to L^2$ that satisfy:

(1) for any $h \in H$, $\|A_T h\|_2 \to 0$ as $T \to \infty$, where $\|\cdot\|_2$ denotes the L^2-norm;
(2) for any $h \in H$ there is a natural $T(h)$ such that $\theta_0 + A_T h \in \mathbf{\Theta}$ for all $T > T(h)$;

(3) for any $h \in H$ and $T > T(h)$ the representation

$$\ln \frac{d\mathbb{P}_{T,\theta_0+A_T h}}{d\mathbb{P}_{T,\theta_0}}(\mathbf{X}_T) = \Delta_T(h,\theta_0) - \frac{1}{2}\|h\|_2^2 + \phi(T,h,\theta_0) \tag{2.1}$$

is valid, where $\Delta_T(h) = \Delta_T(h,\theta_0)$ is a random linear function on H asymptotically (as $T \to \infty$) $N(0, \|h\|_2^2)$—normally distributed for any $h \in H$ and $\phi(T,h,\theta_0) \to 0$ as $T \to \infty$ in $\mathbb{P}_{T,\theta_0}$—probability.

Note that the presence of LAN property depends on the point θ_0, the space L^2 and the family of operators $\{A_T\}$. The choice of $H = H(\theta_0)$ may be rather arbitrary. We need only that $\overline{H} = L^2$.

Definition 2.2 (Condition ($\mathcal{H}$)) We say that a pair of functions (f,g) satisfies *condition* ($\mathcal{H}$), if $f \in \mathbf{\Sigma}_p(\beta)$ for $1 \le p \le 2$ and $\beta > 1/p$, and $g \in L^q$, where q is the conjugate of p: $1/p + 1/q = 1$.

The parametric set $\mathbf{\Theta}$ we will always assume to be a subset of the space L^p ($p \ge 1$) consisting of spectral densities satisfying Muckenhoupt condition ($\mathcal{A}_2$) and belonging to the Hölder class $\mathbf{\Sigma}_p(\beta)$. Define $H = H(\theta)$ to be the linear manifold consisting of bounded functions $h(\lambda)$ such that the pair $(\theta, h\theta^{-1})$ satisfies the condition ($\mathcal{H}$). We also define $A_T : L^2 \to L^2$ by $A_T h = [T^{-1/2}\theta] \cdot h$, that is, A_T is the operator of multiplication by function $T^{-1/2}\theta(\lambda)$. The next theorem, which contains sufficient conditions for LAN, can be proved using the arguments of the proof of Theorem 1 in Ginovyan [12] (cf. Solev and Zerbet [36]).

Theorem 2.1 *Let* $\mathbf{\Theta}$, H *and* A_T *be defined as above. Then the family of distributions* $\{\mathbb{P}_{T,\theta}, \theta \in \mathbf{\Theta}\}$ *satisfies LAN condition at any point* $\theta \in \mathbf{\Theta}$ *in the direction* L^2 *with norming factors* A_T *and*

$$\Delta_T(h) = \frac{T^{1/2}}{4\pi} \int_{-\infty}^{\infty} \left[\frac{I_T(\lambda)}{\theta(\lambda)} - 1 \right] h(\lambda)\, d\lambda, \tag{2.2}$$

where

$$I_T(\lambda) = \frac{1}{2\pi T} \left| \int_0^T X(u) e^{-iu\lambda}\, du \right|^2 \tag{2.3}$$

is the periodogram of the process $X(t)$.

2.2 Characterization of Limiting Distribution. H-Efficiency

We now consider the problem of estimating the value $\Phi(\theta)$ of a known functional $\Phi(\cdot)$ at an unknown point $\theta \in \mathbf{\Theta}$ on the basis of an observation $\mathbf{X}_T$, which has distribution $\mathbb{P}_{T,\theta}$. We assume that the family $\{\mathbb{P}_{T,\theta},\ \theta \in \mathbf{\Theta}\}$ satisfies the LAN condition at a point $\theta_0 = f \in \mathbf{\Theta}$ in the direction L^2 with norming factors A_T. We also assume the functional $\Phi(\theta)$ to be Fréchet differentiable at $f \in L^2$ with derivative $\Phi'(f) := \Phi'(f;\lambda)$, that is, there exists a linear continuous functional $\Phi'_f : L^2 \to \mathbb{R}$:

$$\Phi'_f(\psi) = \int_{-\infty}^{\infty} \Phi'(f;\lambda)\psi(\lambda)\, d\lambda, \quad \psi \in L^2 \tag{2.4}$$

such that for $f, g \in L^2$ we have

$$|\Phi(g) - \Phi(f) - \Phi'_f(g - f)| = o(\|g - f\|_2) \quad \text{as} \quad \|g - f\|_2 \to 0. \tag{2.5}$$

Furthermore, we assume that the derivative $\Phi'(f)$ satisfies the condition: uniformly for $f \in \mathbf{\Theta}$

$$0 < \|\Phi'(f)f\|_2 < \infty. \tag{2.6}$$

We need a version of Hájek-Ibragimov-Khas'minskii convolution theorem for regular estimators.

Recall that (see, e.g., Ibragimov and Khas'minskii [21], Sect. 2.9) an estimator $\widehat{\Phi}_T := \widehat{\Phi}_T(\mathbf{X}_T)$ of $\Phi(\theta)$ is called H-regular at $\theta_0 \in \mathbf{\Theta}$, if for any $h \in H$ there exists not depending on h a proper limit distribution function F of the normed difference $T^{1/2}(\widehat{\Phi}_T - \Phi(\theta_h))$, where $\theta_h = \theta_0 + A_T h$, in the sense of weak convergence

$$\mathcal{L}\{T^{1/2}(\widehat{\Phi}_T - \Phi(\theta_h))|\mathbb{P}_{T,\theta_h}\} \quad \Longrightarrow \quad F \quad \text{as } T \to \infty.$$

The next theorem follows from Theorem 3.1 in Ibragimov and Khas'minskii [23], and Theorem 2.1.

Theorem 2.2 *Let $\widehat{\Phi}_T$ be a H-regular estimator of $\Phi(\theta)$ at $f \in \mathbf{\Theta}$. Assume that the pair $(f, \Phi'(f))$ satisfies (2.6). Then under the assumptions of Theorem 2.1 the limit distribution F of $T^{1/2}(\widehat{\Phi}_T - \Phi(f))$ is a convolution of a probability distribution G and a centered normal distribution with variance $\sigma^2 := 4\pi\|\Phi'(f)f\|_2^2$:*

$$F = N(0, \sigma^2) * G. \tag{2.7}$$

By a well-known lemma of Anderson (see, e.g., Ibragimov and Khas'minskii [21], Sect. 2.10), the distribution F in (2.7) is less concentrated in symmetric intervals than the normal distribution $N(0, \sigma^2)$. This justifies the following definition of H-efficiency (cf. Millar [31], Dahlhaus and Wefelmeyer [5], Kutoyants [25], Sect. 2.1, Ginovyan [14]).

Definition 2.3 Let the family $\{\mathbb{P}_{T,\theta},\ \theta \in \Theta\}$ be LAN at a point $f \in \mathbf{\Theta}$. An estimator $\widehat{\Phi}_T$ of $\Phi(\theta)$ is called *H-asymptotically efficient* at f (in the class of H-regular estimators) with asymptotic variance $\sigma^2 := 4\pi\|\Phi'(f)f\|_2^2$, if

$$\mathcal{L}\{T^{1/2}(\widehat{\Phi}_T - \Phi(\theta_h))|\mathbb{P}_{T,\theta_h}\} \quad \Longrightarrow \quad N(0, \sigma^2) \quad \text{as } T \to \infty,$$

that is, the distribution G in (2.7) is degenerate.

Remark 2.1 This efficiency concept is a nonparametric version of Hájek-efficiency, and admits the same intuitive interpretation: the asymptotic distribution of any regular estimator $\widehat{\Phi}_T$ of $\Phi(\theta)$ is always "more spread out" than the centered normal distribution with variance $4\pi\|\Phi'(f)f\|_2^2$ (cf. Hájek [17], Millar [31]).

Remark 2.2 We also have the following characterization of H-regular and H-asymptotically efficient estimators (cf. Dahlhaus and Wefelmeyer [5], Taniguchi and Kakizawa [39], Chap. 6, Ginovyan [14]): if the family $\{\mathbb{P}_{T,\theta}, \theta \in \Theta\}$ is LAN at $f \in \mathbf{\Theta}$, then an estimator

$\widehat{\Phi}_T$ of $\Phi(f)$ is H-regular and H-asymptotically efficient at f with asymptotic variance $4\pi\|f\Phi'(f)\|_2^2$ if and only if it admits the following stochastic approximation:

$$T^{1/2}\big[\widehat{\Phi}_T-\Phi(f)\big]-\Delta_T\big(f\Phi'(f)\big)=o_P(1)\quad \text{as } T\to\infty, \tag{2.8}$$

where $\Delta_T(f\Phi'(f))$ is defined by (2.2) with $h=f\Phi'(f)$.

2.3 A Lower Bound for the Asymptotic Minimax Risk. IK-Efficiency

Denote by $\mathbf{\Phi}_T$ the set of all estimators of $\Phi(\theta)$ constructed on the basis of an observation $\mathbf{X}_T$, and let $\mathbf{W}$ denote the set of all loss functions $w:\mathbb{R}\to\mathbb{R}$, which are symmetric and non-decreasing on $\mathbb{R}^+:=(0,\infty)$, and satisfy $w(x)\ge 0,\ w(0)=0$.

The next theorem, which is a consequence of Theorem 4.1 in Ibragimov and Khas'minskii [23], and Theorem 2.1, contains a minimax lower bound for risks of all possible estimators $\widehat{\Phi}_T$ of $\Phi(\cdot)$ in the neighborhood of a point $f\in\mathbf{\Theta}$ (cf. Has'minskii and Ibragimov [19], Ginovyan [14]).

Theorem 2.3 *Assume that the pair $(f,\Phi'(f))$ satisfies (2.6). Then under the assumptions of Theorem 2.1, for all $w\in\mathbf{W}$*

$$\liminf_{\delta\to 0}\lim_{T\to\infty}\inf_{\widehat{\Phi}_T\in\mathbf{\Phi}_T}\sup_{\|\theta-f\|_2<\delta}\mathbb{E}_\theta\big\{w\big(T^{1/2}\big(\widehat{\Phi}_T-\Phi(f)\big)\big)\big\}\ge\mathbb{E}w(\xi), \tag{2.9}$$

where ξ is a centered normal random variable with variance $4\pi\|\Phi'(f)f\|_2^2$.

Basing on Theorem 2.3, we define the notion of asymptotically efficient estimators in the spirit of Ibragimov and Khas'minskii (IK-efficiency) (see Ibragimov and Khas'minskii [19, 23]).

Definition 2.4 Let the family $\{\mathbb{P}_{T,\theta},\theta\in\mathbf{\Theta}\}$ be LAN at a point $f\in\mathbf{\Theta}$. An estimator $\widehat{\Phi}_T$ of $\Phi(\theta)$ is called *IK-asymptotically efficient* at f for the loss function $w(x)\in\mathbf{W}$, with asymptotic variance $\sigma^2=4\pi\|\Phi'(f)f\|_2^2$, if

$$\liminf_{\delta\to 0}\lim_{T\to\infty}\sup_{\|\theta-f\|_2<\delta}\mathbb{E}_\theta\big\{w\big(T^{1/2}\big(\widehat{\Phi}_T-\Phi(f)\big)\big)\big\}=\mathbb{E}w(\xi), \tag{2.10}$$

where ξ is as in Theorem 2.3.

Remark 2.3 An estimator $\widehat{\Phi}_T$ of $\Phi(\theta)$ satisfying (2.10) is also called *locally asymptotically minimax* (*LAM*) estimator (see, e.g., Levit [29], Kutoyants [25], Sect. 2.1).

Remark 2.4 Both definitions of efficiency—H- and IK-efficiency—roughly speaking, require from an asymptotically efficient estimator $\widehat{\Phi}_T$ the local uniformity of the convergence of the normed difference $T^{1/2}(\widehat{\Phi}_T-\Phi(f))$ to the centered normal random variable ξ with variance $4\pi\|\Phi'(f)f\|_2^2$, and for bounded loss functions $w(\cdot)$ they are rather close. An attraction of the definition of IK-efficiency over that of H-efficiency is that it compares all the estimators constructed on the basis of an observation $\mathbf{X}_T$, rather than only the regular estimators, while an attraction of the definition of H-efficiency over that of IK-efficiency is that it is concerned with limiting distributions, rather than limits of expectations (cf. Beran [2], Kutoyants [25], Sect. 2.1). For detailed discussion of definitions and relationships of various efficiency concepts we refer to Ibragimov and Khas'minskii [21], Chap. 2.

3 Main Results

In this section we state the main results of the paper: we construct asymptotically efficient estimators for linear and nonlinear smooth spectral functionals, and obtain exact asymptotic bounds for minimax mean square risks of estimators of linear functionals.

3.1 Asymptotically Efficient Estimators

For construction of asymptotically efficient estimators we use a general method developed by Ibragimov and Khas'minskii (see, e.g., Ibragimov and Khas'minskii [19, 23], Goldstein and Khas'minskii [15], and references therein).

First consider the relatively simple case where the estimand functional $\Phi(f)$, $f \in \boldsymbol{\Theta}$, is linear and continuous in $L^p(\mathbb{R})$, $p \geq 1$. It is well-known (see, e.g., Riesz and Nagy [35]) that $\Phi(f)$ admits the representation

$$\Phi(f) = \int_{-\infty}^{\infty} f(\lambda) g(\lambda)\, d\lambda, \tag{3.1}$$

where $g(\lambda) \in L^q,\ 1/p + 1/q = 1$. As an estimator for $\Phi(f)$ we consider the averaged periodogram statistic, that is, the simple "plug-in" statistic:

$$\widehat{\Phi}_T := \Phi(I_T) = \int_{-\infty}^{\infty} I_T(\lambda) g(\lambda)\, d\lambda, \tag{3.2}$$

where $I_T(\lambda)$ is the periodogram of $X(t)$ defined by (2.3).

Let $\mathbf{W}_e$ denote the subset of loss functions $w \in \mathbf{W}$ which for some constants $C_1 > 0$ and $C_2 > 0$ satisfy the condition $w(x) \leq C_1 \exp\{C_2|x|\}$.

Theorem 3.1 *Let* $\Phi(f)$ *and* $\widehat{\Phi}_T$ *be defined by* (3.1) *and* (3.2). *Assume that the pair of functions* (f, g) *satisfies the conditions* $(\mathcal{H})$ *and* $0 < \|fg\|_2 < \infty$ *uniformly for* $f \in \boldsymbol{\Theta}$. *Then the statistic* $\widehat{\Phi}_T$ *is*:

(a) *H-regular and H-asymptotically efficient estimator of* $\Phi(f)$ *with asymptotic variance* $4\pi\|fg\|_2^2$;
(b) *IK-asymptotically efficient estimator of* $\Phi(f)$ *for* $w(x) \in \mathbf{W}_e$ *with asymptotic variance* $4\pi\|fg\|_2^2$.

Example 3.1 (Estimation of unknown covariance function) Let $g(\lambda) = e^{iu\lambda}$, then

$$\Phi(f) = \int_{-\infty}^{\infty} e^{iu\lambda} f(\lambda)\, d\lambda := r(u).$$

Thus, in this special case our problem becomes to the estimation of the covariance function $r(u) := \mathbb{E}[X(t+u)X(t)]$ of the process $X(t)$. By Theorem 3.1 the simple "plug-in" statistic $\widehat{\Phi}_T = \Phi(I_T)$, which coincides with the empirical covariance function $\widehat{r}_T(u)$, $u \in [0, T]$:

$$\widehat{\Phi}_T = \int_{-\infty}^{\infty} e^{iu\lambda} I_T(\lambda)\, d\lambda = \frac{1}{T}\int_0^{T-u} X(t)X(t+u)\, dt := \widehat{r}_T(u),$$

is H- and IK-asymptotically efficient estimator for $r(u)$ with asymptotic variance

$$\sigma_u^2 = 4\pi \int_{-\infty}^{\infty} f^2(\lambda)\cos^2(u\lambda)\, d\lambda.$$

The problem of asymptotically efficient estimation becomes somewhat more complicated for non-linear functionals. In this case the simple "plug-in" statistic $\Phi(I_T)$ is not necessary a consistent estimator for the functional $\Phi(f)$, and hence instead of the periodogram $I_T(\lambda)$, we need to use a suitable sequence of consistent estimators $\widehat{f}_T$ of f (cf. [5, 14, 19, 39]). On the other hand, if $\widehat{f}_T$ is a sequence of consistent estimators for f, the estimators $\Phi(\widehat{f}_T)$, in general, will converge to $\Phi(f)$ too slowly to be asymptotically efficient (cf. [14, 19]).

We consider a sequence $\{\widehat{f}_T\}$ of the so-called "undersmoothed" kernel estimators of the unknown spectral density $f(\lambda)$, and derive conditions under which the "plug-in" statistic $\Phi(\widehat{f}_T)$ is asymptotically efficient estimator for $\Phi(f)$.

Remark 3.1 By "undersmoothed" kernel estimator $\widehat{f}_T$ of f we mean the following (cf. [15, 16]): the bandwidth used in the kernel estimator $\widehat{f}_T$ is not optimal for the estimation of f; rather, we take advantage of the smooth, integral nature of the derivative of the estimand functional and undersmooth it. By choosing a small bandwidth, that is, by undersmoothing, the bias term becomes negligible and the behavior of the estimator is determined by a random term, which, with an appropriate normalization, obeys central limit theorem. A similar approach was applied in [5, 14, 38] for discrete-time processes, and in [15, 16] for efficient estimation of smooth functionals defined on a set of probability density functions.

We assume that $f \in \mathbf{\Sigma}_p(\beta)$, and as an estimator for unknown spectral density f we take the statistic (cf. [5, 33, 38]):

$$\widehat{f}_T(\lambda) = \int_{-\infty}^{\infty} W_T(\lambda - \mu) I_T(\mu)\, d\mu, \tag{3.3}$$

where $I_T(\lambda)$ is the periodogram of $X(t)$ defined by (2.3). For the kernel $W_T(\lambda)$ we set down the following assumptions.

Assumption 3.1 $W_T(\lambda) = M_T W(M_T\lambda)$, where $M_T = O(T^\alpha)$, and $b_T := M_T^{-1}$ is the bandwidth. The choice of the number α $(0 < \alpha < 1)$ will depend on the a priori knowledge about f and Φ.

Assumption 3.2 $W(\lambda)$ is bounded, even, nonnegative function with $W(\lambda) \equiv 0$ for $|\lambda| > 1$ and

$$\int_{-1}^{1} W(\lambda)\, d\lambda = 1, \qquad \int_{-1}^{1} \lambda^k W(\lambda)\, d\lambda = 0, \quad k = 1, 2, \ldots, r,$$

where $r = [\beta]$ is the integer part of β.

We assume the functional $\Phi(\cdot)$ to be Fréchet differentiable in L^2 with derivative $\Phi'(f) := \Phi'(f;\lambda)$ satisfying (2.6) and a Hölder condition: there exist constants $C > 0$ and δ $(0 < \delta \le 1)$ such that for any $f_1, f_2 \in L^2$,

$$\|\Phi'(f_1) - \Phi'(f_2)\| \le C\|f_1 - f_2\|_2^\delta. \tag{3.4}$$

Theorem 3.2 *Let the spectral density* $f(\cdot)$ *and the functional* $\Phi(\cdot)$ *be such that*:

(i) *the pair* $(f, \Phi'(f))$ *satisfies the conditions* $(\mathcal{H})$ *and* (2.6) *uniformly for* $f \in \mathbf{\Theta}$;
(ii) $\Phi(\cdot)$ *satisfies the condition* (3.4) *with* $\delta \ge (2\beta - 1)^{-1}$.

Let the estimator $\widehat{f}_T$ for f be defined by (3.3) with the kernel $W_T(\lambda)$ satisfying Assumptions 3.1 and 3.2 with $\frac{1}{2\beta} < \alpha < \frac{\delta}{\delta+1}$. Then the "plug-in" statistic $\Phi(\widehat{f}_T)$ is:

(a) *H-regular and H-asymptotically efficient estimator of $\Phi(f)$ with asymptotic variance $4\pi\|\Phi'(f)f\|_2^2$;*
(b) *IK-asymptotically efficient estimator of $\Phi(f)$ for $w(x) \in \mathbf{W}_e$ with asymptotic variance $4\pi\|\Phi'(f)f\|_2^2$.*

Example 3.2 Consider the problem of estimation of the integrated squared spectral density functional $\Phi(f)$:

$$\Phi(f) := \|f\|_2^2 = \int_{-\infty}^{\infty} f^2(\lambda)\, d\lambda. \tag{3.5}$$

In this case $\Phi'(f) = 2f$, and it follows from Theorems 3.2 that the "plug-in" statistic

$$\widehat{\Phi}_T = \Phi(\widehat{f}_T) = \int_{-\infty}^{\infty} \big[\widehat{f}_T(\lambda)\big]^2\, d\lambda,$$

where $\widehat{f}_T(\lambda)$ is as in (3.3), is H- and IK-asymptotically efficient estimator for functional (3.5) with asymptotic variance $\sigma^2 = 16\pi\|f^2\|_2^2$.

Remark 3.2 The discrete-time analogs of Theorems 3.1 and 3.2 were proved in Ginovyan [14]. For short-memory discrete-time models H-asymptotically efficient estimators were constructed by Dahlhaus and Wefelmeyer [5], and Millar [31], while IK-asymptotically efficient estimators were constructed by Ibragimov and Khas'minskii [19, 23]), and Ginovyan [8].

3.2 Exact Asymptotic Bounds for the Minimax Mean Square Risk

We return to the problem of estimation of a linear, L^p-continuous functional $\Phi(f)$. If $1 \le p \le 2$ the functional $\Phi(f)$ is continuous in L^2, and so we can apply Theorem 3.1 to construct an asymptotically efficient estimator for $\Phi(f)$. If $p > 2$ we no longer have an efficient estimator, and it becomes of interest to estimate the rate of decrease (as $T \to \infty$) of the minimax risk

$$\inf_{\widehat{\Phi}_T \in \mathbf{\Phi}_T} \sup_{f \in \mathbf{\Sigma}} \mathbb{E}_f\big\{w\big(\widehat{\Phi}_T - \Phi(f)\big)\big\},$$

where $\mathbf{\Sigma}$ is a given class of spectral densities and $\mathbf{\Phi}_T$ is the set of all estimators of $\Phi(f)$ constructed on the basis of an observation $\mathbf{X}_T$. It is clear that the bounds will depend on the number p and the smoothness properties of functions from $\mathbf{\Sigma}$. Below we obtain exact asymptotic bounds for the minimax mean square risk:

$$\Delta_T^2 := \sup_{\|\Phi\|=1} \inf_{\widehat{\Phi}_T} \sup_{f \in \mathbf{\Sigma}_p(\beta)} E_f|\widehat{\Phi}_T - \Phi(f)|^2. \tag{3.6}$$

More precisely, we prove that $\Delta_T^2 \asymp T^{-a}$ $(a > 0)$, where the number a is determined by the parameters p and β. (Here and below the notation $a_T \asymp b_T$ means that the ratio a_T/b_T is asymptotically (as $T \to \infty$) bounded away from 0 and ∞.)

Theorem 3.3 *Let $\Phi(f)$ be a linear L^p-continuous functional, and let Δ_T^2 be as in* (3.6). *The following assertions hold*:

(A) *If $p \ge 2$ and $\beta > 1/p$, then $\Delta_T^2 \asymp T^{-\frac{2p\beta}{p+2p\beta-2}}$.*
(B) *If either $p \ge 2$ and $\beta \le 1/p$ or $1 \le p \le 2$ and $\beta \le 1/2$, then $\Delta_T^2 \asymp T^{-2\beta}$.*
(C) *If $1 \le p \le 2$ and $\beta \ge 1/2$, then $\Delta_T^2 \asymp T^{-1}$.*

Remark 3.3 A similar result for probability density functionals was proved in Ibragimov and Has'minskii [22]. For discrete-time processes asymptotically exact bounds were obtained in Ginovyan [13]. For continuous-time processes asymptotically upper bounds were obtained in Ginovyan [10].

4 Proofs

4.1 Auxiliary Results

In this subsection we present some preliminary results that we use in the proofs of theorems. Denote by $\psi_A(\lambda)$ the Dirichlet singular integral defined for a function $\psi(\lambda) \in L^p(\mathbb{R})$ ($1 \le p < \infty$) by

$$\psi_A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin A(\lambda - x)}{\lambda - x}\psi(x)\,dx. \tag{4.1}$$

Note that $\psi_A(\lambda)$ is an entire analytic function of exponential type A. In the first lemma we collect several properties of function $\psi_A(\lambda)$. By $C(h_1, \dots, h_k)$ we denote constants depending on parameters $h_1, \dots, h_k$.

Lemma 4.1 *The following assertions hold*:

(a) *Let $\psi(\lambda) \in \mathbf{H}_p(\beta)$, $p \ge 1$, $\beta > 0$. Then $\|\psi_A\|_p \le C(p)\|\psi\|_p$ and*

$$\|\psi - \psi_A\|_p \le C(p,\beta)\,A^{-\beta}.$$

(b) *Let $\psi(\lambda) \in L^p$, $p \ge 1$. Then $\|\psi_A\|_q \le 2\,A^{1/p-1/q}\|\psi_A\|_p$, where $p < q \le \infty$.*
(c) *Let $\psi(\lambda) \in \mathbf{H}_p(\beta)$, where $\beta = r + \alpha$, $r \in \mathbb{N}_0$, $0 < \alpha \le 1$ and $p \ge 1$. Then*

$$\|\psi^{(j)}\|_p \le C < \infty, \quad j = \overline{1,r}.$$

(d) *Let $\psi(\lambda) \in \mathbf{H}_p(\beta)$, $p \ge 1$, $\beta > 0$, $q > p$ and $\beta \ne 1/p - 1/q$. Then*

$$\|\psi_A\|_q \le C \cdot \max\{1; A^{1/p-1/q-\beta}\}.$$

Proofs of assertions (a)–(c) can be found in Nikol'skii [32], Sect. 8.10 (see also Butzer and Nessel [4], Chap. 3), for the proof of assertion (d) we refer to Ibragimov and Has'minskii [22].

The next lemma is the well-known Hardy-Littlewood type embedding theorem for the classes $\mathbf{H}_p(\beta)$ (see, e.g., Nikol'skii [32], Sect. 6.3).

Lemma 4.2 *Let $\psi(\lambda) \in \mathbf{H}_p(\beta)$ with $\beta > 0$ and $p \geq 1$. The following assertions hold*:

(a) *If $\beta \leq 1/p$ and $p < p_1 < p/(1-\beta p)$, then $\psi(\lambda) \in \mathbf{H}_{p_1}(\beta - 1/p + 1/p_1)$.*
(b) *If $\beta > 1/p$, then $\psi(\lambda)$ is continuous and $\|\psi\|_\infty < \infty$.*

The proof of the next lemma is similar to that of Theorem 3 in [10].

Lemma 4.3 *Let $\Phi(f)$ and $\widehat{\Phi}_T$ be defined by (3.1) and (3.2). Assume that the pair of functions (f, g) satisfies the conditions $(\mathcal{H})$ and $0 < \|fg\|_2 < \infty$ uniformly for $f \in \mathbf{\Theta}$. Then for all $w(x) \in \mathbf{W}_e$ uniformly for f,*

$$\lim_{T\to\infty} \mathbb{E}_f\big\{w\big(T^{1/2}\big(\widehat{\Phi}_T - \Phi(f)\big)\big)\big\} = \mathbb{E}w(\xi), \tag{4.2}$$

where ξ is a centered normal random variable with variance $4\pi\|fg\|_2^2$.

Lemma 4.4 *Let $f \in \mathbf{\Sigma}_p(\beta)$, and $\psi(\lambda)$ be a continuous even function such that the pair (f, ψ) satisfies the conditions $(\mathcal{H})$ and $0 < \|f\psi\|_2 < \infty$. Let $\widehat{f}_T(\lambda)$ be as in (3.3) with kernel $W_T(\lambda)$ satisfying Assumptions 3.1 and 3.2, where $\frac{1}{2\beta} < \alpha \leq 1$. Then the distribution of the random variable*

$$\eta_T = T^{1/2}\int_{-\infty}^{\infty} \psi(\lambda)\big[\widehat{f}_T(\lambda) - f(\lambda)\big]\,d\lambda \tag{4.3}$$

as $T \to \infty$ tends to the normal distribution $N(0, \sigma^2)$, where

$$\sigma^2 = 4\pi\int_{-\infty}^{\infty} f^2(\lambda)\psi^2(\lambda)\,d\lambda. \tag{4.4}$$

Proof It follows from Lemma 4.3 (see also the proof of Theorem 3 in [10]) that the distribution of the random variable

$$\xi_T := T^{1/2}\int_{-\infty}^{\infty} \psi(\lambda)\big[I_T(\lambda) - f(\lambda)\big]\,d\lambda \tag{4.5}$$

as $T \to \infty$ tends to normal distribution $N(0, \sigma^2)$ with σ^2 as in (4.4). Therefore to complete the proof it is enough to show that

$$|\xi_T - \eta_T| = o_P(1) \quad \text{as } T \to \infty, \tag{4.6}$$

By (3.3) we have

$$\begin{aligned}\int_{-\infty}^{\infty} \psi(\lambda)\widehat{f}_T(\lambda)\,d\lambda &= \int_{-\infty}^{\infty} \psi(\lambda)\left[\int_{-\infty}^{\infty} W_T(\lambda-\mu)I_T(\mu)\,d\mu\right]d\lambda \\ &= \int_{-\infty}^{\infty} \psi(\lambda)\left[\int_{-\infty}^{\infty} W_T(\lambda-\mu)\big[I_T(\mu) - f(\mu)\big]\,d\mu\right]d\lambda \\ &\quad + \int_{-\infty}^{\infty} \psi(\lambda)\left[\int_{-\infty}^{\infty} W_T(\lambda-\mu)f(\mu)\,d\mu\right]d\lambda. \end{aligned} \tag{4.7}$$

By (4.3) and (4.7)

$$\begin{aligned}\eta_T &= T^{1/2}\int_{-\infty}^{\infty}\psi(\lambda)\Big[\int_{-\infty}^{\infty}W_T(\lambda-\mu)\big[I_T(\mu)-f(\mu)\big]\,d\mu\Big]\,d\lambda\\ &\quad + T^{1/2}\int_{-\infty}^{\infty}\psi(\lambda)\Big[\int_{-\infty}^{\infty}W_T(\lambda-\mu)f(\mu)\,d\mu - f(\lambda)\Big]\,d\lambda\\ &:= \eta_T^{(1)} + S_T \quad \text{(say).}\end{aligned} \tag{4.8}$$

Putting $M_T(\lambda-\mu)=t$ and taking into account that by Assumption 3.1, $W(t)=\frac{1}{M_T}W_T(\frac{t}{M_T})$, from (4.8) we obtain

$$\eta_T^{(1)} = T^{1/2}\int_{-\infty}^{\infty}\Psi_T(\lambda)\big[I_T(\lambda)-f(\lambda)\big]\,d\lambda, \tag{4.9}$$

where

$$\Psi_T(\lambda) = \int_{-\infty}^{\infty}\psi\left(\lambda+\frac{t}{M_T}\right)W(t)\,dt. \tag{4.10}$$

Thus, by (4.5), (4.9) and (4.10) we have

$$\eta_T^{(1)} - \xi_T = T^{1/2}\int_{-\infty}^{\infty}A(M_T,\lambda)\big[I_T(\lambda)-f(\lambda)\big]\,d\lambda, \tag{4.11}$$

where

$$A(M_T,\lambda) = \Psi_T(\lambda)-\psi(\lambda) = \int_{-\infty}^{\infty}\psi\left(\lambda+\frac{t}{M_T}\right)W(t)\,dt - \psi(\lambda). \tag{4.12}$$

It follows from the properties of the kernel $W_T(t)$ and the dominated convergence theorem that (cf. [38])

$$\lim_{M_T\to\infty}|A(M_T,\lambda)| = 0.$$

Hence by (4.11) and Lemma 4.3 we have

$$E|\eta_T^{(1)}-\xi_T|^2 = \int_{-\infty}^{\infty}|A(M_T,\lambda)|^2 f^2(\lambda)\,d\lambda \to 0 \quad \text{as } M_T\to\infty,$$

implying that

$$|\eta_T^{(1)}-\xi_T| = o_P(1) \quad \text{as } T\to\infty. \tag{4.13}$$

Next, applying Hölder and Minkowski generalized inequalities we can show that

$$|S_T| \le T^{1/2}\|\psi\|_q(1+\|W_T\|_1)E_{M_T,p}(f), \tag{4.14}$$

where $E_{A,p}(f)$ is the best approximation of function f by entire analytic functions of exponential type A in the metric of L^p. By Lemma 4.1(a), the assumption $f\in\mathbf{\Sigma}_p(\beta)$ implies $E_{A,p}(f)\le CA^{-\beta}$. Hence in view of (4.14)

$$S_T = O\big(T^{1/2}M_T^{-\beta}\big) = O\big(T^{1/2-\alpha\beta}\big) \to 0 \tag{4.15}$$

as $T\to\infty$ because by assumption $\alpha > \frac{1}{2\beta}$.

A combination of (4.8), (4.13) and (4.15) yields (4.6). This completes the proof of Lemma 4.4. □

Lemma 4.5 *Let* $f \in \boldsymbol{\Sigma}_p(\beta)$, *and let* $\widehat{f}_T(\lambda)$ *be as in* (3.3) *with kernel* $W_T(\lambda)$ *satisfying Assumptions* 3.1 *and* 3.2 *with* $\frac{1}{2\beta} < \alpha < \frac{\delta}{\delta+1}$, $\delta > 0$, *then*

$$T^{1/2}\|\widehat{f}_T - f\|_2^{1+\delta} = o_P(1) \quad \text{as } T \to \infty.$$

Proof Along the lines of the proof of Theorem 4 in [10] it can be shown that

$$E|\widehat{f}_T(\lambda) - f(\lambda)|^2 = O\big(M_T T^{-1}\big) + O\big(M_T^{-2\beta}\big) \tag{4.16}$$

uniformly in λ. Using Fubini's theorem from (4.16) we have

$$\|\widehat{f}_T - f\|_2 = O_P\big(M_T^{1/2} T^{-1/2}\big) + O_P\big(M_T^{-\beta}\big). \tag{4.17}$$

Hence taking into account that $M_T = O(T^\alpha)$ from (4.17) we can write

$$T^{1/2}\|\widehat{f}_T - f\|_2^{1+\delta} = O_P\big(T^{\frac{1}{2}+(\frac{\alpha}{2}-\frac{1}{2})(1+\delta)}\big) + O_P\big(T^{\frac{1}{2}-\alpha\beta(1+\delta)}\big). \tag{4.18}$$

The assumptions imply $\frac{1}{2} + (\frac{\alpha}{2} - \frac{1}{2})(1+\delta) < 0$ and $\frac{1}{2} - \alpha\beta(1+\delta) < 0$. Hence both terms in (4.18) are $o_P(1)$ as $T \to \infty$, and the result follows. □

4.2 Proofs of the Theorems

Proof of Theorem 3.1 Since in this case $\Phi'(f) = g$, it follows from (2.2), (3.1) and (3.2) that

$$T^{1/2}\big[\widehat{\Phi}_T - \Phi(f)\big] = T^{1/2}\int_{-\infty}^{\infty}\big[I_T(\lambda) - f(\lambda)\big]g(\lambda)\,d\lambda = \Delta_T(fg). \tag{4.19}$$

So, the assertion (a) of the theorem follows from (4.19) and Remark 2.2, while the assertion (b) follows from Lemma 4.3. Theorem 3.1 is proved. □

Proof of Theorem 3.2 It follows from (2.4), (2.5) and (3.4) that

$$\begin{aligned}&\left|\Phi(\widehat{f}_T) - \Phi(f) - \int_{-\infty}^{\infty}\Phi'(f;\lambda)\big(\widehat{f}_T(\lambda) - f(\lambda)\big)\,d\lambda\right| \\ &\quad \le \|\widehat{f}_T - f\| \sup_{0\le\theta\le1}\big\|\Phi'\big(f + \theta(\widehat{f}_T - f)\big) - \Phi'(f)\big\| \le C\|\widehat{f}_T - f\|^{1+\delta}.\end{aligned}$$

Therefore

$$\begin{aligned}T^{1/2}\big[\Phi(\widehat{f}_T) - \Phi(f)\big] &= T^{1/2}\int_{-\infty}^{\infty}\Phi'(f;\lambda)\big[\widehat{f}_T(\lambda) - f(\lambda)\big]\,d\lambda \\ &\quad + O_P\big(T^{1/2}\|\widehat{f}_T - f\|^{1+\delta}\big).\end{aligned} \tag{4.20}$$

Using the arguments of the proof of Lemma 4.4 with $\psi(\lambda) = \Phi'(f;\lambda)$ (cf. (4.6)), we conclude that as $T \to \infty$

$$T^{1/2}\int_{-\infty}^{\infty}\Phi'(f;\lambda)\big[\widehat{f}_T(\lambda) - f(\lambda)\big]\,d\lambda = T^{1/2}\int_{-\infty}^{\infty}\Phi'(f;\lambda)\big[I_T(\lambda) - f(\lambda)\big]\,d\lambda + o_P(1).$$

Hence, by Lemma 4.5 and (4.20)

$$T^{1/2}\big[\Phi(\widehat{f}_T)-\Phi(f)\big]=T^{1/2}\int_{-\infty}^{\infty}\Phi'(f;\lambda)\big[I_T(\lambda)-f(\lambda)\big]\,d\lambda+o_P(1). \tag{4.21}$$

By (2.2) and (4.21) we have

$$T^{1/2}\big[\Phi(\widehat{f}_T)-\Phi(f)\big]-\Delta_T\big(f\Phi'(f)\big)=o_P(1)\quad \text{as } T\to\infty. \tag{4.22}$$

Comparing (2.8) and (4.22), we conclude that the statistic $\Phi(\widehat{f}_T)$ is an H-regular and H-asymptotically efficient estimator for $\Phi(f)$ with asymptotic variance $4\pi\|f\Phi'(f)\|_2^2$. The assertion (b) of the theorem follows from (4.22) and Lemma 4.3. Theorem 3.2 is proved. □

Proof of Theorem 3.3 The corresponding upper bounds have been obtained in [10], where the following proposition is proved. □

Proposition 4.1 *Under the conditions of Theorem* 3.3

$$\Delta_T^2\le\begin{cases} CT^{-\frac{2p\beta}{p+2p\beta-2}}, & \textit{for}\;\; p\ge 2,\;\beta>1/p,\\ CT^{-2\beta}, & \textit{for}\;\; p\ge 2,\;\beta\le 1/p,\\ CT^{-2\beta}, & \textit{for}\;\; 1\le p\le 2,\;\beta<1/2,\\ CT^{-1}, & \textit{for}\;\; 1\le p\le 2,\;\beta\ge 1/2. \end{cases}$$

Therefore to complete the proof of Theorem 3.3 we need only to establish the corresponding lower bounds for the risk Δ_T^2, which are collected in the next theorem.

Theorem 4.1 *Under the conditions of Theorem* 3.3 *the following assertions hold*:

(A) *If* $p\ge 2$ *and* $\beta>1/p$, *then* $\Delta_T^2\ge cT^{-\frac{2p\beta}{p+2p\beta-2}}$;
(B) *If either* $p\ge 2$ *and* $\beta\le 1/p$ *or* $1\le p\le 2$ *and* $\beta<1/2$, *then* $\Delta_T^2\ge cT^{-2\beta}$;
(C) *If* $1\le p\le 2$ *and* $\beta\ge 1/2$, *then* $\Delta_T^2\ge cT^{-1}$.

Proof We use Stein-Levit method (see, e.g., Ibragimov and Has'minskii [22] and [21], Chap. 6). As an estimator of the linear functional $\Phi(f)$ we take the statistic $\widehat{\Phi}_{T,A}$, defined by (see [10])

$$\widehat{\Phi}_{T,A}=\int_{-\infty}^{\infty}I_T(\lambda)g_A(\lambda)\,d\lambda, \tag{4.23}$$

where $A=A(T)\le T$, $A(T)\to\infty$ as $T\to\infty$, $g_A(\lambda)$ is the Dirichlet singular integral corresponding to the function $g(\lambda)$ defined by (4.1), and $I_T(\lambda)$ is the periodogram given by (2.3). We set

$$\Sigma_p'(\beta)=\big\{f\in\Sigma_p(\beta);\,f(\lambda)\ge c>0\big\}$$

and

$$h(\lambda)=h_A(\lambda):=\frac{g_A(\lambda)}{\sqrt{T}\|fg_A\|_2^2}. \tag{4.24}$$

Let $f(\lambda)$ be some spectral density function from the class $\Sigma'_p(\beta)$ such that $f(\lambda)h_A(\lambda) \in \Sigma_p(\beta)$. Then for sufficiently large values of T the function

$$\theta(\lambda) = \theta_{T,h}(\lambda) := f(\lambda)\big(1 + h_A(\lambda)\big), \quad A = A(T)$$

is a spectral density from the class $\Sigma'_p(\beta)$.

Let $\mathbb{P}_{T,\theta}$ be the distribution of $\mathbf{X}_T = \{X(t),\ 0 \le t \le T\}$ with spectral density $\theta(\lambda)$. By Theorem 2.1 the family of Gaussian distributions $\{\mathbb{P}_{T,\theta},\ \theta \in \Sigma'_p(\beta)\}$ is locally asymptotically normal (LAN) at a point f in the direction of the space L^2. Therefore, we can apply Theorem 2.3 (see also Theorem 4.1 in [22]) for loss function $w(x) = x^2$, to obtain

$$\sup_{\theta \in \Sigma_p(\beta)} \inf_{\widehat{\Phi}_T} \mathbb{E}_\theta |\widehat{\Phi}_T - \Phi(\theta)|^2 \ge \frac{c}{T} \int_{-\infty}^{\infty} f^2(\lambda)\, g_A^2(\lambda)\, d\lambda, \tag{4.25}$$

where $\widehat{\Phi}_T$ is an arbitrary estimator of functional $\Phi(\theta)$, constructed on the basis of $\mathbf{X}_T$ and c is some positive constant.

Thus, to complete the proof of Theorem 4.1 we need to choose $A = A(T)$ to satisfy:

(1) the function $f(\lambda)\, h_A(\lambda)$ belongs to the class $\Sigma_p(\beta)$;
(2) the right-hand side of inequality (4.25) has the form T^{-a}, where the number a is specified by theorem.

We prove the assertions (A) and (B), the assertion (C) can be proved similarly.
We first prove part (A). Assume that $f(\lambda) \in \Sigma_p(\beta)$, where $p \ge 2$ and $\beta > 1/p$. We show that $f(\lambda)h(\lambda) \in \Sigma_p(\beta)$, where $h(\lambda) = h_A(\lambda)$ is as in (4.24). Let $\beta = \alpha + r$, where $r \in \mathbb{N}_0$ and $0 < \alpha \le 1$. Applying Leibnitz formula to compute the derivative $(f h_A)^{(r)}$, we find

$$\begin{aligned} J_r &:= \big\| (f h_A)^{(r)}(\cdot + \delta) - (f h_A)^{(r)}(\cdot) \big\|_p \\ &\le C \sum_{k=0}^{r} \big\| f^{(k)}(\cdot + \delta) h_A^{(r-k)}(\cdot + \delta) - f^{(k)}(\cdot) h_A^{(r-k)}(\cdot) \big\|_p. \end{aligned} \tag{4.26}$$

First consider the case where $r \ge 1$. We have

$$\begin{aligned} &\big(f h_A^{(r)}\big)(\cdot + \delta) - \big(f h_A^{(r)}\big)(\cdot) \\ &\quad = f(\cdot + \delta)\big[h_A^{(r)}(\cdot + \delta) - h_A^{(r)}(\cdot)\big] + h_A^{(r)}(\cdot)\big[f(\cdot + \delta) - f(\cdot)\big] \end{aligned}$$

and

$$\begin{aligned} &f^{(k)}(\cdot + \delta) h_A^{(r-k)}(\cdot + \delta) - f^{(k)}(\cdot) h_A^{(r-k)}(\cdot) \\ &\quad = f^{(k)}(\cdot + \delta)\big[h_A^{(r-k)}(\cdot + \delta) - h_A^{(r-k)}(\cdot)\big] + h_A^{(r-k)}(\cdot)\big[f^{(k)}(\cdot + \delta) - f^{(k)}(\cdot)\big]. \end{aligned}$$

Hence applying Hölder and Minkowski inequalities from (4.26) we obtain

$$\begin{aligned} J_r &\le C \big\| f(\cdot + \delta) h_A^{(r)}(\cdot + \delta) - f(\cdot) h_A^{(r)}(\cdot) \big\|_p \\ &\quad + C \sum_{k=1}^{r} \big\| f^{(k)}(\cdot + \delta) h_A^{(r-k)}(\cdot + \delta) - f^{(k)}(\cdot) h_A^{(r-k)}(\cdot) \big\|_p \\ &\le C \|f\|_\infty \big\| h_A^{(r)}(\cdot + \delta) - h_A^{(r)}(\cdot) \big\|_p \end{aligned}$$

$$
\begin{aligned}
&+ C\|h_A^{(r)}\|_\infty \big\| f(\cdot+\delta) - f(\cdot)\big\|_p \\
&+ C\sum_{k=1}^{r} \|f^{(k)}\|_p \big\|h_A^{(r-k)}(\cdot+\delta) - h_A^{(r-k)}(\cdot)\big\|_\infty \\
&+ C\sum_{k=1}^{r} \|h_A^{(r-k)}\|_\infty \big\| f^{(k)}(\cdot+\delta) - f^{(k)}(\cdot)\big\|_p. \qquad (4.27)
\end{aligned}
$$

Since $f(\lambda) \in \mathbf{H}_p(\beta)$, $\beta = \alpha + r$, by Lemma 4.1(c), we get

$$
\|f^{(k)}\|_p \le C < \infty, \quad k = 1, 2, \dots, r. \qquad (4.28)
$$

Further, because $r \ge 1$ we have $\beta > 1/p$, and hence by Lemma 4.2(b) we obtain

$$
\|f\|_\infty \le C < \infty. \qquad (4.29)
$$

The function $h(\lambda) = h_A(\lambda)$ is an entire function of exponential type A. Hence by Bernstein inequality (see, e.g., [4], Sect. 3.5)

$$
\|h_A^{(k)}\|_s \le 2^k A^k \|h_A\|_s, \quad 1 \le s \le \infty, \qquad (4.30)
$$

and by the inequality (see Lemma 4.1(b))

$$
\|h_A\|_s \le C A^{\frac{1}{t}-\frac{1}{s}} \|h_A\|_t, \quad t < s \le \infty, \qquad (4.31)
$$

we have

$$
\|h_A^{(k)}\|_\infty \le 2^k A^k \|h_A\|_\infty \le 2^k A^{k+\frac{1}{q}} \|h_A\|_q. \qquad (4.32)
$$

From the inequalities (4.30)–(4.32) for $q_1 > q$ and $k \le r$, we obtain

$$
\begin{aligned}
\big\|h_A^{(k)}(\cdot+\delta) - h_A^{(k)}(\cdot)\big\|_{q_1} &= \left\| \int_x^{x+\delta} h_A^{(k+1)}(y)\,dy \right\|_{q_1} \\
&\le \min\big(\delta \big\|h_A^{(k+1)}\big\|_{q_1}, 2\big\|h_A^{(k)}\big\|_{q_1}\big) \\
&\le \min\big(\delta A^{k+1+1/q-1/q_1}\|h_A\|_q,\ A^{k+1/q-1/q_1}\|h_A\|_q\big). \qquad (4.33)
\end{aligned}
$$

Therefore, for $q_1 > q$ and $k \le r$

$$
\big\|h_A^{(k)}(\cdot+\delta) - h_A^{(k)}(\cdot)\big\|_{q_1} \le C \cdot \delta^\alpha A^{\beta+1/q-1/p} \|h_A\|_q. \qquad (4.34)
$$

In view of (4.26)–(4.29), (4.32) and (4.34), from (4.27) we find for $r \ge 1$

$$
J_r = \big\|(f h_A)^{(r)}(\cdot+\delta) - (f h_A)^{(r)}(\cdot)\big\|_p \le C \cdot \delta^\alpha A^{\beta+1/q-1/p} \|h_A\|_q. \qquad (4.35)
$$

Now let $r = 0$. We have

$$
\begin{aligned}
J_0 := \big\|(f h_A)(\cdot+\delta) - (f h_A)(\cdot)\big\|_p &\le \|f\|_\infty \|h_A(\cdot+\delta) - h_A(\cdot)\|_p \\
&\quad + \|h_A\|_\infty \|f(\cdot+\delta) - f(\cdot)\|_p \\
&\le C \cdot \big\{\|h_A(\cdot+\delta) - h_A(\cdot)\|_p + \delta^\alpha \|h_A\|_\infty\big\}. \qquad (4.36)
\end{aligned}
$$

By the inequality (4.34) we have

$$\left\|h_A(\cdot+\delta)-h_A(\cdot)\right\|_p \le C\cdot\delta^\alpha A^{\beta+1/q-1/p}\|h_A\|_q. \tag{4.37}$$

Next, taking into account that $\beta\ge 1/p$, from (4.32) we get

$$\|h_A\|_\infty \le C\cdot A^{\frac{1}{q}}\|h_A\|_q \le C\cdot A^{1/q+\beta-1/p}\|h_A\|_q. \tag{4.38}$$

A combination of (4.36)–(4.38) yields

$$J_0=\left\|(fh_A)(\cdot+\delta)-(fh_A)(\cdot)\right\|_p \le C\cdot\delta^\alpha A^{\beta+1/q-1/p}\|h_A\|_q. \tag{4.39}$$

It follows from (4.24), (4.35) and (4.39) that for all $r\in\mathbb{N}_0$, $p\ge 2$ and $\beta>1/p$

$$J_r=\left\|(fh_A)^{(r)}(\cdot+\delta)-(fh_A)^{(r)}(\cdot)\right\|_p \le C\cdot M_{A,T}\delta^\alpha, \tag{4.40}$$

where

$$M_{A,T}:=T^{-1/2}\cdot A^{\beta+1/q-1/p}\|fg_A\|_2^{-2}. \tag{4.41}$$

Therefore, the assertion $f(\lambda)h_A(\lambda)\in\Sigma_p(\beta)$ will be fulfilled, if we can choose A to make $M_{A,T}$ given by (4.41), as small as needed. We set

$$g_A(\lambda)=A^{-\frac{1}{p}}\cdot\frac{\sin A\lambda}{\lambda}. \tag{4.42}$$

Since $\int_{-\infty}^{\infty}\frac{\sin^2 A\lambda}{\lambda^2}\,d\lambda=A\pi$, in view of (4.42) and the assumption $f(\lambda)\ge c>0$, we can write

$$\begin{aligned}\int_{-\infty}^{\infty}g_A^2(\lambda)f^2(\lambda)\,d\lambda &= A^{-\frac{2}{p}}\int_{-\infty}^{\infty}\frac{\sin^2 A\lambda}{\lambda^2}f^2(\lambda)\,d\lambda\\ &\ge c\cdot A^{-\frac{2}{p}}\int_{-\infty}^{\infty}\frac{\sin^2 A\lambda}{\lambda^2}\,d\lambda=\pi c\cdot A^{1-\frac{2}{p}}.\end{aligned} \tag{4.43}$$

Choosing $A=T^{p/(p-2+2p\beta)}$, and taking into account that $\frac{1}{q}=1-\frac{1}{p}$, from (4.41) and (4.43) we find

$$M_{A,T}\le C\cdot\frac{T^{-1/2}\cdot A^{\beta+1/q-1/p}}{A^{1-2/p}}=T^{\frac{2-p}{2(2p\beta+p-2)}}. \tag{4.44}$$

By assumption $p\ge 2$, so from (4.40) and (4.44) we conclude $f(\lambda)h_A(\lambda)\in\Sigma_p(\beta)$. Further, for $A=T^{p/(p-2+2p\beta)}$, from (4.25) and (4.43) we find

$$\Delta_T^2\ge\frac{c}{T}\int_{-\infty}^{\infty}g_A^2(\lambda)f^2(\lambda)\,d\lambda\ge c\cdot T^{-1}A^{1-2/p}\ge T^{-\frac{2p\beta}{2p\beta+p-2}}.$$

This completes the proof of the assertion (A).

Now we proceed to prove the assertion (B). Let $f(\lambda)\in\Sigma_p(\beta)$, where now $p\ge 2$ and $\beta\le 1/p$. We first show that $f(\lambda)h_A(\lambda)\in\Sigma_p(\beta)$, where $h(\lambda)=h_A(\lambda)$ is defined by (4.24). In this case necessarily $r=0$ and $\beta=\alpha\le 1$. Hence we only need to estimate the quantity

$\|(fh_A)(\cdot+\delta)-(fh_A)(\cdot)\|_p$. Denoting by $f_A(\lambda)$ the Dirichlet integral of function $f(\lambda)$ and applying Minkowski and Hölder inequalities, we find

$$\begin{aligned}J_0 := \left\|(fh_A)(\cdot+\delta)-(fh_A)(\cdot)\right\|_p &\le \|h_A\|_\infty \left\|f(\cdot+\delta)-f(\cdot)\right\|_p \\ &+ \left\|f(\cdot)-f_A(\cdot)\right\|_p \left\|h_A(\cdot+\delta)-h_A(\cdot)\right\|_\infty \\ &+ \|f_A\|_\infty \left\|h_A(\cdot+\delta)-h_A(\cdot)\right\|_p. \end{aligned} \tag{4.45}$$

Applying the inequality (4.31) for $s=\infty$ and $t=q$, we obtain

$$\|h_A\|_\infty \left\|f(\cdot+\delta)-f(\cdot)\right\|_p \le C\cdot\delta^\alpha A^{1/q}\|h_A\|_q. \tag{4.46}$$

By the second inequality in Lemma 4.1(a)

$$\left\|f(\cdot)-f_A(\cdot)\right\|_p \le C\cdot A^{-\beta} = C\cdot A^{-\alpha}.$$

Hence, applying the inequality (4.33) for $k=0$ and $q_1=\infty$, we find

$$\begin{aligned}&\left\|f(\cdot)-f_A(\cdot)\right\|_p \left\|h_A(\cdot+\delta)-h_A(\cdot)\right\|_\infty \\ &\quad\le C\cdot A^{-\alpha}\min\left(\delta A^{1+1/q}\|h_A\|_q,\, 2A^{1/q}\|h_A\|_q\right) \le C\cdot\delta^\alpha A^{1/q}\|h_A\|_q. \end{aligned} \tag{4.47}$$

According to Lemma 4.1(d)

$$\|f_A\|_\infty \le C\cdot A^{1/p-\beta} = C\cdot A^{1/p-\alpha}.$$

Hence, applying (4.33) for $k=0$ and $q_1=p$, we get

$$\begin{aligned}&\|f_A\|_\infty \left\|h_A(\cdot+\delta)-h_A(\cdot)\right\|_p \\ &\quad\le C\cdot A^{1/p-\alpha}\min\left(\delta A^{1+1/q-1/p}\|h_A\|_q, 2A^{1/q-1/p}\|h_A\|_q\right) \\ &\quad\le C\cdot\delta^\alpha A^{1/q}\|h_A\|_q. \end{aligned} \tag{4.48}$$

A combination of (4.45), (4.46), (4.47) and (4.48) yields

$$J_0 = \left\|(fh)(\cdot+\delta)-(fh_A)(\cdot)\right\|_p \le C\cdot\delta^\alpha A^{1/q}\|h_A\|_q. \tag{4.49}$$

It follows from (4.24) and (4.49) that

$$\left\|(fh_A)(\cdot+\delta)-(fh_A)(\cdot)\right\|_p \le C\cdot\delta^\alpha M_{A,T}, \tag{4.50}$$

where

$$M_{A,T} := T^{-1/2}\cdot A^{1/q}\|fg_A\|_2^{-2}. \tag{4.51}$$

Setting $g_A(\lambda)=A^{-\beta}\cdot\frac{\sin A\lambda}{\lambda}$, and taking into account that $\int_{-\infty}^{\infty}\frac{\sin^2 A\lambda}{\lambda^2}\,d\lambda = A\pi$ and $f(\lambda)\ge c>0$, we find

$$\int_{-\infty}^{\infty} g_A^2(\lambda) f^2(\lambda)\, d\lambda = A^{-2\beta} \int_{-\infty}^{\infty} \frac{\sin^2 A\lambda}{\lambda^2} f^2(\lambda)\, d\lambda$$
$$\geq c \cdot A^{-2\beta} \int_{-\infty}^{\infty} \frac{\sin^2 A\lambda}{\lambda^2} = \pi c \cdot A^{1-2\beta}. \tag{4.52}$$

Taking $A = T$, from (4.51) and (4.52) we obtain

$$M_{A,T} \leq C \cdot T^{-\frac{1}{2}+\frac{1}{q}-1+2\beta} = C \cdot T^{2(\beta-\frac{1}{p})+\frac{2-p}{2p}}. \tag{4.53}$$

By assumption $p \geq 2$ and $\beta \leq 1/p$, so from (4.50) and (4.53) we conclude $f(\lambda)h_A(\lambda) \in \Sigma_p(\beta)$. For $A = T$, from (4.25) and (4.52) we get

$$\Delta_T^2 \geq \frac{C_0}{T} \int_{-\infty}^{\infty} g_A^2(\lambda) f^2(\lambda)\, d\lambda \geq C \cdot T^{-1} A^{1-2\beta} \geq C \cdot T^{-2\beta}.$$

Thus, for $p \geq 2$ and $\beta \leq 1/p$ the assertion (B) of the theorem is proved.

Now we prove the assertion (B) for $1 \leq p \leq 2$ and $\beta \leq 1/2$. By the previous arguments, we only have to prove an analog of inequality (4.49).

Applying Minkowski and Hölder inequalities, we get the following analog of (4.45):

$$\begin{aligned}
&\big\| (f h_A)(\cdot + \delta) - (f h_A)(\cdot) \big\|_p \\
&\quad \leq \|h_A\|_\infty \big\| f(\cdot + \delta) - f(\cdot) \big\|_p + \big\| f(\cdot) - f_A(\cdot) \big\|_p \big\| h_A(\cdot + \delta) - h_A(\cdot) \big\|_\infty \\
&\qquad + \|f_A\|_{2q/(q-2)} \big\| h_A(\cdot + \delta) - h_A(\cdot) \big\|_2 := J_1 + J_2 + J_3.
\end{aligned} \tag{4.54}$$

The quantities J_1 and J_2 coincide with the first and second terms in (4.45) respectively. Therefore, according to (4.46) and (4.47) we have

$$J_1 \leq C \cdot \delta^\alpha A^{1/q} \|h_A\|_q \quad \text{and} \quad J_2 \leq C \cdot \delta^\alpha A^{1/q} \|h_A\|_q. \tag{4.55}$$

Now we estimate J_3. Setting $q_1 = \frac{2q}{q-2}$, we have $q_1 > p = \frac{q}{q-1}$ and $\frac{1}{p} - \frac{1}{q_1} - \beta = \frac{1}{2} - \beta > 0$. Hence, by Lemma 4.1(d) and Lemma 4.2(a)

$$\|f_A\|_{q_1} \leq C \cdot A^{\frac{1}{p}-\frac{1}{q_1}-\beta}. \tag{4.56}$$

Further, since $q \geq 2$, from (4.31) for $s = 2$ and $t = q$, we obtain

$$\|h_A\|_2 \leq C \cdot A^{\frac{1}{q}-\frac{1}{2}} \|h_A\|_q. \tag{4.57}$$

Using the inequalities (4.56), (4.57) and (4.30) for $s = 2$ and $k = 1$, we find

$$\begin{aligned}
J_3 &= \|f_A\|_{q_1} \big\| h_A(\cdot + \delta) - h_A(\cdot) \big\|_2 \leq C \cdot A^{\frac{1}{p}-\frac{1}{q_1}-\beta} \min\big(\delta \|h_A'\|_2, 2\|h_A\|_2\big) \\
&\leq C \cdot A^{\frac{1}{p}-\frac{q-2}{2q}-\beta} \min(A\delta, 1) \|h_A\|_2 \leq C \cdot A^{\frac{1}{p}-\frac{q-2}{2q}-\beta} A^{\frac{1}{q}-\frac{1}{2}} \|h_A\|_q \\
&\leq C \cdot \delta^\alpha A^{1/q} \|h_A\|_q.
\end{aligned} \tag{4.58}$$

A combination of (4.54), (4.55) and (4.58) yields (4.49). The rest of the proof follows the previous case. Thus, the assertion (B) is proved. This completes the proof of Theorem 4.1. □

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