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Prediction Error for Continuous-Time Stationary Processes with Singular Spectral Densities

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Abstract The paper considers the mean square linear prediction problem for some classes of continuous-time stationary Gaussian processes with spectral densities possessing singularities. Specifically, we are interested in estimating the rate of decrease to zero of the relative prediction error of a future value of the process using the finite past, compared with the whole past, provided that the underlying process is nondeterministic and is "close" to white noise. We obtain explicit expressions and asymptotic formulae for relative prediction error in the cases where the spectral density possess either zeros (the underlying model is an anti-persistent process), or poles (the model is a long memory processes). Our approach to the problem is based on the Krein's theory of continual analogs of orthogonal polynomials and the continual analogs of Szegö theorem on Toeplitz determinants. A key fact is that the relative prediction error can be represented explicitly by means of the so-called "parameter function" which is a continual analog of the Verblunsky coefficients (or reflection parameters) associated with orthogonal polynomials on the unit circle. To this end first we discuss some properties of Krein's functions, state continual analogs of Szegö "weak" theorem, and obtain formulae for the resolvents and Fredholm determinants of the corresponding Wiener-Hopf truncated operators.

Keywords Stationary Gaussian process · Singular spectral density · Prediction error · Parameter function · Wiener-Hopf truncated operator · Szegö theorem

Mathematics Subject Classification (2000) 60G25 · 62M20 · 60G10 · 47B35

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1 Introduction

1.1 The Model

1.1.1 Motivation

Methods for solving mean square linear prediction problem for the ordinary short memory continuous-time stationary processes goes back to the classical works by A. Kolmogorov, N. Wiener, M. Krein and others. However, many recent studies indicated that data in a large number of fields (e.g. in economics and finance) along with short memory display intermediate memory and/or long-range dependence. Moreover, practice often gives rise to what are not ordinary stochastic processes but generalized ones. A simple model example of such type processes is the continuous-time white noise process $\varepsilon(t)$, which can be thought of as the derivative of the Wiener process (Brownian motion) $W(t)$. Since the Wiener process is continuous but nowhere differentiable the derivative $W'(t)$ does not exist in ordinary sense, and hence the white noise $\varepsilon(t) = W'(t)$ is not an ordinary process. In fact, $\varepsilon(t)$ is a generalized stochastic processes whose definition is given below. Thus, it is important to consider the prediction problem for such processes.

1.1.2 Short, Intermediate and Long Memory Processes

Let $\{X(t);\ t \in \mathbb{R}\}$ be a real-valued, centered, mean-continuous, wide sense stationary process defined on a probability space $(\Omega, \mathcal{F}, P)$ with covariance function $r(t)$ and spectral density $f(\lambda)$, that is, $E[X(t)] = 0$, $E|X(t)|^2 < \infty$, $E|X(t) - X(s)|^2 \to 0$ as $t \to s$, and

$$E[X(t)X(s)] = r(t-s) = \int_{-\infty}^{+\infty} e^{i(t-s)\lambda} f(\lambda) d\lambda, \quad t, s \in \mathbb{R}, \tag{1.1}$$

where $E[\cdot]$ stands for the expectation operator with respect to measure P.

A *short memory* processes is defined to be a stationary processes possessing a spectral density $f(\lambda)$ which is bounded above and below: $0 < C_1 \le f(\lambda) \le C_2 < \infty$, where C_1 and C_2 are absolute constants. An *intermediate memory* (or *anti-persistent*) process is a stationary processes whose spectral density $f(\lambda)$ has zeros. A stationary processes is said to be a *long memory* processes, if its spectral density $f(\lambda)$ has poles.

In this paper we consider stationary models with spectral densities of the form

$$f(\lambda) = g(\lambda) \prod_{k=1}^{n} \left[\frac{(\lambda - \omega_k)^2}{(\lambda - \omega_k)^2 + 1} \right]^{\nu_k}, \quad k = \overline{1, n}, \tag{1.2}$$

where $g(\lambda)$ is the short memory component, $\{\omega_k,\ k = \overline{1, n}\}$ are distinct real numbers, and either $\nu_k = m_k \in \mathbb{N} := \{1, 2, \dots\}$ or $\nu_k = \alpha_k \in (-1/2, 1/2) \setminus \{0\}$, $k = \overline{1, n}$.

Observe that if $\nu_k = m_k \in \mathbb{N}$ or $\nu_k = \alpha_k \in (0, 1/2)$ the model (1.2) displays an intermediate memory (anti-persistent) process, while for $\nu_k = \alpha_k \in (-1/2, 0)$ it displays a long memory process, and, as a special case, includes the continuous-time FARIMA(p, d, q) model (with $n = 1$, $\nu_1 = \alpha_1 = d$). Notice also that the model with spectral density (1.2) is obtained from a short memory model with spectral density $g(\lambda)$ by a linear transformation with transfer function

$$\varphi(\lambda) = \prod_{k=1}^{n} \left[\frac{\lambda - \omega_k}{\lambda - \omega_k + i} \right]^{\nu_k}.$$

1.1.3 Generalized Stationary Processes

From the physical point of view, the concept of an ordinary stochastic process $Y(t)$ is related to measurements of random quantities at certain moments of time, without taking the values at other moments of time into account. However, in many cases, it is impossible to localize the measurements to a single point of time. As pointed out by I. Gel'fand and N. Vilenkin (see [10], p. 243), every actual measurement is accomplished by means of a device which has a certain inertia and hence instead of the actually measuring the ordinary process $Y(t)$ it measures certain averaged value $X(\varphi) := \int_{-\infty}^{+\infty} \varphi(t)Y(t)dt$, where $\varphi(t)$ is a function characterizing the device. Moreover, small changes in φ cause small changes in $X(\varphi)$. As a result we obtain a continuous linear functional which leads to the concept of generalized stochastic process.

Let $D = D(\mathbb{R})$ be the space of infinitely differentiable real-valued functions $\varphi(t)$ $(t \in \mathbb{R})$ with finite support (that is, the function $\varphi(t)$ vanish outside some closed interval, called the support of $\varphi(t)$ and denoted by $\mathrm{supp}\{\varphi\}$). The topology in D is defined as follows: we say that a sequence of functions $\varphi_n(t) \in D$ converges to a function $\varphi(t) \in D$ as $n \to \infty$, and write $\varphi_n \Rightarrow \varphi$ if $\mathrm{supp}\{\varphi_n\} \subset [a, b]$ for all $n = 1, 2, \ldots$ and $\varphi_n^{(k)}(t) \to \varphi^{(k)}(t)$ as $n \to \infty$, uniformly in $t \in [a, b]$ for all $k = 0, 1, \ldots$.

Definition 1.1 A generalized wide sense stationary process $\{X(\varphi),\ \varphi \in D\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is a random linear functional, such that $E|X(\varphi)|^2 < \infty$ and the stationarity conditions

$$\begin{aligned} m(\varphi) &:= (X(\varphi), 1) = m(\tau_t\varphi), \quad \varphi \in D, \\ R(\varphi, \psi) &:= (X(\varphi), X(\psi)) = R(\tau_t\varphi, \tau_t\psi), \quad \varphi, \psi \in D, \end{aligned} \tag{1.3}$$

holds, where τ_t is the shift operator: $[\tau_t\varphi](s) = \varphi(s + t)$, and $(\cdot, \cdot)$ is the inner product in the space $L^2(\Omega) = L^2(\Omega, \mathcal{F}, P) := \{\xi = \xi(\omega) :\ E|\xi|^2 < \infty\}$, defined by

$$(\xi, \eta) = E[\xi\overline{\eta}], \quad \xi, \eta \in L^2(\Omega).$$

We assume that the process $X(\varphi)$ is mean-continuous, that is, $E|X(\varphi_n) - X(\varphi)|^2 \to 0$ as $\varphi_n \Rightarrow \varphi$, and possesses a spectral density function $f(\lambda)$, $\lambda \in \mathbb{R}$.

So the covariance functional (1.3) is continuous in each of the arguments and admits the spectral representation (see [10]):

$$R(\varphi, \psi) = \int_{-\infty}^{+\infty} \hat{\varphi}(\lambda)\overline{\hat{\psi}(\lambda)} f(\lambda)\, d\lambda, \quad \varphi, \psi \in D, \tag{1.4}$$

where $\hat{\varphi}$ and $\hat{\psi}$ are the Fourier transforms of functions φ and ψ, respectively, and for some $p \geq 0$,

$$\int_{-\infty}^{+\infty} \frac{f(\lambda)}{(1 + \lambda^2)^p}\, d\lambda < \infty. \tag{1.5}$$

Definition 1.2 A real-valued generalized process $\{X(\varphi),\ \varphi \in D\}$ is called Gaussian with mean functional $m(\varphi)$ and covariance functional $R(\varphi, \psi)$ if its characteristic functional $\Phi(\varphi) := E[\exp\{iX(\varphi)\}]$ has the form

$$\Phi(\varphi) = \exp\left\{im(\varphi) - \frac{1}{2}R(\varphi, \psi)\right\}, \quad \varphi, \psi \in D. \tag{1.6}$$

Definition 1.3 A real-valued stationary Gaussian generalized process $\varepsilon(\varphi)$ with zero mean and covariance function equal to delta-function (the generalized function defined as $\delta(\varphi) = \varphi(0)$) is called white noise.

The white noise is the derivative of the Wiener process and its characteristic functional is given by

$$\Phi_\varepsilon(\varphi) = \exp\left\{-\frac{1}{2}\int_0^\infty \varphi^2(t)dt\right\}, \quad \varphi \in D. \tag{1.7}$$

Since the delta-function is the Fourier transform of Lebesgue measure the spectral density of white noise is $f(\lambda) = 1$.

Remark 1.1 If $Y(t)$ is a real-valued ordinary mean-continuous stationary process such that $E|Y(t)|^2 \le p(t)$ for some polynomial $p(t)$, then the integral

$$X(\varphi) = \int_{-\infty}^{+\infty} \varphi(t)Y(t)dt \tag{1.8}$$

is well defined and determines a real-valued generalized stationary process $X(\varphi)$. On the other hand, the generalized stationary process $X(\varphi)$ given by (1.8) uniquely determines the ordinary process $Y(t)$, that is, if $Y_1(t)$ and $Y_2(t)$ are two ordinary mean-continuous stationary processes generating by (1.8) the same generalized stationary process $X(\varphi)$, then $Y_1(t) = Y_2(t)$ with probability 1 for each $t \in \mathbb{R}$. In this case will say that the generalized stationary process $X(\varphi)$ is an ordinary process, and will identify the processes $X(\varphi)$ and $Y(t)$. Observe also that for ordinary stationary processes the condition (1.5) is satisfied with $p = 0$.

1.2 The Prediction Problem

Let $H := H(X) \subset L^2(\Omega)$ be the Hilbert space generated by the process $\{X(\varphi);\ \varphi \in \mathbb{D}\}$. For $a, b \in \mathbb{R}$, $-\infty \le a \le b \le \infty$, denote by $H_a^b := H_a^b(X)$ the subspace of H spanned by the random variables $X(\varphi)$ with $\operatorname{supp}\{\varphi\} \subset [a, b]$. Denote by $P_{[a,b]}$ the operator of orthogonal projection of $H(X)$ onto the subspace $H_a^b(X)$, and let $P_{[a,b]}^\perp$ be the orthogonal projection of $H(X)$ onto the orthogonal complement of $H_a^b(X)$, that is, $P_{[a,b]}^\perp \xi = \xi - P_{[a,b]}\xi$ for $\xi \in H(X)$. Then for any $T > 0$ the projection $P_{[-T,0]}X(\varphi)$ may be regarded as the best mean square linear predictor of the random variable $X(\varphi)$ based on the past of length $T : H_{-T}^0(X)$, and

$$\sigma^2(f;T) = \mathbb{E}|P_{[-T,0]}^\perp X(\varphi)|^2 = \mathbb{E}|X(\varphi) - P_{[-T,0]}X(\varphi)|^2$$

as its prediction error. Similarly,

$$\sigma^2(f) = \mathbb{E}|P_{[-\infty,0]}^\perp X(\varphi)|^2$$

may be regarded as the prediction error of $X(\varphi)$ based on the infinite past $H_{-\infty}^0(X)$.

The well-known Kolmogorov-Krein alternative states (see, e.g., [26] and [16], p. 57): *Either*

$$\int_{-\infty}^{+\infty} \frac{\log f(\lambda)}{1+\lambda^2} > -\infty \quad \Leftrightarrow \quad \sigma^2(f) > 0 \tag{1.9}$$

or

$$\int_{-\infty}^{+\infty} \frac{\log f(\lambda)}{1+\lambda^2} = -\infty \quad \Leftrightarrow \quad \sigma^2(f) = 0. \tag{1.10}$$

A generalized stationary process $\{X(\varphi);\ \varphi \in \mathbb{D}\}$ is called regular (or nondeterministic) if its spectral density $f(\lambda)$ satisfies (1.9), and is called singular (or deterministic) if $f(\lambda)$ satisfies (1.10). We assume that our process $X(\varphi)$ is nondeterministic, and set

$$\widetilde{\delta}(f;T) = \sigma^2(f;T) - \sigma^2(f). \tag{1.11}$$

The quantity $\widetilde{\delta}(f;T)$ is called the *relative prediction error* of the random variable $X(\varphi)$ using the past of length T, compared with the whole past. It is clear that $\widetilde{\delta}(f;T) \geq 0$ and

$$\widetilde{\delta}(f;T) \to 0 \quad \text{as } T \to \infty.$$

Let now H_1 and H_2 be two subspaces of H and P_1 and P_2 be the orthogonal projection operators in H onto H_1 and H_2, respectively. Consider the functional

$$\tau(H_1, H_2) = \mathrm{tr}[P_1 P_2 P_1], \tag{1.12}$$

where $\mathrm{tr}[A]$ stands for the trace of an operator A.

Observe (see, e.g., [16], p. 113), that the subspaces H_1 and H_2 are orthogonal if and only if $P_1 P_2 P_1 = 0$, and the functional $\tau(H_1, H_2)$ estimates how far the subspaces H_1 and H_2 are from being mutually orthogonal. It is clear that $\tau(H_1, H_2) = \tau(H_2, H_1)$.

The operator $P_1 P_2 P_1$ is called the canonical correlation operator corresponding to the pair of subspaces (H_1, H_2), and was introduced into the theory of stochastic processes by I.M. Gel'fand and A.M. Yaglom [11]. The relationship between this operator and the prediction problem was discussed in [11, 34]. The operator $P_1 P_2 P_1$ plays a crucial role in characterization of different regularity classes of stationary Gaussian processes. An extensive discussion of this question can be found in [16], Sect. IV.2.

For $T, s \in \mathbb{R}^+ := (0, \infty)$ we set

$$\tau(f;T,s) = \tau(H_{-T}^0, H_0^s), \tag{1.13}$$

$$\tau(f;s) = \tau(H_{-\infty}^0, H_0^s), \tag{1.14}$$

$$\delta(f;T,s) = \tau(f;s) - \tau(f;T,s). \tag{1.15}$$

It is clear that $\delta(f;T,s)$ is nonnegative and for any fixed s tends to zero as $T \to \infty$. The quantity $\delta(f;T,s)$ provides a natural measure of accuracy of prediction of the random variable $\xi \in H_0^s$ by the observed values $\eta \in H_{-T}^0$ (the past of length T), compared with their prediction by the observed values $\eta \in H_{-\infty}^0$ (the whole past). It is also natural to call the quantity

$$\delta(f;T) = \lim_{s\to 0} \frac{1}{s} \delta(f;T,s) \tag{1.16}$$

the *relative prediction error* of the random variable $\xi \in H_0^s$ using the past of length T, compared with the whole past. Clearly $\delta(f;T) \geq 0$ and

$$\delta(f;T) \to 0 \quad \text{as } T \to \infty.$$

Observe (see, e.g., [11, 22]) that the relative prediction errors $\widetilde{\delta}(f;T)$ and $\delta(f;T)$ defined by (1.11) and (1.16), respectively have the same asymptotic behavior as $T\to\infty$.

In this paper, we are interested in estimating the rate of decrease of $\delta(f;T)$ to zero as $T\to\infty$, depending on the properties of spectral density $f(\lambda)$, provided that the underlying process $X(\varphi)$ is nondeterministic and is "close" to white noise, meaning that the spectral density $f(\lambda)$ of $X(\varphi)$ satisfies (1.9) and the condition (see, e.g., [22, 23, 32]):

$$f_a(\lambda):=f(\lambda)-1\in L^1(\mathbb{R}). \tag{1.17}$$

The problem of asymptotic behavior of the mean square prediction error for continuous-time stationary processes was considered by A. Arimoto [2, 3], E. Hayashi [15], A. Inoue and Y. Kasahara [18, 19], Y. Kasahara [20], Seghier [29]. These authors used the so-called "projecting back and forth onto infinite past and future" approach based on the Dym-Seghier formula and von Neumann's alternating projection theorem (see [8, 29], and [25], Sect. 9.6.3).

Our approach to the problem is based on the M.G. Krein's theory of continual analogs of orthogonal polynomials on the unit circle and the continual analogs of G. Szegö's celebrated theorem on Toeplitz determinants (see, e.g., [1, 6, 7, 14, 21, 30], and references therein). A key fact is that the relative prediction error defined by (1.16) can be represented explicitly by means of the so-called "parameter function" which is a continual analog of the Verblunsky coefficients (or reflection parameters) associated with orthogonal polynomials on the unit circle (see Theorem 3.1 and Remarks 3.1 and 3.2). To this end first we discuss some properties of Krein's functions, state continual analogs of Szegö "weak" theorem, and obtain formulae for the resolvents and Fredholm determinants of the corresponding Wiener-Hopf truncated operators.

Some aspects of this approach were developed in papers by V. Solev [30–32] and N. Mesropian [22, 23] (see also N. Babayan [4], M. Ginovyan [12], and M. Ginovyan and L. Mikaelyan [13]). In particular, in [4] and [23] it was proved that for short memory processes (the spectral density $f(\lambda)$ is bounded above and below) the asymptotic behavior of prediction error $\delta(f;T)$ is determined by the differential (smoothness) properties of the spectral density $f(\lambda)$, and were presented necessary and sufficient conditions for exponential and power rates of decrease of $\delta(f;T)$ to zero as $T\to\infty$.

Here we are concerned with the asymptotic behavior of the mean square prediction error for continuous-time stationary processes with singular spectral densities. The spectral density of the underlying process is allowed to possess either zeros (the model is an anti-persistent process), or poles (the model is a long memory processes). We obtain either explicit expressions or asymptotic formulae for prediction error $\delta(f;T)$. The results show that the asymptotic relation

$$\delta(f;T)\sim\frac{1}{T}\quad \text{as } T\to\infty \tag{1.18}$$

is valid whenever the spectral density $f(\lambda)$ of the underlying process possesses at least one singularity (zero or pole) of power type.

Notice that the asymptotic relation (1.18), for some classes of continuous-time stationary processes that display intermediate or long memory was established in [13, 18, 19] and [20].

The remaining of the paper is organized as follows: In Sect. 2 we state the main results of the paper—Theorems 2.1–2.6. In Sect. 3 we discuss continual analogs of Szegö weak theorem and derive formulae for prediction error $\delta(f;T)$. In Sect. 4 we obtain explicit expressions for the resolvents of Wiener-Hopf truncated operators for two special classes of generating functions. Section 5 contains formulae for the Fredholm determinants of the corresponding Wiener-Hopf truncated operators. In Sect. 6 we prove the main results.

2 Main Results

The main results of this paper are the following theorems. Notice that Theorems 2.1–2.3 contain explicit expression for the prediction error $\delta(f;T)$, while Theorems 2.4–2.6 describe the asymptotic behavior of $\delta(f;T)$ as $T \to \infty$.

Theorem 2.1 *Let $f(\lambda)$ be the spectral density of a continuous-time stationary Gaussian process defined by*

$$f(\lambda) = \frac{(\lambda-\omega)^2}{(\lambda-\omega)^2+1}, \tag{2.1}$$

where ω is a real number. Then for any $T > 0$

$$\delta(f;T) = \frac{1}{T+2}. \tag{2.2}$$

Theorem 2.2 *Let $f(\lambda)$ be the spectral density of a continuous-time stationary Gaussian process defined by*

$$f(\lambda) = \frac{(\lambda-\omega)^2}{\lambda^2+\mu^2}, \tag{2.3}$$

where ω is any real number and μ is a positive number. Then for any $T > 0$

$$\delta(f;T) = \frac{\omega^2+\mu^2}{T(\omega^2+\mu^2)+2\mu}. \tag{2.4}$$

Theorem 2.3 *Let $f(\lambda)$ be the spectral density of a continuous-time stationary Gaussian process defined by*

$$f(\lambda) = \left[\frac{(\lambda-\omega)^2}{(\lambda-\omega)^2+1}\right]^2, \tag{2.5}$$

where ω is a real number. Then for any $T > 0$

$$\delta(f;T) = \frac{4\,(T^3+12T^2+48T+60)}{(T+4)(T^3+12T^2+48T+48)}. \tag{2.6}$$

Theorem 2.4 *Let $f(\lambda)$ be the spectral density of a continuous-time stationary Gaussian process defined by*

$$f(\lambda) = \left[\frac{(\lambda-\omega)^2}{(\lambda-\omega)^2+1}\right]^m, \quad m \in \mathbb{N}, \tag{2.7}$$

where ω is a real number. Then

$$\delta(f;T) \sim m^2/T, \quad \text{as } T \to \infty. \tag{2.8}$$

Here and in what follows the notation $a(t) \sim b(t)$ as $t \to \infty$ means that $b(t) \neq 0$ for sufficiently large t, and $\lim_{t\to\infty}(a(t)/b(t)) = 1$.

We will say that a function $g(\lambda)$ is *regular* if it is a spectral density of a short memory process, that is,

$$0 < C_1 \le g(\lambda) \le C_2 < \infty, \tag{2.9}$$

where C_1 and C_2 are absolute constants, and $g(\lambda)$ possesses usual smoothness properties (see, e.g., [5]).

Theorem 2.5 *Let $X(t)$ be a continuous-time stationary Gaussian process with spectral density $f(\lambda)$ defined by*

$$f(\lambda)=g(\lambda)\prod_{k=1}^{n}\left[\frac{(\lambda-\omega_k)^2}{(\lambda-\omega_k)^2+1}\right]^{m_k}, \quad m_k\in\mathbb{N},\ k=\overline{1,n}, \tag{2.10}$$

where $g(\lambda)$ is regular and ω_k, $k=\overline{1,n}$, are distinct real numbers. Then

$$\delta(f;T)\sim\frac{1}{T}\sum_{k=1}^{n}m_k^2 \quad \text{as } T\to\infty. \tag{2.11}$$

Theorem 2.6 *Let $X(t)$ be a continuous-time stationary Gaussian process with spectral density $f(\lambda)$ defined by*

$$f(\lambda)=g(\lambda)\prod_{k=1}^{n}\left[\frac{(\lambda-\omega_k)^2}{(\lambda-\omega_k)^2+1}\right]^{\alpha_k}, \quad \alpha_k\in(-1/2,1/2)\setminus\{0\},\ k=\overline{1,n}, \tag{2.12}$$

where $g(\lambda)$ is regular and ω_k, $k=\overline{1,n}$, are distinct real numbers. Then

$$\delta(f;T)\sim\frac{1}{T}\sum_{k=1}^{n}\alpha_k^2 \quad \textit{as } T\to\infty. \tag{2.13}$$

Remarks

1. Special cases of Theorems 2.1–2.3 were considered by Dym and McKean ([9], Sect. 6.10, Example 2), and Ginovyan and Mikaelyan [13].
2. Recall that a continuous-time stationary ARMA(p,q) process is defined to be a process possessing a rational spectral density:

$$f(\lambda)=\frac{|Q(i\lambda)|^2}{|P(i\lambda)|^2}, \tag{2.14}$$

   where $Q(z)=\sum_{k=0}^{q}a_kz^k$ and $P(z)=\sum_{k=0}^{p}b_kz^k$ are polynomials that have no roots in the right half-plane. In this context, the models in Theorems 2.1 and 2.2 are ARMA(1, 1)-processes, in Theorem 2.3 it is an ARMA(2, 2)-process, and in Theorem 2.4 the model is an ARMA(m,m)-process.
3. An explicit expression for prediction error $\delta(f;T)$ can also be obtained for general continuous-time stationary ARMA(p,q) processes with spectral density (2.14), where $P(z)$ is different from zero, while $Q(z)$ is allowed to possess real zeros (see Remark 5.1).
4. Let $X(t)$ and $Y(t)$ be two stationary process with spectral density functions $f_X(\lambda)$ and $f_Y(\lambda)$, respectively. We say that the process $Y(t)$ is obtained from $X(t)$ by a linear transformation with transfer (or spectral characteristic) function $\varphi(\lambda)$, if

$$f_Y(\lambda)=|\varphi(\lambda)|^2f_X(\lambda). \tag{2.15}$$

In this context, the models in Theorems 2.5 and 2.6 are obtained from a short memory process with spectral density $g(\lambda)$, by a linear transformation with transfer functions $\varphi_1(\lambda)$ and $\varphi_2(\lambda)$, respectively given by

$$\varphi_1(\lambda) = \prod_{k=1}^{n} \left[\frac{\lambda - \omega_k}{\lambda - \omega_k + i} \right]^{m_k}, \quad m_k \in \mathbb{N}, \tag{2.16}$$

and

$$\varphi_2(\lambda) = \prod_{k=1}^{n} \left[\frac{\lambda - \omega_k}{\lambda - \omega_k + i} \right]^{\alpha_k}, \quad \alpha_k \in (-1/2, 1/2) \setminus \{0\}. \tag{2.17}$$

5. Observe that the model in Theorem 2.6 displays a long memory process for $-1/2 < \alpha_k < 0$, and, as a special case, includes the continuous-time FARIMA(p, d, q) model (with $n = 1$, $\alpha_1 = d$). Asymptotic behavior of prediction error for this model was discussed in [20].
6. Theorem 2.6 is a continual counterpart of the result by I. Ibragimov and V. Solev [17].
7. The basic conclusion that can be made from Theorems 2.1–2.6 is that the asymptotic relation

$$\delta(f; T) \sim \frac{1}{T} \quad \text{as } T \to \infty \tag{2.18}$$

is valid, if the spectral density $f(\lambda)$ of the underlying process $X(t)$ has the form $f(\lambda) = f_0(\lambda)g(\lambda)$, where $g(\lambda)$ is regular and $f_0(\lambda)$ possesses at least one singularity (zero or pole) of power type.

3 Formulae for Prediction Error $\delta(f; T)$

In this section we derive formulae for prediction error $\delta(f; T)$. To this end first we define and discuss some properties of *Krein functions*, which are continual analogs of the polynomials orthogonal on the unit circle, associated with spectral density (weight function) $f(\lambda)$, and state continual analogs of Szegö weak theorem (see, e.g., [1, 7, 21, 30, 32]).

3.1 Krein Functions

Under the condition (1.17) the Fourier transformation $H(f; t)$ of function $f(\lambda) - 1$ given by

$$H(t) := H(f; t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} [f(\lambda) - 1] e^{-it\lambda} \, d\lambda \tag{3.1}$$

is well-defined. In the Hilbert space $L^2(0, r)$ consider the following integral operator generated by the function $f(\lambda)$:

$$(T_r(f)\varphi)(t) = \varphi(t) + \int_0^r H(t - s)\varphi(s) \, ds, \quad 0 < t < r. \tag{3.2}$$

The operator $T_r(f)$ is called Wiener-Hopf truncated operator (or Toeplitz truncated operator), and the generating function $f(\lambda)$ is called the symbol of $T_r(f)$.

Observe that for any $r>0$ the operator $T_r(f)$ is self-adjoint and compact. Moreover, it is well-known (see [1, 21]) that for any positive number r for the Hermitian kernel $H(f;t-s)$ $(0\le t,s\le r)$ there exists a Hermitian resolvent:

$$\Gamma_r(t,s)=\overline{\Gamma_r(s,t)},\quad 0\le t,s\le r,\ r>0, \tag{3.3}$$

satisfying the equation

$$\Gamma_r(t,s)+\int_0^r H(t-u)\Gamma_r(u,s)\,du=H(t-s),\quad 0\le t,s\le r. \tag{3.4}$$

Also, the function $\Gamma_r(t,s)$ is jointly continuous in r,t,s, and moreover, it is continuously differentiable in r and satisfies the conditions:

$$\Gamma_r(t,s)=\Gamma_r(r-s,r-t), \tag{3.5}$$

$$\frac{\partial}{\partial r}\Gamma_r(t,s)=-\Gamma_r(t,r)\Gamma_t(r,s), \tag{3.6}$$

where $0\le t,s\le r$ and $0\le r<\infty$.

We set $\Gamma_r(t)=\Gamma_r(t,0)$ and for $0\le r<\infty$ define

$$\begin{aligned} P(r,\lambda)&=e^{ir\lambda}-\int_0^r \Gamma_r(t)e^{-i\lambda(r-t)}\,dt\\ &=e^{ir\lambda}\left(1-\int_0^r \Gamma_r(t)e^{-it\lambda}\,dt\right). \end{aligned} \tag{3.7}$$

The functions $\{P(r,\lambda)\}_{r\ge 0}$, called Krein functions, were originally introduced by M.G. Krein (see [21]), as continual analogs of the orthogonal polynomials on the unit circle.

Notice that the functions $\{P(r,\lambda)\}_{r\ge 0}$ are of exponential type exactly r, and possess the following orthonormality property

$$\int_{-\infty}^{+\infty}\overline{P(\lambda,s)}P(\lambda,t)f(\lambda)\,d\lambda=\delta(t-s). \tag{3.8}$$

Thus, the Krein functions are already normalized, and they correspond to the monic orthogonal polynomials in the discrete case.

We also introduce the “reverse” function $P_*(r,\lambda)$, i.e., the [*]-transformation of $P(r,\lambda)$:

$$\begin{aligned} P_*(r,\lambda)&=[*](P(r,\lambda))=e^{ir\lambda}\overline{P(r,\lambda)}\\ &=1-\int_0^r\overline{\Gamma_r(s)}e^{is\lambda}\,ds=1-\int_0^r\Gamma_r(0,s)e^{is\lambda}\,ds. \end{aligned} \tag{3.9}$$

Notice that the functions $\{P_*(r,\lambda)\}_{r\ge 0}$ are of exponential type not greater than r.

Definition 3.1 The function $a(r)$ defined by

$$a(r)=\Gamma_r(0,r),\quad r>0, \tag{3.10}$$

is called parameter function associated with the system of Krein functions $\{P(r,\lambda)\}_{r\ge 0}$.

Remark 3.1 The parameter function $a(r)$ associated with Krein functions is a continual analog of the Verblunsky coefficients (or reflection parameters) associated with orthogonal polynomials on the unit circle.

Remark 3.2 The parameter function $a(r)$ plays a key role in prediction theory. In Theorem 3.1 we show that the prediction error $\delta(f;T)$ can be represented explicitly by means of the function $a(r)$. Observe, however, that even for simple models to find $a(r)$ is not easy question. The following trivial case is well-known (see, e.g., [7, 28]): if the underlying model is a white-noise model, in which case $f(\lambda)\equiv 1$, then $\Gamma_r(t,s)=0$, $a(r)=0$, $P(r,\lambda)=\exp(i\lambda r)$, and $P_*(r,\lambda)=1$. In Sect. 4 we will compute the functions $\Gamma_r(t,s)$ and $a(r)$ for two non-trivial models, specified by the spectral densities (2.1) and (2.5).

We list some properties of the parameter function $a(r)$ (see [1, 21, 32]).

1. It follows from (3.4), (3.6) and (3.10) that $a(r)$ is continuous and satisfies the equality

$$\Gamma_r(0,0)=\Gamma_0(0,0)-\int_0^r |a(t)|^2\,dt = H(0)-\int_0^r |a(t)|^2\,dt. \tag{3.11}$$

2. From (3.5), (3.6), (3.7), (3.9) and (3.10) we obtain the following system of differential equations, called Krein system, which is a continual analog of the Szegö-Levinson-Durbin recursions in the discrete case

$$\frac{\partial}{\partial r}P(r,\lambda)=i\lambda P(r,\lambda)-\overline{a(r)}P_*(r,\lambda), \qquad P(0,\lambda)=1, \tag{3.12}$$

$$\frac{\partial}{\partial r}P_*(r,\lambda)=-a(r)P(r,\lambda), \qquad P_*(0,\lambda)=1. \tag{3.13}$$

3. The conditions $a(t)\in L^2(\mathbb{R})$ and (1.9) are equivalent.
4. The following equality holds:

$$\int_r^\infty |a(f;t)|^2\,dt=\int_{-\infty}^{+\infty}|\Pi(\lambda)-P_*(r,\lambda)|^2 f(\lambda)\,d\lambda, \tag{3.14}$$

where $\Pi(\lambda)$ is an outer function from H^{2+} (the Hardy class in the upper half-plane), with boundary values satisfying $|\Pi(\lambda)|^{-2}=f(\lambda)$ a.e. on $\mathbb{R}$.

3.2 Continual Analogs of Szegö Weak Theorem

Assume that $f_a=f-1\in L^1(\mathbb{R})$, and consider the Wiener-Hopf truncated operator $T_r(f)$ defined by (3.2). It is known (see, e.g., [6, 32]) that under the above assumption the operator $T_r(f)-I$ is a trace class operator, and hence the Fredholm determinant $D(f;r)=\det(T_r(f))=\det(I+T_r(f_a))$ is well-defined. The next result is a version of continual analog of the Szegö weak theorem in terms of parameter function $a(t)$.

Lemma 3.1 (Szegö weak theorem. First version) *If $a(t)\in L^2(\mathbb{R})$, then*

$$\lim_{r\to\infty}\frac{d}{dr}[\ln D(f;r)]=\frac{1}{2\pi}\int_{-\infty}^{+\infty} f_a(t)\,dt-\int_0^\infty |a(t)|^2\,dt. \tag{3.15}$$

Proof By Akhiezer formula (see [1]) we have for any $r>0$

$$\ln D(f;r)=\ln\det(I+T_r(f_a))=\int_0^r \Gamma_u(0,0)\,du. \tag{3.16}$$

On the other hand, by (3.1) and (3.11)

$$\Gamma_u(0,0)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}[f(t)-1]\,dt-\int_0^r |a(t)|^2\,dt. \tag{3.17}$$

From (3.16) and (3.17) we obtain

$$\frac{d}{dr}[\ln D(f;r)]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}[f(t)-1]\,dt-\int_0^r |a(t)|^2\,dt. \tag{3.18}$$

Taking limit in (3.18) as $r\to\infty$, we obtain (3.15). □

If the function $f_a=f-1$ is not necessarily in $L^1(\mathbb{R})$, but is in $L^1(\mathbb{R})+L^2(\mathbb{R})$, then the corresponding operators $T_r(f)-I$ will be Hilbert-Schmidt operators, and, instead of ordinary Fredholm determinants $D(f;r)$, we use the so-called 2-regularized Fredholm determinants defined by (see [6], p. 6, 597)

$$D_2(f;r)=D(f;r)\exp\{-\mathrm{tr}\,[T_r(f)]\}. \tag{3.19}$$

The next result is a version of the Szegö weak theorem in terms of the 2-regularized Fredholm determinants.

Lemma 3.2 (Szegö weak theorem. Second version) *If* $a(t)\in L^2(\mathbb{R})$, *then*

$$\lim_{r\to\infty}\frac{d}{dr}[\ln D_2(f;r)]=-\int_0^\infty |a(t)|^2\,dt. \tag{3.20}$$

Proof It is known (see [32]) that the Wiener-Hopf truncated operator $T_r(f_a)$ is a trace class operator only when $f_a\in L^1(\mathbb{R})$, and in this case

$$\mathrm{tr}[T_r(f)]=\frac{r}{\pi}\int_{-\infty}^{+\infty} f_a(t)\,dt. \tag{3.21}$$

From (3.18), (3.19) and (3.21) we easily obtain

$$\frac{d}{dr}[\ln D_2(f;r)]=-\int_0^r |a(t)|^2\,dt. \tag{3.22}$$

Taking limit in (3.22) as $r\to\infty$, we obtain (3.20). □

3.3 Formulae for Prediction Error $\delta(f;r)$

We assume that the conditions (1.9) and (1.17) are fulfilled. The next two theorems contain formulae for the prediction error $\delta(f;r)$.

Theorem 3.1 *The following equalities hold*:

$$\delta(f;r) = \int_r^\infty |a(f;t)|^2\,dt \tag{3.23}$$

$$= \int_{-\infty}^{+\infty} |\Pi(\lambda) - P_*(r,\lambda)|^2 f(\lambda)\,d\lambda \tag{3.24}$$

$$= \frac{d}{dr}[\ln D(f;r)] - \lim_{r\to\infty} \frac{d}{dr}[\ln D(f;r)] \tag{3.25}$$

$$= \frac{d}{dr}[\ln D_2(f;r)] - \lim_{r\to\infty} \frac{d}{dr}[\ln D_2(f;r)]. \tag{3.26}$$

Theorem 3.2 *Let* $\ln f \in L^1(\mathbb{R})$, *then the following equality holds*:

$$\delta(f;r) = \frac{d}{dr}[\ln D(f;r)] - \ln G(f), \tag{3.27}$$

where $G(f)$ *is the geometric mean of function* f *defined by*

$$G(f) = \exp\left\{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \ln f(t)\,dt\right\}. \tag{3.28}$$

Corollary 3.1 (Szegö weak theorem. Third version) *Under the conditions of Theorem* 3.2

$$\lim_{r\to\infty} \frac{d}{dr}[\ln D(f;r)] = \ln G(f). \tag{3.29}$$

Proof of Theorem 3.1 The proof of the equality (3.23) can be found in [22] (see, also, [32]). The equality (3.24) follows from (3.14) and (3.23). To prove (3.25), observe that by (3.15) and (3.18)

$$\begin{aligned}
&\frac{d}{dr}[\ln D(f;r)] - \lim_{r\to\infty} \frac{d}{dr}[\ln D(f;r)] \\
&\quad = \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} f_a(t)\,dt - \int_0^r |a(t)|^2\,dt\right] \\
&\qquad - \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} f_a(t)\,dt - \int_0^\infty |a(t)|^2\,dt\right] \\
&\quad = \int_r^\infty |a(t)|^2\,dt.
\end{aligned} \tag{3.30}$$

So, the result follows from (3.23) and (3.30). Finally, the equality (3.26) follows from (3.20), (3.22) and (3.23). □

Proof of Theorem 3.2 It is known (see [32]) that

$$\int_0^\infty |a(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f_a(t)\,dt - \frac{1}{2\pi}\int_{-\infty}^{+\infty} \ln f(t)\,dt. \tag{3.31}$$

Hence, using Lemma 3.1, (3.23), (3.28) and (3.31), we obtain

$$\begin{aligned}\delta(f;r) &= \frac{d}{dr}[\ln D(f;r)] - \lim_{r\to\infty}\frac{d}{dr}[\ln D(f;r)]\\ &= \frac{d}{dr}[\ln D(f;r)] - \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} f_a(t)\,dt - \int_0^{\infty}|a(t)|^2\,dt\right]\\ &= \frac{d}{dr}[\ln D(f;r)] - \frac{1}{2\pi}\int_{-\infty}^{+\infty}\ln f(t)\,dt\\ &= \frac{d}{dr}[\ln D(f;r)] - \ln G(f).\end{aligned}$$

□

4 Inverses of the Wiener-Hopf Truncated Operators

In this section we show that the Wiener-Hopf truncated operators $T_r(f)$ generated by the spectral densities (symbols) $f(\lambda)$ defined by (2.1) and (2.5) are invertible and obtain explicit expressions for the corresponding resolvents $\Gamma_r(f;t,s)$ and parameter functions $a(f;r)$.

Theorem 4.1 *The operators $T_r(f)$ $(0<r<\infty)$ with symbol (2.1) are invertible in the spaces $L_2(0,r)$ and*

$$\left(\left[T_r(f)\right]^{-1}\varphi\right)(t) = \varphi(t) - \int_0^r \Gamma_r(t,s)\varphi(s)\,ds, \tag{4.1}$$

where

$$\Gamma_r(t,s) := \Gamma_r(f:t,s) = e^{-it\omega}\left[\frac{(1+t)(1+s)}{r+2} - 1 - \min\{t,s\}\right]e^{is\omega}. \tag{4.2}$$

Theorem 4.2 *The operators $T_r(f)$ $(0<r<\infty)$ with symbol (2.5) are invertible in the spaces $L_2(0,r)$ and*

$$\left(\left[T_r(f)\right]^{-1}\varphi\right)(t) = \varphi(t) - \int_0^r \Gamma_r(t,s)\varphi(s)\,ds, \tag{4.3}$$

with $\Gamma_r(t,s)=\Gamma_r(s,t)$ and for $s\le t$,

$$\Gamma_r(t,s) = \Gamma_r(f;t,s) = Q_r(t,s) + \frac{1}{\Delta}\left[\psi_{r1}(t)\widetilde{\psi}_{r1}(s) + \psi_{r2}(t)\widetilde{\psi}_{r2}(s)\right], \tag{4.4}$$

where

$$\begin{aligned}Q_r(t,s) &= t - s + 2 - (t-2)(s-2)(t-r)\\ &\quad + \frac{1}{2}(t+s-4)(t^2-r^2) - \frac{1}{3}(t^3-r^3),\end{aligned} \tag{4.5}$$

$$\Delta = \frac{1}{192}(r+4)(r^3+12r^2+48r+48), \tag{4.6}$$

$$\psi_{r1}(t) = U\phi_1 = -\frac{1}{4}\left[t^2 - 2rt - 4t + r^2 + 4r + 2\right], \tag{4.7}$$

$$\psi_{r2}(t) = U\phi_2 = -\frac{1}{12}\big[t^3 - 3(r^2+4r+6)t + 2(r^3+6r^2+12r+6)\big], \tag{4.8}$$

$$\widetilde{\psi}_{r1}(s) = \frac{1}{96}\big[(r^2+4r)s^3 - (2r^3+12r^2+24r+24)s^2 + (r^4+8r^3+30r^2+72r+96)s - (4r^3+36r^2+96r+48)\big], \tag{4.9}$$

$$\widetilde{\psi}_{r2}(s) = -\frac{r+4}{96}\big[2s^3 - 3rs^2 - (12r+36)s + (r^3+12r^2+42r+24)\big]. \tag{4.10}$$

Remark 4.1 It follows from (3.10) and (4.2) that, if the spectral density $f(\lambda)$ is given by (2.1), then for the corresponding parameter function $a(r)$ we have

$$a(r) = \Gamma_r(f;0,r) = \frac{1}{r+2}. \tag{4.11}$$

Similarly, from (3.10) and (4.3)–(4.9), we find that if the spectral density $f(\lambda)$ is given by (2.5), then the corresponding parameter function $a(r)$ has the form

$$a(r) = \Gamma_r(f;0,r) = \frac{2(r+6)(r^2+6r+12)}{(r+4)(r^3+12r^2+48r+48)}. \tag{4.12}$$

To prove Theorems 4.1 and 4.2 we need two lemmas.

Lemma 4.1 *If a Wiener-Hopf truncated operator $T_r(f)$ generated by the symbol $f(\lambda)$ is invertible and $\Gamma_r(f;t,s)$ is it's resolvent, then the Wiener-Hopf truncated operators $T_r(f_\omega)$ generated by the symbols $f_\omega(\lambda) = f(\lambda-\omega)$ ($\omega \in \mathbb{R}$) are also invertible and the corresponding resolvent functions are given by*

$$\Gamma_r(f_\omega;t,s) = e^{-it\omega}\Gamma_r(f;t,s)e^{is\omega}. \tag{4.13}$$

Proof If

$$f(\lambda) = 1 + \int_{-\infty}^{+\infty} H(t)e^{it\lambda}\,dt,$$

then

$$f_\omega(\lambda) = 1 + \int_{-\infty}^{+\infty} H(t)e^{it(\lambda-\omega)}\,dt = 1 + \int_{-\infty}^{+\infty} H_\omega(t)e^{it\lambda}\,dt$$

with $H_\omega(t) = H(t)e^{-it\omega}$, and for the corresponding Wiener-Hopf operators we have

$$T_r(f_\omega) = U_\omega T_r(f)U_\omega^{-1}, \tag{4.14}$$

where

$$(U_\omega h)(t) = e^{-it\omega}h(t), \quad h(t) \in L_2(0,r)$$

is a unitary operator in $L_2(0,r)$, and

$$(U_\omega^{-1}h)(t) = e^{it\omega}h(t). \tag{4.15}$$

It follows from (4.14) that

$$T_r^{-1}(f_\omega) = U_\omega T_r^{-1}(f)U_\omega^{-1}. \tag{4.16}$$

From (4.15) and (4.16) we obtain (4.13). □

Let $f_i(\lambda)$ $(i=1,2)$ be two symbols such that $f_i(\lambda)-1 \in L^1(\mathbb{R})$, and let $H(f_i;t)$ be the Fourier transformations of functions $f_i(\lambda)-1$ defined by (3.1). Consider the Wiener-Hopf and Hankel operators $T(f_i)$ and $H(f_i)$ with symbols f_i $(i=1,2)$:

$$(T(f)\varphi)(t) = \varphi(t) + \int_0^\infty H(f_i;t-s)\varphi(s)\,ds,$$
$$(H(f_i)\varphi)(t) = \int_0^\infty H(f_i;t+s)\varphi(s)\,ds.$$

The next result is a continual analog of Widom's formula for Toeplitz operators (see, e.g., [24, 33]).

Lemma 4.2 *The following equality holds*:

$$T_r(f_1f_2) = T_r(f_1)T_r(f_2) + P_rH(f_1)H(\widetilde{f_2})P_r + Q_rH(\widetilde{f_1})H(f_2)Q_r, \tag{4.17}$$

where P_r is the orthogonal projector in $L_2(0,\infty)$ onto $L_2(0,r)$, that is,

$$(P_r\varphi)(t) = \begin{cases} \varphi(t), & \text{if } t \in [0,r], \\ 0, & \text{if } t \notin [0,r], \end{cases} \tag{4.18}$$

$Q_r\varphi(t) = (P_r\varphi)(r-t)$, and $\widetilde{f_i(\lambda)} = f_i(-\lambda)$.

Proof of Theorem 4.1 By Lemma 4.1, without loss of generality, we prove the result for $\omega = 0$. We have

$$\frac{\lambda}{\lambda+i} = 1 - \int_0^\infty e^{-t}e^{i\lambda t}dt, \quad (\operatorname{Im}\lambda \ge 0) \tag{4.19}$$

$$\frac{\lambda}{\lambda-i} = 1 - \int_{-\infty}^0 e^{t}e^{i\lambda t}dt, \quad (\operatorname{Im}\lambda \le 0) \tag{4.20}$$

$$\frac{\lambda+i}{\lambda} = 1 + \int_0^\infty e^{i\lambda t}dt, \quad (\operatorname{Im}\lambda > 0) \tag{4.21}$$

$$\frac{\lambda-i}{\lambda} = 1 + \int_{-\infty}^0 e^{i\lambda t}dt, \quad (\operatorname{Im}\lambda < 0). \tag{4.22}$$

We set

$$T = T_r\left[\frac{\lambda^2}{\lambda^2+1}\right], \qquad H = H\left[\frac{\lambda}{\lambda+i}\right], \qquad T_\pm = T_r\left[\frac{\lambda}{\lambda\pm i}\right]. \tag{4.23}$$

From (4.17) and (4.19)–(4.23) we obtain

$$(T_+ f)(t) = f(t) - \int_0^t e^{-(t-s)} f(s)ds, \tag{4.24}$$

$$(T_- f)(t) = f(t) - \int_t^r e^{t-s} f(s)ds \tag{4.25}$$

and

$$(T_+^{-1} f)(t) = f(t) + \int_0^t g(s)ds, \tag{4.26}$$

$$(T_-^{-1} f)(t) = f(t) + \int_t^r g(s)ds. \tag{4.27}$$

Taking into account that the Hankel operator H with symbol $\frac{\lambda}{\lambda-i}$ is trivial and $\widetilde{\frac{\lambda}{\lambda-i}} = \frac{\lambda}{\lambda+i}$, from (4.17) we obtain

$$T = T_+T_- + P_r H^2 P_r = (I + P_r H^2 P_r T_-^{-1} T_+^{-1}) T_+ T_-. \tag{4.28}$$

For the Hankel operator H we have

$$(Hf)(t) = -(f, e^{-s})e^{-t} \tag{4.29}$$

and

$$(H^2 f)(t) = -(Hf, e^{-s})e^{-t} = -(f, He^{-s})e^{-t} = -\frac{1}{2}(f, e^{-s})e^{-t}, \tag{4.30}$$

where $(\cdot,\cdot)$ stands for the inner product in $L_2(0,r)$.

Setting $T_-^{-1}T_+^{-1} = U$, by (4.28) we obtain

$$T^{-1} = U(I + P_r H^2 P_r U)^{-1}. \tag{4.31}$$

In view of (4.26) and (4.27)

$$(Uf)(t) = (T_-^{-1}T_+^{-1})(t) = f(t) + \int_0^r Q_r(t,s) f(s)ds, \tag{4.32}$$

where

$$Q_r(t,s) = 1 + r - t \quad \text{for } s \le t, \tag{4.33}$$

and $Q_r(t,s) = Q_r(s,t)$. Next, observe that

$$((I + P_r H^2 P_r U)f)(t) = f(t) + (f, 1 + r - t)\frac{1}{2}e^{-t}.$$

Therefore

$$((I + P_r H^2 P_r U)^{-1} f)(t) = f(t) - (Kf)(t), \tag{4.34}$$

where

$$(Kf)(t) = -\frac{1}{r+2}(f, 1 + r - s)e^{-t}.$$

By (4.32)–(4.34) we have $T^{-1} = U + UK$, and

$$(UK)f = -\frac{1}{r+2}(f, 1+r-s)Ue^{-t} = -\frac{1}{r+2}(f, 1+r-s)(1+r-t). \tag{4.35}$$

Finally, we obtain

$$(T_r^{-1})f(t) = f(t) - \int_0^r \Gamma_r(t,s)f(s)ds, \tag{4.36}$$

where

$$\Gamma_r(t,s) = \Gamma_r(t,s) = \frac{(1+t)(1+s)}{r+2} - 1 - s \quad \text{for } s \le t \tag{4.37}$$

and $\Gamma_r(t,s) = \Gamma_r(s,t)$. This implies (4.2) with $\omega = 0$. Theorem 4.1 is proved. □

Proof of Theorem 4.2 First observe that

$$\left(\frac{\lambda}{\lambda+i}\right)^2 = 1 + \int_0^\infty (t-2)e^{-t}e^{i\lambda t}dt, \quad (\operatorname{Im}\lambda \ge 0), \tag{4.38}$$

$$\left(\frac{\lambda}{\lambda-i}\right)^2 = 1 + \int_{-\infty}^0 (-t-2)e^{t}e^{i\lambda t}dt, \quad (\operatorname{Im}\lambda \le 0), \tag{4.39}$$

$$\left(\frac{\lambda+i}{\lambda}\right)^2 = 1 + \int_0^\infty (t+2)e^{i\lambda t}dt, \quad (\operatorname{Im}\lambda > 0), \tag{4.40}$$

$$\left(\frac{\lambda-i}{\lambda}\right)^2 = 1 + \int_{-\infty}^0 (-t+2)e^{i\lambda t}dt, \quad (\operatorname{Im}\lambda < 0). \tag{4.41}$$

Since the Hankel operator H with symbol $(\frac{\lambda}{\lambda-i})^2$ is trivial, from (4.17) we obtain

$$\begin{aligned} T_r\left[\left(\frac{\lambda^2}{\lambda^2+1}\right)^2\right] &= T_r\left[\left(\frac{\lambda}{\lambda+i}\right)^2\left(\frac{\lambda}{\lambda-i}\right)^2\right] \\ &= T_r\left[\left(\frac{\lambda}{\lambda+i}\right)^2\right]T_r\left[\left(\frac{\lambda}{\lambda-i}\right)^2\right] \\ &\quad + P_rH\left[\left(\frac{\lambda}{\lambda+i}\right)^2\right]H\left[\left(\frac{\lambda}{\lambda+i}\right)^2\right]P_r. \end{aligned} \tag{4.42}$$

We set

$$T = T_r\left[\left(\frac{\lambda^2}{\lambda^2+1}\right)^2\right], \qquad H = H\left[\left(\frac{\lambda}{\lambda+i}\right)^2\right], \qquad T_\pm = T_r\left[\left(\frac{\lambda}{\lambda\pm i}\right)^2\right].$$

Taking into account $\widetilde{(\frac{\lambda}{\lambda-i})^2} = (\frac{\lambda}{\lambda+i})^2$, the equality (4.42) can be written as follows

$$T = T_+T_- + P_rH^2P_r,$$

or

$$T = (I + P_rH^2P_rT_-^{-1}T_+^{-1})T_+T_-. \tag{4.43}$$

From (4.19) and (4.39) we obtain

$$(T_+ f)(t) = f(t) + \int_0^t (t - s - 2)e^{-t+s} f(s)ds, \tag{4.44}$$

$$(T_- f)(t) = f(t) + \int_t^r (-t + s - 2)e^{t-s} f(s)ds. \tag{4.45}$$

It follows from (4.17) that

$$T_r\left[\left(\frac{\lambda}{\lambda+i}\right)^2\right] T_r\left[\left(\frac{\lambda+i}{\lambda}\right)^2\right] = P_r,$$

$$T_r\left[\left(\frac{\lambda}{\lambda-i}\right)^2\right] T_r\left[\left(\frac{\lambda-i}{\lambda}\right)^2\right] = P_r,$$

where P_r is as in (4.18). Hence the operators T_+ and T_- are invertible in $L_2(0, r)$, so by (4.40) and (4.42)

$$(T_+^{-1} f)(t) = f(t) + \int_0^t (t - s + 2) f(s)ds, \tag{4.46}$$

$$(T_-^{-1} f)(t) = f(t) + \int_t^r (-t + s + 2) f(s)ds. \tag{4.47}$$

The operator H^2 in (4.43) is two-dimensional, and can be represented in the form

$$(H^2 f)(t) = (f, \phi_1)\phi_1 + (f, \phi_2)\phi_2,$$

where $(\cdot, \cdot)$ stands for the inner product in $L_2(0, r)$, while ϕ_1 and ϕ_2 are defined by

$$\phi_1(t) = \frac{1}{2}(t-1)e^{-t}, \qquad \phi_2(t) = \left(\frac{1}{2}t - 1\right)e^{-t}. \tag{4.48}$$

By (4.43)

$$T^{-1} = T_-^{-1} T_+^{-1} (I + P_r H^2 P_r T_-^{-1} T_+^{-1})^{-1}.$$

Hence to compute T^{-1} it is enough to invert the operator $I + P_r H^2 P_r T_-^{-1} T_+^{-1}$.

Setting $T_-^{-1} T_+^{-1} = U$, in view of (4.46) and (4.47) we obtain

$$(Uf)(t) = f(t) + \int_0^r Q_r(t, s) f(s)ds,$$

where

$$Q_r(t, s) = t - s + 2 + (2 - t)(2 - s)(r - t) + \frac{1}{2}(r^2 - t^2)(4 - t - s) + \frac{1}{3}(r^3 - t^3) \tag{4.49}$$

for $s \le t$ and $Q_r(t, s) = Q_r(s, t)$. Next, observe that the operator

$$I + P_r H^2 P_r T_-^{-1} T_+^{-1} = P_r + P_r H^2 P_r U$$

can be represented as follows

$$(P_r + P_r H^2 P_r U) f = f + (f, U\phi_1)\phi_1 + (f, U\phi_2)\phi_2.$$

Therefore

$$(P_r + P_r H^2 P_r U)^{-1} f = f - Kf,$$

where K is a two-dimensional operator. We have

$$Kf = \frac{1}{\Delta}\big[(1 + (\phi_2, \psi_{r2})(f, \psi_{r1}) - (\phi_2, \psi_{r1})(f, \psi_{r2})\big]\phi_1$$
$$+ \frac{1}{\Delta}\big[(1 + (\phi_1, \psi_{r1})(f, \psi_{r2}) - (\phi_1, \psi_{r2})(f, \psi_{r1})\big]\phi_2, \qquad (4.50)$$

where $\psi_{r1} = U\phi_1$, $\psi_{r2} = U\phi_2$, and

$$\Delta = \big[1 + (\phi_1, \psi_{r1})\big]\big[1 + (\phi_2, \psi_{r2})\big] - (\phi_1, \psi_{r2})(\phi_2, \psi_{r1}),$$

where ϕ_1 and ϕ_2 are defined by (4.48). By direct computations we get

$$\psi_{r1} = U\phi_1 = -\frac{1}{4}\big[t^2 - 2rt - 4t + r^2 + 4r + 2\big], \qquad (4.51)$$

$$\psi_{r2} = U\phi_2 = -\frac{1}{12}\big[t^3 - 3(r^2 + 4r + 6)t + 2(r^3 + 6r^2 + 12r + 6)\big], \qquad (4.52)$$

$$\Delta = \frac{1}{192}(r + 4)(r^3 + 12r^2 + 48r + 48). \qquad (4.53)$$

Therefore $\Delta \neq 0$ for $r > 0$, which is necessary and sufficient for invertibility of the operator $(P_r + P_r H^2 P_r U)$.

Thus, we have proved that for all $r > 0$ the operator $T_r[(\frac{\lambda^2}{\lambda^2+1})^2]$ is invertible, and its inverse can be represented in the form

$$T^{-1} = U - UK. \qquad (4.54)$$

It is easy to see that the operator UK is two-dimensional and can be written in the form

$$UKf = \frac{1}{\Delta}(f, \widetilde{\psi}_{r1})U\phi_1 + (f, \widetilde{\psi}_{r2})U\phi_2 = (f, \widetilde{\psi}_{r1})\psi_{r1} + (f, \widetilde{\psi}_{r2})\psi_{r2}, \qquad (4.55)$$

where

$$\widetilde{\psi}_{r1} = [1 + (\phi_2, \psi_{r2})]\,\psi_{r1} - (\phi_2, \psi_{r1})\psi_{r2}$$
$$= \frac{1}{96}\big[(r^2 + 4r)t^3 - (2r^3 + 12r^2 + 24r + 24)t^2$$
$$+ (r^4 + 8r^3 + 30r^2 + 72r + 96)t - (4r^3 + 36r^2 + 96r + 48)\big] \qquad (4.56)$$

and

$$\widetilde{\psi}_{r2} = [1 + (\phi_1, \psi_{r1})]\,\psi_{r2} - (\phi_1, \psi_{r2})\psi_{r1}$$
$$= -\frac{r+4}{96}\big[2t^3 - 3rt^2 - (12r + 36)t + (r^3 + 12r^2 + 42r + 24)\big]. \qquad (4.57)$$

From (4.51)–(4.57) after some algebra we obtain

$$(T_r^{-1})f(t) = f(t) - \int_0^r \Gamma_r(t,s)f(s)ds, \qquad (4.58)$$

with $\Gamma_r(t,s)=\Gamma_r(s,t)$ given by

$$\Gamma_r(t,s)=-Q_r(t,s)+\psi_{r1}(t)\widetilde{\psi}_{r1}(s)+\psi_{r2}(t)\widetilde{\psi}_{r2}(s)\quad \text{for } s\le t, \tag{4.59}$$

where $Q_r(t,s)$, $\psi_{r1}(t)$, $\psi_{r2}(t)$, $\widetilde{\psi}_{r1}(s)$ and $\widetilde{\psi}_{r2}(s)$ are defined by (4.49), (4.51), (4.52), (4.56) and (4.57) respectively. Theorem 4.2 is proved. □

5 Determinants of the Wiener-Hopf Truncated Operators

This section contains either explicit expressions or asymptotic formulae for the Fredholm determinants $D(f;r)$ (or 2-regularized Fredholm determinants $D_2(f;r)$) of the Wiener-Hopf truncated operators $T_r(f)$ generated by the spectral densities (symbols) $f(\lambda)$, specified in Theorems 2.1–2.6.

Theorem 5.1 *Let the symbol $f(\lambda)$ of the Wiener-Hopf truncated operator $T_r(f)$ be as in (2.1). Then $T_r(f)-I$ is a trace class operator and*

$$D(f;r)=e^{-r}(1+r/2). \tag{5.1}$$

Proof The result follows from Akhiezer formula (see (3.16)) and (4.2). □

Theorem 5.2 *Let the symbol $f(\lambda)$ of the Wiener-Hopf truncated operator $T_r(f)$ be as in (2.3). Then $T_r(f)-I$ is a Hilbert-Schmidt operator and*

$$D_2(f;r)=e^{-r(\omega^2+\mu^2)/(2\mu)}\left(1+\frac{r(\omega^2+\mu^2)}{2\mu}\right). \tag{5.2}$$

Proof For the proof we refer to [5] (see, also, [6], Sect. 10.13). □

Theorem 5.3 *Let the symbol $f(\lambda)$ of the Wiener-Hopf truncated operator $T_r(f)$ be as in (2.5). Then $T_r(f)-I$ is a trace class operator and*

$$D(f;r)=\frac{e^{-2r}}{12}\left[\left(\frac{r}{2}\right)^4+8\left(\frac{r}{2}\right)^3+24\left(\frac{r}{2}\right)^2+30\left(\frac{r}{2}\right)+12\right]. \tag{5.3}$$

Proof The result follows from Akhiezer formula (see (3.16)) and Theorem 4.2 (see, also, [5, 24]). □

Theorem 5.4 *Let the symbol $f(\lambda)$ of the Wiener-Hopf truncated operator $T_r(f)$ be as in (2.7). Then $T_r(f)-I$ is a trace class operator and*

$$D(f;r)\sim C(m)\cdot e^{-mr}(r/2)^{m^2}\quad \text{as } r\to\infty, \tag{5.4}$$

where $C(m)$ is a constant depending on m.

Proof The proof can be found in [5] (see, also, [6], Sect. 10.13). □

Remark 5.1 An explicit expression for $D(f;r)$ can also be obtained for general rational symbols of the form (2.14), where $P(z)$ is different from zero, while $Q(z)$ is allowed to possess real zeros (see [5] and [6], p. 599).

For the next two results we refer to [6], Sect. 10.13.

Theorem 5.5 *Let the symbol $f(\lambda)$ of the Wiener-Hopf truncated operator $T_r(f)$ be as in (2.10). Then $T_r(f)-I$ is a trace class operator and*

$$D(f;r)\sim C(g,m,\omega)\cdot e^{-r\sum_{k=1}^{n}m_k}r^{\sum_{k=1}^{n}m_k^2}[G(g)]^r \quad \text{as } r\to\infty, \tag{5.5}$$

where $G(g)$ is the geometric mean of function g (see (3.28)), and $C(g,m,\omega)$ is a constant depending on g, $m=(m_1,\dots,m_n)$ and $\omega=(\omega_1,\dots,\omega_n)$.

Theorem 5.6 *Let the symbol $f(\lambda)$ of the Wiener-Hopf truncated operator $T_r(f)$ be as in (2.12). Then $T_r(f)-I$ is a trace class operator and*

$$D(f;r)\sim C(g,\alpha,\omega)\cdot e^{-r\sum_{k=1}^{n}\alpha_k}r^{\sum_{k=1}^{n}\alpha_k^2}[G(g)]^r \quad \text{as } r\to\infty, \tag{5.6}$$

where $G(g)$ is the geometric mean of function g (see (3.28)), and $C(g,\alpha,\omega)$ is a constant depending on g, $\alpha=(\alpha_1,\dots,\alpha_n)$ and $\omega=(\omega_1,\dots,\omega_n)$.

6 Proof of Theorems 2.1–2.6

Proof of Theorem 2.1 The result follows (3.23) and (4.11):

$$\delta(f;T)=\int_T^\infty |a(r)|^2\,dr=\int_T^\infty \frac{1}{(r+2)^2}\,dr=\frac{1}{T+2}.$$

Notice that the result can also be deduced from Theorems 3.2 and 5.1. Indeed, taking into account the elementary equality

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\ln\frac{\lambda^2}{\lambda^2+1}\,d\lambda=-1, \tag{6.1}$$

by (2.1), (3.27), (3.28) and (5.1) we obtain

$$\begin{aligned}\delta(f;T)&=\frac{d}{dT}[\ln D(f;T))]-\ln G(f)\\&=\frac{d}{dT}[-T+\ln(1+T/2))]-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\ln\frac{\lambda^2}{\lambda^2+1}\,d\lambda=\frac{1}{T+2}.\end{aligned}$$ □

Proof of Theorem 2.2 The result follows Theorems 3.2 and 5.2. Indeed, by (3.26) and (5.2) we have

$$\begin{aligned}\delta(f;T)&=\frac{d}{dT}[\ln D_2(f;T)]-\lim_{T\to\infty}\frac{d}{dr}[\ln D_2(f;T)]\\&=\left[-\frac{\omega^2+\mu^2}{2\mu}+\frac{\omega^2+\mu^2}{T(\omega^2+\mu^2)+2\mu}\right]-\left[-\frac{\omega^2+\mu^2}{2\mu}\right]\\&=\frac{\omega^2+\mu^2}{T(\omega^2+\mu^2)+2\mu}.\end{aligned}$$ □

Proof of Theorem 2.3 The result follows (3.23) and (4.12):

$$\delta(f;T) = \int_T^\infty |a(r)|^2\,dr = \int_T^\infty \left[\frac{2(r+6)(r^2+6r+12)}{(r+4)(r^3+12r^2+48r+48)}\right]^2 dr$$
$$= \frac{4\,(T^3+12T^2+48T+60)}{(T+4)(T^3+12T^2+48T+48)}.$$

Notice that the result can also be deduced from Theorems 3.2 and 5.3. Indeed, by (2.5), (3.27), (5.3) and (6.1) we have

$$\delta(f;T) = \frac{d}{dT}[\ln D(f;T)] - \ln G(f)$$
$$= \frac{d}{dT}\big[-2T + \ln[(T/2)^4 + 8(T/2)^3 + 24(T/2)^2 + 30(T/2) + 12] - \ln 12\big] + 2$$
$$= \frac{4\,(T^3+12T^2+48T+60)}{(T+4)(T^3+12T^2+48T+48)}.$$

□

Proof of Theorem 2.4 The result follows Theorems 3.2 and 5.4. Indeed, by (2.7), (3.27), (3.28), (5.4) and (6.1) we obtain

$$\delta(f;T) = \frac{d}{dT}[\ln D(f;T)] - \ln G(f)$$
$$\sim \frac{d}{dT}[-mT + m^2\ln T + C] - [-m] = \frac{m^2}{T}.$$

□

Proof of Theorem 2.5 The result follows Theorems 3.2 and 5.5. Indeed, by (2.10), (3.27), (3.28), (5.5) and (6.1) we obtain

$$\delta(f;T) = \frac{d}{dT}[\ln D(f;T)] - \ln G(f)$$
$$\sim \frac{d}{dT}\left[-T\sum_{k=1}^n m_k + T\ln G(g) + \left(\sum_{k=1}^n m_k^2\right)\ln T + C\right]$$
$$- \left[-\sum_{k=1}^n m_k + \ln G(g)\right] = \frac{1}{T}\cdot\sum_{k=1}^n m_k^2.$$

□

Proof of Theorem 2.6 The result follows Theorems 3.2 and 5.6. Indeed, by (2.12), (3.27), (3.28), (5.6) and (6.1) we obtain

$$\delta(f;T) = \frac{d}{dT}[\ln D(f;T)] - \ln G(f)$$
$$\sim \frac{d}{dT}\left[-T\sum_{k=1}^n \alpha_k + T\ln G(g) + \left(\sum_{k=1}^n \alpha_k^2\right)\ln T + C\right]$$
$$- \left[-\sum_{k=1}^n \alpha_k + \ln G(g)\right] = \frac{1}{T}\cdot\sum_{k=1}^n \alpha_k^2.$$

□

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