ON THE CENTRAL LIMIT THEOREM FOR
TOEPLITZ QUADRATIC FORMS OF STATIONARY SEQUENCES[*]

M. S. GINOVYAN[†] AND A. A. SAHAKYAN[†]

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Abstract. Let $X(t)$, $t = 0, \pm 1, \ldots$, be a real-valued stationary Gaussian sequence with a spectral density function $f(\lambda)$. The paper considers the question of applicability of the central limit theorem (CLT) for a Toeplitz-type quadratic form Q_n in variables $X(t)$, generated by an integrable even function $g(\lambda)$. Assuming that $f(\lambda)$ and $g(\lambda)$ are regularly varying at $\lambda = 0$ of orders α and β, respectively, we prove the CLT for the standard normalized quadratic form Q_n in a critical case $\alpha + \beta = \frac{1}{2}$.

We also show that the CLT is not valid under the single condition that the asymptotic variance of Q_n is separated from zero and infinity.

Key words. stationary Gaussian sequence, spectral density, Toeplitz-type quadratic forms, central limit theorem, asymptotic variance, slowly varying functions

1. Introduction. Let $X(t)$, $t = 0, \pm 1, \pm 2, \ldots$, be a centered ($\mathbf{E} X(t) = 0$) real-valued stationary Gaussian sequence with a spectral density $f(\lambda)$ and a covariance function $r(t)$, i.e.,

$$(1.1) \qquad r(t) = \int_{-\pi}^{\pi} e^{i\lambda t} f(\lambda) \, d\lambda.$$

We consider a question concerning an asymptotic distribution (as $n \to \infty$) of the following Toeplitz-type quadratic form of the process $X(t)$:

$$(1.2) \qquad Q_n = \sum_{k,j=1}^{n} a(k-j) \, X(k) \, X(j),$$

where

$$(1.3) \qquad a(k) = \int_{-\pi}^{\pi} e^{i\lambda k} g(\lambda) \, d\lambda, \qquad k = 0, \pm 1, \pm 2, \ldots,$$

are the Fourier coefficients of some real, even, integrable function $g(\lambda)$, $\lambda \in \mathbb{T} = [-\pi, \pi]$. We will refer to $g(\lambda)$ as a generating function for the quadratic form Q_n. Throughout the paper the functions $f(\lambda)$ and $g(\lambda)$ are assumed to be 2π-periodic.

The limit distribution of the random variables (1.2) is completely determined by the spectral density $f(\lambda)$ and the generating function $g(\lambda)$, and depending on their properties it can be either Gaussian (that is, Q_n with an appropriate normalization obeys the central limit theorem (CLT)), or non-Gaussian.

We naturally raise the following two questions:

(a) Under what conditions on $f(\lambda)$ and $g(\lambda)$ will the limit distribution of Q_n be Gaussian?

(b) Describe the limit distribution of Q_n if it is non-Gaussian.

[†]Institute of Mathematics of the National Academy of Sciences of Armenia, Marshal Bagramyan Ave. 24-B, 375019, Yerevan, Armenia (mamgin.55@gmail.com; sart@instmath.sci.am).

In this paper we essentially discuss question (a). This question goes back to the classical monograph by Grenander and Szegö [9], where they considered this problem as an application of their theory of the asymptotic behavior of the trace of products of truncated Toeplitz matrices.

Later this problem was studied by Ibragimov [11] and Rosenblatt [12], in connection with statistical estimation of the spectral ($F(\lambda)$) and covariance ($r(t)$) functions, respectively. Since 1986, there has been a renewed interest in questions (a) and (b), related to the statistical inferences for long-range dependent processes (see, e.g., the papers by Avram [1], Fox and Taqqu [4], Giraitis and Surgailis [8], Terrin and Taqqu [14], Taniguchi [17], and the monograph by Taniguchi and Kakizawa [18]). In the papers [1], [4], and [8] sufficient conditions for quadratic form Q_n to obey the CLT were obtained.

To state the corresponding results, we need some notation: By $\widetilde{Q}_n$ we denote the normalized quadratic form

$$(1.4) \qquad \widetilde{Q}_n = \frac{1}{\sqrt{n}} \left(Q_n - \mathbf{E} Q_n \right).$$

The notation

$$(1.5) \qquad \widetilde{Q}_n \Longrightarrow N(0, \sigma^2)$$

will mean that the distribution of the random variable $\widetilde{Q}_n$ converges (as $n \to \infty$) to the centered normal distribution with variance σ^2.

By $T_n(f)$ and $T_n(g)$ we denote the $n \times n$ Toeplitz matrices generated by functions f and g, respectively, i.e.,

$$(1.6) \qquad T_n(f) = \left\| r(k-j) \right\|_{k,j=\overline{1,n}} \quad \text{and} \quad T_n(g) = \left\| a(k-j) \right\|_{k,j=\overline{1,n}},$$

where $r(k)$ and $a(k)$ are as in (1.1) and (1.3), respectively. By C, M, C_k, M_k we denote constants that can vary from line to line.

THEOREM A (Avram). *Let the spectral density $f(\lambda)$ and the generating function $g(\lambda)$ be such that $f(\lambda) \in L^{p_1}(\mathbb{T})$, $g(\lambda) \in L^{p_2}(\mathbb{T})$, where $p_1, p_2 \geqq 1$ and $1/p_1 + 1/p_2 \leqq \frac{1}{2}$. Then (1.5) holds with σ^2 given by*

$$(1.7) \qquad \sigma^2 = 16\pi^3 \int_{-\pi}^{\pi} f^2(\lambda) \, g^2(\lambda) \, d\lambda.$$

Remark 1.1. For $p_1 = p_2 = \infty$, Theorem A was first established by Grenander and Szegö [9, Theorem 11.6], while the case $p_1 = 2$, $p_2 = \infty$ was proved by Ibragimov [11] and Rosenblatt [12].

THEOREM B (Fox and Taqqu). *Assume that the following conditions hold:*

(a) *The discontinuities of $f(\lambda)$ and $g(\lambda)$ have Lebesgue measure zero, and $f(\lambda)$ and $g(\lambda)$ are bounded on $[\delta, \pi]$ for all $\delta > 0$;*

(b) *there exist $\alpha < 1$ and $\beta < 1$ such that $\alpha + \beta < \frac{1}{2}$,*

$$(1.8) \qquad f(\lambda) \sim |\lambda|^{-\alpha} L_1(\lambda) \qquad \text{as} \quad \lambda \to 0,$$
$$(1.9) \qquad g(\lambda) \sim |\lambda|^{-\beta} L_2(\lambda) \qquad \text{as} \quad \lambda \to 0,$$

where $L_1(\lambda)$ and $L_2(\lambda)$ are slowly varying at $\lambda = 0$ functions.

Then (1.5) holds with variance σ^2 as in (1.7).

The proofs of Theorems A and B in [1] and [4] are based on the well-known representation of the kth order cumulant $\chi_k(\cdot)$ of the quadratic form $\widetilde{Q}_n$ (see, e.g., [9], [11])

$$\chi_k(\widetilde{Q}_n) = \begin{cases} 0 & \text{for} \quad k = 1, \\ n^{-k/2} 2^{k-1} (k-1)! \operatorname{tr}\left[T_n(f)\,T_n(g)\right]^k & \text{for} \quad k \geqq 2, \end{cases}$$

where $\operatorname{tr}[A]$ stands for the trace of a matrix A.

A different approach, applied in [8], extended Theorems A and B to linear sequences. In the Gaussian case the corresponding result can be formulated as follows.

THEOREM C (Giraitis and Surgailis). *Assume that as $n \to \infty$*

$$(1.10) \qquad \chi_2(\widetilde{Q}_n) = \frac{2}{n} \operatorname{tr}\left[T_n(f)\,T_n(g)\right]^2 \longrightarrow 16\pi^3 \int_{-\pi}^{\pi} f^2(\lambda)\,g^2(\lambda)\,d\lambda < \infty.$$

Then (1.5) holds with variance σ^2 as in (1.7).

In [1] and [4] (see also [8]) it was established that each of the conditions of Theorems A and B implies (1.10). Unfortunately (1.10) is not an explicit condition. In [8] the following explicit sufficient condition was also obtained.

THEOREM D (Giraitis and Surgailis). *Let $f \in L^2(\mathbb{T})$, $g \in L^2(\mathbb{T})$, $fg \in L^2(\mathbb{T})$ and for $\mu \to 0$*

$$(1.11) \qquad \int_{-\pi}^{\pi} f^2(\lambda)\,g^2(\lambda - \mu)\,d\lambda \longrightarrow \int_{-\pi}^{\pi} f^2(\lambda)\,g^2(\lambda)\,d\lambda.$$

Then (1.5) holds with variance σ^2 as in (1.7).

In the same paper [8] Giraitis and Surgailis conjectured that (1.10) holds under the single condition that the integral on the right-hand side of (1.10) is finite. In [6] one of the authors answered this conjecture negatively. To state this result, consider the functions

$$(1.12) \qquad f_0(\lambda) = \begin{cases} \left(\dfrac{2^s}{s^2}\right)^{1/p} & \text{if} \quad 2^{-s-1} \leqq \lambda \leqq 2^{-s}, \quad s = 2m, \\ 0 & \text{if} \quad 2^{-s-1} \leqq \lambda \leqq 2^{-s}, \quad s = 2m+1, \end{cases}$$

and

$$(1.13) \qquad g_0(\lambda) = \begin{cases} \left(\dfrac{2^s}{s^2}\right)^{1/q} & \text{if} \quad 2^{-s-1} \leqq \lambda \leqq 2^{-s}, \quad s = 2m+1, \\ 0 & \text{if} \quad 2^{-s-1} \leqq \lambda \leqq 2^{-s}, \quad s = 2m, \end{cases}$$

where m is a positive integer and $p, q \geqq 1$.

It is easy to see that $f_0(\lambda) \in L^p(\mathbb{T})$, $g_0(\lambda) \in L^q(\mathbb{T})$, $f_0(\lambda)\,g_0(\lambda) \in L^r(\mathbb{T})$ for every r and $\sigma^2 = 16\pi^3 \int_{-\pi}^{\pi} f_0^2(\lambda)\,g_0^2(\lambda)\,d\lambda = 0$. On the other hand, in [6] it was proved that for $1/p + 1/q > 1$

$$(1.14) \qquad \chi_2(\widetilde{Q}_n) = \frac{2}{n} \operatorname{tr}\left(T_n(f_0)\,T_n(g_0)\right)^2 \longrightarrow \infty \qquad \text{as} \quad n \to \infty,$$

and thereby the convergence in (1.10) breaks down. In [6] it was conjectured that the condition

$$0 < \int_{-\pi}^{\pi} f^2(\lambda)\,g^2(\lambda)\,d\lambda < \infty$$

implies the convergence in (1.10).

The problem (b), that is, description of the limit distribution of the quadratic form Q_n if it is non-Gaussian, was considered by Terrin and Taqqu in [14] and [15]. Let

$$f(\lambda) = |\lambda|^{-\alpha} L_1(\lambda) \quad \text{and} \quad g(\lambda) = |\lambda|^{-\beta} L_2(\lambda),$$

where $L_1(\lambda)$ and $L_2(\lambda)$ are slowly varying at 0 bounded functions. In [14] and [15] it was proved that if $\alpha < 1$, $\beta < 1$, and $\alpha + \beta > \frac{1}{2}$, then the random variable

$$(1.15) \qquad \widehat{Q}_n = \frac{1}{n^{\alpha+\beta} L_1(1/n)\, L_2(1/n)}\,(Q_n - \mathbf{E}Q_n)$$

converges in distribution to some non-Gaussian random variable $Y(\alpha, \beta)$, which can be represented as a double Wiener–Itô integral.

Note that the slowly varying functions $L_1(\lambda)$ and $L_2(\lambda)$ are of importance because they provide great flexibility in the choice of functions $f(\lambda)$ and $g(\lambda)$. In [14] it was proved that they influence only the normalization in (1.15) and not the limit $Y(\alpha, \beta)$. In this paper we show that in the critical case $\alpha + \beta = \frac{1}{2}$ the limit distribution of the standard normalized quadratic form Q_n depends on the functions $L_1(\lambda)$ and $L_2(\lambda)$.

The critical case $\alpha + \beta = \frac{1}{2}$ was partially investigated by Terrin and Taqqu in [16]. Starting from $Y(\alpha, \beta)$, which exists only when $\alpha + \beta > \frac{1}{2}$, they showed that when $0 < \alpha < 1, 0 < \beta < 1$ the random variable $(\alpha+\beta-\frac{1}{2})Y(\alpha, \beta)$ converges in distribution to a Gaussian random variable as $\alpha + \beta$ approaches $\frac{1}{2}$.

In this paper, assuming that $f(\lambda)$ and $g(\lambda)$ are regularly varying at $\lambda = 0$ of orders α and β, respectively, we prove the CLT for standard normalized quadratic form Q_n in the critical case $\alpha + \beta = \frac{1}{2}$. We also show that the CLT for Q_n is not valid under the single condition that the asymptotic variance of Q_n is separated from zero and infinity.

2. Main results. Let SV be a class of slowly varying at zero functions $u(\lambda)$ satisfying $u(\lambda) \in L^\infty(\mathbf{R})$, $\lim_{\lambda \to 0} u(\lambda) = 0$, $u(\lambda) = u(-\lambda)$, and $0 < u(\lambda) < u(\mu)$ for $0 < \lambda < \mu$.

THEOREM 2.1. *Let*

$$(2.1) \qquad\qquad\qquad f(\lambda) \leqq |\lambda|^{-\alpha} L_1(\lambda)$$

and

$$(2.2) \qquad\qquad\qquad \left|g(\lambda)\right| \leqq |\lambda|^{-\beta} L_2(\lambda),$$

where

$$(2.3) \qquad \begin{array}{c} \alpha < 1, \quad \beta < 1, \quad \alpha + \beta \leqq \dfrac{1}{2}, \\[2mm] L_i(\lambda) \in SV, \quad \lambda^{-(\alpha+\beta)} L_i(\lambda) \in L^2(\mathbb{T}), \qquad i = 1, 2. \end{array}$$

Then (1.5) *holds with variance* σ^2 *as in* (1.7).

Remark 2.1.   Examples of the spectral density $f(\lambda)$ and the generating function $g(\lambda)$ satisfying the conditions of Theorem 2.1 provide the functions

$$f(\lambda) = |\lambda|^{-\alpha}\, |\log|\lambda||^{-\gamma} \quad \text{and} \quad g(\lambda) = |\lambda|^{-\beta}\, |\log|\lambda||^{-\gamma},$$

where $\alpha < 1$, $\beta < 1$, $\alpha + \beta \leqq \frac{1}{2}$, and $\gamma > \frac{1}{2}$.

Remark 2.2. The functions $f(\lambda)$ and $g(\lambda)$ in Theorem 2.1 have singularities at the point $\lambda = 0$ and are bounded in any neighborhood of this point. It is clear that the choice of the point $\lambda = 0$ is not essential, and instead any point $\lambda_0 \in [-\pi, \pi]$ can be taken. Using the properties of the product of the Toeplitz matrices $T_n(f)$ and $T_n(g)$ it can be shown that the assertion of Theorem 2.1 remains valid in the case where $f(\lambda)$ and $g(\lambda)$ have singularities of the form (2.1), (2.2) at the same finite number of points of the segment $[-\pi, \pi]$.

For the functions $f, g \in L^1(\mathbb{T})$ we denote

$$(2.4) \qquad \varphi(t_1, t_2, t_3) = \int_{-\pi}^{\pi} f(u)\, g(u - t_1)\, f(u - t_2)\, g(u - t_3)\, du.$$

THEOREM 2.2. *If the function* $\varphi(t_1, t_2, t_3) \in L^2(\mathbb{T}^3)$ *is continuous at* $(0, 0, 0)$, *then* (1.5) *holds with variance* σ^2 *as in* (1.7).

PROPOSITION 2.1. *Theorem* 2.2 *implies Theorems* A *and* D.

Remark 2.3. For the functions $f(\lambda) = \lambda^{-3/4}$ and $g(\lambda) = \lambda^{3/4}$ satisfying the conditions of Theorem B, the function $\varphi(t_1, t_2, t_3)$ is not defined for $t_2 = 0$, $t_1 \neq 0$, $t_3 \neq 0$. This shows that Theorem 2.2 generally does not imply Theorem B.

The next result shows that the condition of positiveness and finiteness of the asymptotic variance of the quadratic form Q_n is not sufficient for Q_n to obey the CLT.

PROPOSITION 2.2. *There exist a spectral density* $f(\lambda)$ *and a generating function* $g(\lambda)$ *such that*

$$(2.5) \qquad 0 < \int_{-\pi}^{\pi} f^2(\lambda)\, g^2(\lambda)\, d\lambda < \infty$$

and

$$(2.6) \qquad \lim_{n \to \infty} \sup \chi_2(\widetilde{Q}_n) = \lim_{n \to \infty} \sup \frac{2}{n} \operatorname{tr}\big[T_n(f)\, T_n(g)\big]^2 = \infty;$$

that is, the condition (2.5) *does not guarantee convergence in* (1.10).

3. Preliminaries. Recall (see [3], [13]) that a positive function $u(x)$ is called slowly varying at zero if

$$\lim_{x \to 0} \frac{u(\lambda x)}{u(x)} = 1$$

for any $\lambda > 0$. We list some properties of slowly varying functions, which we will use below. The following property is well known (see, e.g., [13]).

LEMMA 3.1. *Let* $u(x)$ *and* $v(x)$, $x \in \mathbf{R}$, *be slowly varying at zero functions. Then*
(a) *for any* $p < 1$

$$\int_0^y x^{-p} u(x)\, dx = O\big(y^{1-p} u(y)\big) \qquad as \quad y \to 0;$$

(b) *the function* $x^p u(x)$ *increases on some interval* $(0, \delta)$ *if* $p > 0$, *and decreases if* $p < 0$;
(c) $u(x) v(x)$ *and* $u(x)/v(x)$ *are slowly varying at zero functions.*

LEMMA 3.2. *For any functions* $u, v \in SV$ *and numbers* $p, q < 1$, $p + q > 1$, *there exists a constant* $M > 0$ *such that*

$$(3.1) \qquad \int_{\mathbb{T}} |x|^{-p} |x - y|^{-q} u(x)\, v^{-1}(x - y)\, dx \leqq M |y|^{1-p-q} u(y)\, v^{-1}(y), \qquad y \in \mathbb{T}.$$

Proof. Denote

$$Q(x,y) = |x|^{-p}|x-y|^{-q}u(x)\,v^{-1}(x-y).$$

It is not hard to check that for any $\delta > 0$

$$\sup_{|y|>\delta} \int_{\mathbb{T}} Q(x,y)\,dx < \infty \quad \text{and} \quad \min_{|y|>\delta} y^{1-p-q}u(y)\,v^{-1}(y) > 0.$$

Therefore it is enough to prove (3.1) for $y \in (-\delta, \delta)$ with sufficiently small $\delta > 0$. Applying Lemma 3.1(a) we obtain

$$\int_{0<|x|<|y|/2} Q(x,y)\,dx \leqq \left(\frac{|y|}{2}\right)^{-q} v^{-1}\left(\frac{y}{2}\right) \int_{0<|x|<|y|/2} |x|^{-p}u(x)\,dx$$

$$(3.2) \qquad\qquad\qquad\qquad \leqq Cy^{1-p-q}u(y)\,v(y),$$

$$\int_{|y|/2<|x|<2|y|} Q(x,y)\,dx \leqq \left(\frac{|y|}{2}\right)^{-p} u(2|y|) \int_{|y|/2<|x|<2|y|} |x-y|^{-q}v^{-1}(x-y)\,dx$$

$$\leqq C|y|^{-p}u(|y|) \int_{0<|x|<4|y|} |x|^{-q}\,v^{-1}(x)\,dx$$

$$(3.3) \qquad\qquad\qquad\qquad \leqq Cy^{1-p-q}u(y)\,v(y),$$

and

$$\int_{2|y|<|x|<\pi} Q(x,y)\,dx \leqq |y|^{-p}v^{-1}(y) \int_{2|y|<|x|<\pi} |x|^{-q}u(x)\,dx$$

$$(3.4) \qquad\qquad\qquad\qquad \leqq Cy^{1-p-q}u(y)\,v(y).$$

From (3.2)–(3.4) we obtain (3.1). Lemma 3.2 is proved.

The following lemma can be proved similarly.

LEMMA 3.3. *Given functions $u, w \in SV$ satisfying*

$$\int_{\mathbb{T}} x^{-1}u(x)\,w^{-3}(x)\,dx < \infty,$$

for any $q \in (0,1)$ there exists a constant $M > 0$ such that

$$\int_{\mathbb{T}} |x|^{-1}|x-y|^{-q}u(x)\,w^{-2}(x)\,w^{-1}(x-y)\,dx \leqq M|y|^{-q}w^{-3}(y), \qquad y \in \mathbb{T}.$$

We denote by $D_n(x)$ the Dirichlet kernel

$$(3.5) \qquad\qquad\qquad D_n(x) = \frac{\sin(nx/2)}{\sin(x/2)}.$$

It is not hard to see that

$$(3.6) \qquad \left|D_n(x)\right| \leqq \min\{n,\ |x|^{-1}\} \quad \text{and} \quad \left|D_n(x)\right| \leqq Cn\psi_n(x), \qquad x \in \mathbb{T},$$

where $\psi_n(x) = (1 + n|x|)^{-1}$.

LEMMA 3.4. *For any function* $w \in SV$ *and a number* $t \in (0,1)$ *there exists a constant* $M > 0$ *such that*

$$\left|D_n(x)\right| \leqq M w(n^{-1})\, n^t |x|^{t-1} w^{-1}(x).$$

Proof. According to Lemma 3.1(b) the functions $x^{t-1} w^{-1}(x)$ and $x^{-t} w(x)$ are decreasing in some interval $(0, \delta)$. Since

$$\min\left\{ w(n^{-1})\, n^t |x|^{t-1} w^{-1}(x) \right\} > 0,$$

we can assume that $n^{-1} < \delta$ and $|x| < \delta$. Now, if $|x| \leqq n^{-1}$, then $n^{1-t} w^{-1}(1/n) \leqq x^{t-1} w^{-1}(x)$ and (3.6) imply

$$|D_n(x)| \leqq n = w(n^{-1})\, n^t n^{1-t} w^{-1}(n^{-1}) \leqq w(n^{-1})\, n^t |x|^{t-1} w^{-1}(x).$$

The proof in the case $|x| > n^{-1}$ is similar. Lemma 3.4 is proved.

The following lemma was proved in [8].

LEMMA 3.5. *For any* $\delta \in (0,1)$ *there exists a constant* $C_\delta > 0$ *such that*

$$n \int_{\mathbb{T}} \psi_n(x-y)\, \psi_n(x-z)\, dx \leqq C_\delta \psi_n^{1-\delta}(y-z), \qquad y, z \in \mathbb{T}.$$

Denote

$$(3.7) \qquad \Phi_n(x_1, x_2, x_3) = \frac{1}{(2\pi)^3 n}\, D_n(x_1)\, D_n(x_2)\, D_n(x_3)\, D_n(x_1 + x_2 + x_3),$$

where $D_n(x)$ is as in (3.5). For given $\alpha \in (0, \pi)$ we set

$$\mathbb{E}_\alpha = \left\{ |\mathbf{x}| \leqq \alpha \right\} = \{(x_1, x_2, x_3);\ |x_k| \leqq \alpha,\ k = 1, 2, 3\},$$

$$\mathbb{E}_\alpha^c = \left\{ |\mathbf{x}| \leqq \pi \right\} \setminus \left\{ |\mathbf{x}| \leqq \alpha \right\}.$$

LEMMA 3.6. *The kernel* $\Phi_n(\mathbf{x})$ *defined by* (3.7) *with* $\mathbf{x} = (x_1, x_2, x_3)$ *possesses the following properties*:

(a) $\int_{\mathbb{T}^3} \Phi_n(\mathbf{x})\, d\mathbf{x} = 1$;

(b) $\sup_n \int_{\mathbb{T}^3} |\Phi_n(\mathbf{x})|\, d\mathbf{x} = C_1 < \infty$;

(c) *for any* $\varepsilon \in (0, \pi)$ *we have* $\lim_{n \to \infty} \int_{\mathbb{E}_\varepsilon^c} \left|\Phi_n(\mathbf{x})\right| d\mathbf{x} = 0$;

(d) *for any* $\delta > 0$ *there exists a positive constant* M_δ *such that*

$$(3.8) \qquad \int_{\mathbb{E}_\delta^c} \Phi_n^2(\mathbf{x})\, d\mathbf{x} \leqq M_\delta, \qquad n = 1, 2, \dots.$$

Proof. Proofs of (a)–(c) can be found in [2, Lemma 3.1]. To prove (d) first observe that

$$(3.9) \quad \int_{\mathbb{T}} D_n^2(x)\, dx \leqq C n \quad \text{and} \quad \left|D_n(x)\right| \leqq C_\delta \qquad \text{as} \quad |x| > \delta,\ n = 1, 2, \dots,$$

where $D_n(x)$ is the Dirichlet kernel, while C and C_δ are some positive constants. We have

$$\int_{\mathbb{E}_\delta^c} \Phi_n^2(\mathbf{x})\, d\mathbf{x} \leqq \int_{|x_1| > \delta} \Phi_n^2(\mathbf{x})\, d\mathbf{x} + \int_{|x_2| > \delta} \Phi_n^2(\mathbf{x})\, d\mathbf{x} + \int_{|x_3| > \delta} \Phi_n^2(\mathbf{x})\, d\mathbf{x}$$

$$(3.10) \qquad =: I_1 + I_2 + I_3.$$

Clearly, it is enough to estimate I_1. We have

$$I_1 \leqq \int_{|x_1|>\delta, |x_2|>\delta/3} \Phi_n^2(\mathbf{x}) \, d\mathbf{x} + \int_{|x_1|>\delta, |x_3|>\delta/3} \Phi_n^2(\mathbf{x}) \, d\mathbf{x}$$

(3.11)
$$+ \int_{|x_1|>\delta, |x_2|\leqq\delta/3, |x_3|\leqq\delta/3} \Phi_n^2(\mathbf{x}) \, d\mathbf{x} =: I_1^{(1)} + I_1^{(2)} + I_1^{(3)}.$$

Using (3.9) we obtain

(3.12) $$I_1^{(1)} \leqq C_\delta \frac{1}{n^2} \int_{\mathbb{T}^3} D_n^2(x_3) \, D_n^2(x_1 + x_2 + x_3) \, dx_1 \, dx_2 \, dx_3 \leqq M_\delta.$$

Likewise,

(3.13) $$I_1^{(2)} \leqq M_\delta.$$

Now, observing that in the integral $I_1^{(3)}$ the integration region is such that $|x_1 + x_2 + x_3| > \delta/3$, from (3.9) we find

(3.14) $$I_1^{(3)} \leqq C_\delta \frac{1}{n^2} \int_{\mathbb{T}^3} D_n^2(x_2) \, D_n^2(x_3) \, dx_1 \, dx_2 \, dx_3 \leqq M_\delta.$$

From (3.12)–(3.14) we obtain (3.11). Lemma 3.6 is proved.

LEMMA 3.7. *Let the function $\Psi(\mathbf{u}) \in L^2(\mathbb{T}^3)$ be continuous at $\mathbf{0} = (0,0,0)$. Then*

(3.15) $$\lim_{n \to \infty} \int_{\mathbb{T}^3} \Psi(\mathbf{u}) \, \Phi_n(\mathbf{u}) \, d\mathbf{u} = \Psi(\mathbf{0}),$$

where $\mathbf{u} = (u_1, u_2, u_3)$ and the function $\Phi_n(\mathbf{u})$ is defined by (3.7).

Proof. By Lemma 3.6(a) we have

(3.16) $$R_n := \int_{\mathbb{T}^3} \Psi(\mathbf{u}) \, \Phi_n(\mathbf{u}) \, d\mathbf{u} - \Psi(\mathbf{0}) = \int_{\mathbb{T}^3} \left[\Psi(\mathbf{u}) - \Psi(\mathbf{0}) \right] \Phi_n(\mathbf{u}) \, d\mathbf{u}.$$

For any $\varepsilon > 0$, a $\delta > 0$ can be chosen to satisfy

(3.17) $$\left| \Psi(\mathbf{u}) - \Psi(\mathbf{0}) \right| < \frac{\varepsilon}{C_1} \qquad \text{if} \quad \mathbf{u} \in \mathbb{E}_\delta,$$

where C_1 is the constant from Lemma 3.6(b). We represent $\Psi = \Psi_1 + \Psi_2$ such that

(3.18) $$\|\Psi_1\|_2 \leqq \frac{\varepsilon}{\sqrt{M_\delta}} \quad \text{and} \quad \|\Psi_2\|_\infty < \infty,$$

where M_δ is the constant from Lemma 3.6(d). Applying Lemma 3.6(b)–(d) and (3.16)–(3.18) for sufficiently large n we obtain

$$|R_n| \leqq \int_{\mathbb{E}_\delta} \left| \Psi(\mathbf{u}) - \Psi(\mathbf{0}) \right| \left| \Phi_n(\mathbf{u}) \right| d\mathbf{u}$$

$$+ \int_{\mathbb{E}_\delta^c} \left| \Psi_1(\mathbf{u}) \right| \left| \Phi_n(\mathbf{u}) \right| d\mathbf{u} + \int_{\mathbb{E}_\delta^c} \left| \Psi_2(\mathbf{u}) - \Psi(\mathbf{0}) \right| \left| \Phi_n(\mathbf{u}) \right| d\mathbf{u}$$

$$\leqq \frac{\varepsilon}{C_1} \int_{\mathbb{E}_\delta} \left| \Phi_n(\mathbf{u}) \right| d\mathbf{u} + \|\Psi_1\|_2 \left[\int_{\mathbb{E}_\delta^c} \Phi_n^2(\mathbf{u}) \, d\mathbf{u} \right]^{1/2} + C_2 \int_{\mathbb{E}_\delta^c} \left| \Phi_n(\mathbf{u}) \right| d\mathbf{u} \leqq 3 \, \varepsilon.$$

This together with (3.16) implies (3.15). Lemma 3.7 is proved.

4. Proofs of main results.

Proof of Theorem 2.1. For $f, g \in L^1(\mathbb{T})$ and $\mathbf{x} = (x_1, x_2, x_3, x_4)$ we set

$$F(\mathbf{x}) = f(x_1) f(x_2) g(x_3) g(x_4),$$

and let

$$H_n(\mathbf{x}) = G_n(x_1 - x_3) G_n(x_2 - x_3) G_n(x_4 - x_1) G_n(x_4 - x_2),$$

where

$$(4.1) \qquad G_n(u) = \sum_{k=1}^n e^{iku} = e^{iu(n+1)/2} D_n(u).$$

It is easy to check that

$$(4.2) \qquad \operatorname{tr}\big(T_n(f) \, T_n(g)\big)^2 = \int_{\mathbb{T}^4} F(\mathbf{x}) \, H_n(\mathbf{x}) \, d\mathbf{x}.$$

By Theorem B it is enough to consider the case $\alpha + \beta = \frac{1}{2}$. Thus, by Theorem C we need to prove that

$$(4.3) \qquad \lim_{n \to \infty} \frac{1}{n} \int_{\mathbb{T}^4} F(\mathbf{x}) \, H_n(\mathbf{x}) \, d\mathbf{x} = 8\pi^3 \int_{\mathbb{T}} f^2(x) \, g^2(x) \, dx,$$

provided that

$$(4.4) \qquad f(x) \leqq |x|^{-\alpha} L(x), \quad |g(x)| \leqq |x|^{-\beta} L(x), \qquad x \in \mathbb{T},$$

where $L = L_1 + L_2 \in SV$ and

$$(4.5) \qquad \alpha < 1, \quad \beta < 1, \quad \alpha + \beta = \frac{1}{2}, \qquad \int_{\mathbb{T}} x^{-1} L^2(x) \, dx < \infty.$$

If $\alpha, \beta \geqq 0$, then (4.4) implies $f \in L^{1/\alpha}(\mathbb{T})$, $g \in L^{1/\beta}(\mathbb{T})$, and Theorem 2.1 follows from Theorem A. Assuming $\beta < 0$, from (4.5) we have

$$(4.6) \qquad \frac{1}{2} < \alpha < 1, \qquad -\frac{1}{2} < \beta < 0.$$

For $\varepsilon \in (0, 1)$, we set

$$f_\varepsilon(x) = \begin{cases} 0 & \text{if} \quad |x| < \varepsilon, \\ f(x) & \text{if} \quad \varepsilon \leqq |x| \leqq \pi, \end{cases}$$

and

$$\mathbb{T}_{i,\varepsilon} = \big\{ \mathbf{x} \in \mathbb{T}^4 \colon |x_i| < \varepsilon \big\}, \qquad i = 1, 2.$$

We have

$$\frac{1}{n} \int_{\mathbb{T}^4} F(\mathbf{x}) \, H_n(\mathbf{x}) \, d\mathbf{x} = J_n^1 + J_n^2,$$

where

$$J_n^1 := \frac{1}{n} \int_{\mathbb{T}^4} f_\varepsilon(x_1) f_\varepsilon(x_2) \, g(x_3) \, g(x_4) \, H_n(\mathbf{x}) \, d\mathbf{x}$$

and

$$|J_n^2| \leqq \frac{1}{n} \int_{\mathbb{T}_{1,\varepsilon}} |F(\mathbf{x}) H_n(\mathbf{x})| \, d\mathbf{x} + \frac{1}{n} \int_{\mathbb{T}_{2,\varepsilon}} |F(\mathbf{x}) H_n(\mathbf{x})| \, d\mathbf{x} =: I_n^1 + I_n^2.$$

Since $f_\varepsilon, g \in L^\infty(\mathbb{T})$, we have

$$\lim_{n \to \infty} J_n^1 = 8\pi^3 \int_{\mathbb{T}} f_\varepsilon^2(x) \, g^2(x) \, dx.$$

The last integral tends to $\int_{\mathbb{T}} f^2(x) \, g^2(x) \, dx$ as $\varepsilon \to 0$; hence (4.3) follows from

(4.7)
$$\lim_{\varepsilon \to 0, n \to \infty} (I_n^1 + I_n^2) = 0.$$

It is enough to prove (4.7) for I_n^1. Set

$$B_{i,j} = \left\{ \mathbf{x} \in \mathbb{T}^4 \colon |x_i| \leqq \frac{|x_j|}{2} \right\}, \qquad i = 1, 2, \quad j = 3, 4,$$

$$B = \left\{ \mathbf{x} \in \mathbb{T}^4 \colon |x_1| < \varepsilon, \; |x_i| > \frac{|x_j|}{2}, \; i = 1, 2, \; j = 3, 4 \right\}.$$

Then we have

(4.8)
$$I_n^1 \leqq \frac{1}{n} \sum_{i=1}^{2} \sum_{j=3}^{4} \int_{B_{i,j}} F(\mathbf{x}) H_n(\mathbf{x}) \, d\mathbf{x} + \frac{1}{n} \int_B F(\mathbf{x}) H_n(\mathbf{x}) \, d\mathbf{x}.$$

Let $w \in SV$ be a function satisfying

(4.9)
$$\int_{\mathbb{T}} x^{-1} L^2(x) \, w^{-4}(x) \, dx < \infty.$$

Since

$$\frac{|x_3|}{2} < |x_1 - x_3| < 2|x_3| \qquad \text{as} \quad \mathbf{x} \in B_{1,3},$$

the bounds (4.4) and Lemma 3.4 imply

$$A_{1,3} := \frac{1}{n} \int_{B_{1,3}} F(\mathbf{x}) \, G_n(\mathbf{x}) \, d\mathbf{x}$$

$$\leqq Cw^4 \left(\frac{1}{n} \right) \int_{B_{1,3}} |x_1|^{-\alpha} |x_2|^{-\alpha} |x_3|^{-\beta} |x_4|^{-\beta} L(x_1) \, L(x_2) \, L(x_3) \, L(x_4)$$

$$\times |x_1 - x_3|^{-3/4} |x_2 - x_3|^{-3/4} |x_1 - x_4|^{-3/4} |x_2 - x_4|^{-3/4}$$

$$\times w^{-1}(x_1 - x_3) \, w^{-1}(x_2 - x_3) \, w^{-1}(x_1 - x_4) \, w^{-1}(x_2 - x_4) \, d\mathbf{x}$$

$$\leqq Cw^4 \left(\frac{1}{n} \right) \int_{\mathbb{T}^2} |x_2|^{-\alpha} |x_4|^{-\beta} |x_2 - x_4|^{-3/4} L(x_2) \, L(x_4) \, w^{-1}(x_2 - x_4) \, dx_2$$

$$\times \int_{\mathbb{T}} |x_1|^{-\alpha} |x_1 - x_4|^{-3/4} L(x_1) \, w^{-1}(x_1 - x_4) \, dx_1$$

$$\times \int_{\mathbb{T}} |x_3|^{-\beta - 3/4} |x_2 - x_3|^{-3/4} L(x_3) \, w^{-1}(x_3) \, w^{-1}(x_2 - x_3) \, dx_3 \, dx_2 \, dx_4.$$

Applying first Lemma 3.2, then Lemma 3.3, we obtain

$$A_{1,3} \leqq Cw^4\left(\frac{1}{n}\right) \int_{\mathbb{T}^2} |x_2|^{-\alpha}|x_4|^{-\beta}|x_2 - x_4|^{-3/4}L(x_2)\,L(x_4)\,w^{-1}(x_2 - x_4)$$

$$\times |x_4|^{-\alpha+1/4}L(x_4)\,w^{-1}(x_4)\,|x_2|^{-\beta-1/2}L(x_2)\,w^{-2}(x_2)\,dx_2\,dx_4$$

$$= Cw^4\left(\frac{1}{n}\right)\int_{\mathbb{T}} |x_4|^{-1/4}L^2(x_4)\,w^{-1}(x_4)$$

$$\times \int_{\mathbb{T}} |x_2|^{-1}|x_2 - x_4|^{-3/4}L^2(x_2)\,w^{-2}(x_2)w^{-1}(x_2 - x_4)\,dx_2\,dx_4$$

$$(4.10) \quad \leqq Cw^4\left(\frac{1}{n}\right)\int_{\mathbb{T}} |x_4|^{-1}L^2(x_4)\,w^{-4}(x_4)\,dx_4 = o(1),$$

as $n \to \infty$. Similarly we can prove that all the integrals in the first sum in (4.8) tend to zero as $n \to \infty$. To estimate the last integral in (4.8) we use (4.4) and Lemma 3.5 to obtain

$$A := \frac{1}{n}\int_B \big|F(\mathbf{x})\,H_n(\mathbf{x})\big|\,d\mathbf{x}$$

$$\leqq Cn^3 \int_B |x_1|^{-\alpha}|x_2|^{-\alpha}|x_3|^{-\beta}|x_4|^{-\beta}L(x_1)\,L(x_2)\,L(x_3)\,L(x_4)$$

$$\times \psi_n(x_1 - x_3)\,\psi_n(x_2 - x_3)\,\psi_n(x_1 - x_4)\,\psi_n(x_2 - x_4)\,d\mathbf{x}$$

$$\leqq Cn^3 \int_{(-2\varepsilon,2\varepsilon)^2} |x_3|^{-1/2}|x_4|^{-1/2}L(x_3)\,L(x_4)$$

$$\times \int_{\mathbb{T}} \psi_n(x_1 - x_3)\,\psi_n(x_1 - x_4)\,L(x_1)\,dx_1$$

$$\times \int_{\mathbb{T}} \psi_n(x_2 - x_3)\,\psi_n(x_2 - x_4)\,L(x_2)\,dx_2\,dx_3\,dx_4$$

$$\leqq Cn \int_{(-2\varepsilon,2\varepsilon)} |x_3|^{-1/2}L(x_3)\int_{\mathbb{T}} |x_4|^{-1/2}\psi_n^{3/2}(x_3 - x_4)\,L(x_4)\,dx_4\,dx_3$$

$$(4.11) \quad \leqq C\int_{-2n\varepsilon}^{2n\varepsilon} |y|^{-1/2}\,L\left(\frac{y}{n}\right)\int_{-\infty}^{\infty} \frac{|x|^{-1/2}}{(1+|x-y|)^{3/2}}\,L\left(\frac{x}{n}\right)\,dx\,dy.$$

Let us prove that

$$(4.12) \quad \int_{-\infty}^{\infty} \frac{|x|^{-1/2}}{(1+|x-y|)^{3/2}}L\left(\frac{x}{n}\right)\,dx \leqq Cy^{-1/2}L\left(\frac{y}{n}\right), \qquad y \in \mathbb{T}.$$

Indeed, for $y \in \mathbf{T}$

$$\int_{|x|\leqq|y|} \frac{|x|^{-1/2}}{(1+|x-y|)^{3/2}}L\left(\frac{x}{n}\right)\,dx \leqq CL\left(\frac{y}{n}\right)\int_{\mathbb{T}} |x|^{-1/2}\,dx$$

$$(4.13) \qquad\qquad\qquad\qquad\qquad\qquad \leqq CL\left(\frac{y}{n}\right) \leqq Cy^{-1/2}\,L\left(\frac{y}{n}\right).$$

According to Lemma 3.1 the function $t^{-1/2}L(t)$ is decreasing on some interval $(0,\delta)$. Hence, assuming without loss of generality that $n > \pi/\delta$, we have for $|x| > |y|$

$$|x|^{-1/2}L\left(\frac{x}{n}\right) = n^{-1/2}\left(\frac{|x|}{n}\right)^{-1/2}L\left(\frac{x}{n}\right) \leqq n^{-1/2}\left(\frac{|y|}{n}\right)^{-1/2}L\left(\frac{y}{n}\right) = |y|^{-1/2}L\left(\frac{y}{n}\right).$$

Therefore

$$\int_{|x|>|y|}\frac{|x|^{-1/2}}{(1+|x-y|)^{3/2}}L\left(\frac{x}{n}\right)dx \leqq C|y|^{-1/2}L\left(\frac{y}{n}\right)\int_{-\infty}^{\infty}\frac{1}{(1+|x|)^{3/2}}\,dx$$

$$\text{(4.14)} \hspace{4cm} \leqq C|y|^{-1/2}L\left(\frac{y}{n}\right).$$

From (4.13), (4.14) we obtain (4.12), and from (4.11), (4.12), and (4.5)

$$\text{(4.15)} \qquad A \leqq C\int_{-2n\varepsilon}^{2n\varepsilon}|y|^{-1}L^2\left(\frac{y}{n}\right)dy = C\int_{-2\varepsilon}^{2\varepsilon}|t|^{-1}L^2(t)\,dt = o(\varepsilon),$$

as $\varepsilon \to 0$. A combination of (4.8), (4.10), and (4.15) yields (4.7). Theorem 2.1 is proved.

Proof of Theorem 2.2. By the change of variables $x_1 = u$, $x_1 - x_3 = u_1$, $x_3 - x_2 = u_2$, and $x_2 - x_4 = u_3$ from (4.2) we obtain

$$\text{tr}\big(T_n(f)\,T_n(g)\big)^2 = \int_{\mathbb{T}^4} G_n(u_1)\,G_n(u_2)\,G_n(u_3)\,G_n(-u_1-u_2-u_3)$$

$$\times f(u)\,g(u-u_1)\,f(u-u_1-u_2)$$

$$\text{(4.16)} \hspace{3cm} \times g(u-u_1-u_2-u_3)\,du_1\,du_2\,du_3\,du_4,$$

where $G_n(u)$ is as in (4.1). Taking into account the equality

$$e^{iu_1(n+1)/2}\,e^{iu_2(n+1)/2}\,e^{iu_3(n+1)/2}\,e^{-i(u_1+u_2+u_3)(n+1)/2} = 1$$

and that $D_n(u)$ is an even function, from (4.16) we obtain

$$\text{(4.17)} \quad \text{tr}\big(T_n(f)\,T_n(g)\big)^2 = 8\pi^3\int_{\mathbb{T}^3}\Psi(u_1,u_2,u_3)\,\Phi_n(u_1,u_2,u_3)\,du_1\,du_2\,du_3,$$

where $\Phi_n(u_1,u_2,u_3)$ is defined by (3.7), $\Psi(u_1,u_2,u_3) = \varphi(u_1, u_1+u_2, u_1+u_2+u_3)$, and $\varphi(u_1,u_2,u_3)$ is defined by (2.4). By Theorem C and (4.17) we need to prove that

$$\text{(4.18)} \hspace{2.5cm} \lim_{n\to\infty}\int_{\mathbb{T}^3}\Psi(\mathbf{u})\,\Phi_n(\mathbf{u})\,d\mathbf{u} = \int_{\mathbb{T}}f^2(x)\,g^2(x)\,dx.$$

Now, since the functions $\varphi(u_1,u_2,u_3)$ and $\Psi(u_1,u_2,u_3) = \varphi(u_1, u_1+u_2, u_1+u_2+u_3)$ are square integrable and continuous at $(0,0,0)$ simultaneously, and

$$\Psi(0,0,0) = \int_{\mathbb{T}}f^2(x)\,g^2(x)\,dx,$$

from Lemma 3.7 we obtain (4.18). Theorem 2.2 is proved.

Proof of Proposition 2.1.　To show that Theorem 2.2 implies Theorem A it is enough to prove that the function

$$(4.19) \quad \varphi(\mathbf{t}) := \int_{\mathbb{T}} f_0(u)\, f_1(u - t_1)\, f_2(u - t_2)\, f_3(u - t_3)\, du, \quad \mathbf{t} = (t_1, t_2, t_3),$$

belongs to $L^2(\mathbb{T}^3)$ and is continuous at $(0, 0, 0)$, provided that

$$(4.20) \qquad f_i \in L^{p_i}(\mathbb{T}), \quad 1 \leqq p_i \leqq \infty, \quad i = 0, 1, 2, 3, \qquad \sum_{i=0}^{3} \frac{1}{p_i} \leqq 1.$$

It follows from the Hölder inequality and (4.20) that

$$\left| \varphi(\mathbf{t}) \right| \leqq \prod_{i=0}^{3} \| f_i \|_{L^{p_i}(\mathbb{T})}, \qquad \mathbf{t} = (t_1, t_2, t_3) \in \mathbb{T}^3.$$

Therefore, $\varphi(\mathbf{t}) \in L^2(\mathbb{T}^3)$. To prove the continuity of $\varphi(\mathbf{t})$ at the point $(0, 0, 0)$ we consider three cases.

Case 1. $p_i < \infty$, $i = 0, 1, 2, 3$. For an arbitrary $\varepsilon > 0$ we find $\delta > 0$ satisfying (see (4.20))

$$(4.21) \qquad \left\| f_i(u - t) - f_i(u) \right\|_{p_i} \leqq \varepsilon, \qquad i = 1, 2, 3, \quad \text{if} \quad |t| \leqq \delta.$$

We fix $\mathbf{t} = (t_1, t_2, t_3)$ with $|\mathbf{t}| < \delta$ and denote $\bar{f}_i(u) = f_i(u + t_i) - f_i(u)$, $i = 1, 2, 3$. Then (4.21) implies $\| \bar{f}_i \|_{p_i} \leqq \varepsilon$, $i = 1, 2, 3$, and hence

$$\varphi(\mathbf{t}) = \int_{\mathbb{T}} f_0(u) \prod_{i=1}^{3} \left(\bar{f}_i(u) + f_i(u) \right) du = \varphi(0, 0, 0) + W,$$

where the quantity W is a sum of five integrals. Each of them contains at least one function $\bar{f}_i$ and can be estimated as the following integral:

$$\left| \int_{\mathbb{T}} f_0(u)\, \bar{f}_1(u)\, f_2(u)\, f_3(u)\, du \right| \leqq \| f_0 \|_{p_0} \| \bar{f}_1 \|_{p_1} \| f_2 \|_{p_2} \| f_3 \|_{p_3} \leqq C\varepsilon.$$

Case 2. $p_i \leqq \infty$, $i = 0, 1, 2, 3$, $\sum_{i=0}^{3} 1/p_i < 1$. There exist finite numbers $p_i' < p_i$, $i = 0, 1, 2, 3$, $\sum_{i=0}^{3} 1/p_i' \leqq 1$, for which we have $f_i \in L^{p_i}(\mathbb{T})$. Hence the function φ is continuous at $(0, 0, 0)$ as in Case 1.

Case 3. $p_i \leqq \infty$, $i = 0, 1, 2, 3$, $\sum_{i=0}^{3} 1/p_i = 1$.

At least one of the numbers p_i is finite. Suppose, without loss of generality, that $p_0 < \infty$. For any $\varepsilon > 0$ we find functions f_0', f_0'' such that

$$(4.22) \qquad f_0 = f_0' + f_0'', \quad f_0' \in L^\infty, \quad \| f_0'' \|_{p_0} < \varepsilon.$$

Then

$$(4.23) \qquad \varphi(\mathbf{t}) = \varphi'(\mathbf{t}) + \varphi''(\mathbf{t}),$$

where the functions φ' and φ'' are defined as φ in (4.19) with f_0 replaced by f_0' and f_0'', respectively. It follows from (4.22) that φ' is continuous at $(0, 0, 0)$ (see

Case 2), while for φ'', an application of the Hölder inequality yields $|\varphi''(\mathbf{t})| \leqq C\varepsilon$. Hence, for sufficiently small $|\mathbf{t}|$

$$\left|\varphi(\mathbf{t}) - \varphi(0,0,0)\right| \leqq \left|\varphi'(\mathbf{t}) - \varphi'(0,0,0)\right| + \left|\varphi''(\mathbf{t}) - \varphi''(0,0,0)\right| \leqq (C+1)\,\varepsilon,$$

and the result follows.

Now we prove that Theorem 2.2 implies Theorem D. To this end it is enough to show that the function

$$\varphi(\mathbf{t}) = \int_{\mathbb{T}} f(u)\,g(u - t_1)\,f(u - t_2)\,g(u - t_3)\,du, \qquad \mathbf{t} = (t_1, t_2, t_3) \in \mathbb{T}^3,$$

is continuous at $(0,0,0)$, provided that f and g satisfy the conditions of Theorem D, i.e., $f \in L_2(\mathbb{T})$, $g \in L_2(\mathbb{T})$, $fg \in L_2(\mathbb{T})$, and (1.11) holds.

Since

$$\varphi^2(\mathbf{t}) \leqq 2\pi \int_{\mathbb{T}} f^2(u)\,g^2(u - t_1)\,f^2(u - t_2)\,g^2(u - t_3)\,du,$$

we have

$$\int_{\mathbb{T}^3} \varphi^2(\mathbf{t})\,d\mathbf{t} \leqq \int_{\mathbb{T}} \left[\int_{\mathbb{T}} g^2(u - t_1)\,dt_1 \int_{\mathbb{T}} f^2(u - t_2)\,dt_2 \int_{\mathbb{T}} g^2(u - t_3)\,dt_3 \right] f^2(u)\,du$$

$$= \|f\|_2^4\,\|g\|_2^4 < \infty.$$

Now we prove the continuity of $\varphi(\mathbf{t})$ at the point $(0,0,0)$. Let ε be an arbitrary positive number. We denote

$$E_K = \{u \in \mathbb{T}: \, |f(u)| \leqq K\}, \quad f_1(u) = \chi_{E_K}(u)\,f(u), \quad f_2(u) = f(u) - f_1(u),$$

where $K > 0$ is chosen to satisfy

$$\int_{\mathbb{T} \setminus E_K} f^2(u)\,g^2(u)\,du \leqq \varepsilon.$$

Then

$$(4.24) \qquad f = f_1 + f_2, \quad \|f_1\|_\infty \leqq K, \quad \int_{\mathbb{T}} f_2^2(u)\,g^2(u)\,du \leqq \varepsilon.$$

Consider the decomposition

$$\varphi(\mathbf{t}) = \int_{\mathbb{T}} f_1(u)\,g(u - t_1)\,f_1(u - t_2)\,g(u - t_3)\,du$$

$$+ \int_{\mathbb{T}} f_2(u)\,g(u - t_1)\,f(u - t_2)\,g(u - t_3)\,du$$

$$(4.25) \qquad + \int_{\mathbb{T}} f_1(u)\,g(u - t_1)\,f_2(u - t_2)\,g(u - t_3)\,du =: \varphi_1(\mathbf{t}) + \varphi_2(\mathbf{t}) + \varphi_3(\mathbf{t}).$$

We estimate the functions $\varphi_k(\mathbf{t})$, $k = 1, 2, 3$, separately. We have

$$\varphi_1(\mathbf{t}) = \int_{\mathbb{T}} f_1(u)\,g(u - t_1)\,f_1(u - t_2)\,\big[g(u - t_3) - g(u)\big]\,du$$

$$+ \int_{\mathbb{T}} f_1(u)\,g(u)\,f_1(u - t_2)\,\big[g(u - t_1) - g(u)\big]\,du$$

$$(4.26) \qquad + \int_{\mathbb{T}} f_1(u)\,g^2(u)\,f_1(u - t_2)\,du =: I_1 + I_2 + I_3.$$

Using the Hölder inequality, from (4.24) we get

(4.27) $$\left|I_1\right| \leqq K^2 \|g\|_2 \left\|g(u+t_3)-g(u)\right\|_2 = o(1) \qquad \text{as} \quad t_3 \to 0.$$

Similarly

(4.28) $$\left|I_2\right| = o(1) \qquad \text{as} \quad t_1 \to 0$$

and in view of (4.24) we have

$$\left|I_3 - \int_{\mathbb{T}} \varphi(0,0,0)\right| = \left|\int_{\mathbb{T}} f_1(u+t_2)\, g^2(u+t_2)\, f_1(u)\, du - \int_{\mathbb{T}} f_1^2(u)\, g^2(u)\, du\right|$$

$$+ \left|\int_{\mathbb{T}} f_2^2(u)\, g^2(u)\, du\right|$$

(4.29) $$\leqq K\left\|f_1(u+t_2)\, g^2(u+t_2) - f_1(u)\, g_1^2(u)\right\|_1 + \varepsilon = o(1) + \varepsilon$$

as $t_2 \to 0$. From (4.26)–(4.29) for sufficiently small $|\mathbf{t}|$ we obtain

(4.30) $$\left|\varphi_1(\mathbf{t}) - \varphi(0,0,0)\right| \leqq 2\varepsilon.$$

Next, for $\varphi_2(\mathbf{t})$ we have

$$\left|\varphi_2(\mathbf{t})\right|^2 \leqq \int_{\mathbb{T}} f_2^2(u)\, g^2(u-t_1)\, du \int_{\mathbb{T}} f_2^2(u-t_2)\, g^2(u-t_3)\, du$$

$$= \left|\int_{\mathbb{T}} f^2(u)\, g^2(u-t_1)\, du - \int_{\mathbb{T}} f_1^2(u)\, g^2(u-t_1)\, du\right|$$

$$\times \int_{\mathbb{T}} f^2(u)\, g^2(u+t_2-t_3)\, du$$

$$\longrightarrow \left|\int_{\mathbb{T}} f^2(u)\, g^2(u)\, du - \int_{\mathbb{T}} f_1^2(u)\, g^2(u)\, du\right| \int_{\mathbb{T}} f^2(u)\, g^2(u)\, du,$$

as $|\mathbf{t}| \to 0$. Therefore, in view of (4.24) for sufficiently small $|\mathbf{t}|$

(4.31) $$\left|\varphi_2(\mathbf{t})\right| \leqq \varepsilon \int_{\mathbb{T}} f^2(u)\, g^2(u)\, du.$$

Similarly we can prove that for sufficiently small $|\mathbf{t}|$

(4.32) $$\left|\varphi_3(\mathbf{t})\right| \leqq \varepsilon \int_{\mathbb{T}} f^2(u)\, g^2(u)\, du.$$

A combination of (4.25) and (4.30)–(4.32) yields

$$\lim_{\mathbf{t}\to 0} \varphi(\mathbf{t}) = \varphi(0,0,0).$$

This completes the proof of Proposition 2.1.

Proof of Proposition 2.2.   We construct functions $f(\lambda)$ and $g(\lambda)$ satisfying the conditions (2.5) and (2.6). Let $p \geqq 2$ be fixed; we choose a number $q > 1$ satisfying $1/p + 1/q > 1$. For such p and q consider the functions $f_0(\lambda)$ and $g_0(\lambda)$ defined by (1.12) and (1.13), respectively. For an arbitrary finite positive constant C we set $g_\pm(\lambda) = g_0(\lambda) \pm C$.

Since the functions $f_0(\lambda)$ and $g_0(\lambda)$ have disjoint supports, we have

$$\int_{-\pi}^{\pi} f_0^2(\lambda)\, g_\pm^2(\lambda)\, d\lambda = \int_{-\pi}^{\pi} f_0^2(\lambda)\, (g_0(\lambda) \pm C)^2\, d\lambda = C^2 \int_{-\pi}^{\pi} f_0^2(\lambda)\, d\lambda < \infty,$$

and hence (2.5) is fulfilled. Next, by (1.14)

$$(4.33) \qquad \frac{1}{n}\, \mathrm{tr}\big(T_n(f_0)\, T_n(g_0)\big)^2 \longrightarrow \infty \qquad \text{as} \quad n \to \infty,$$

and by Theorem A with $p_1 = p \geqq 2$ and $p_2 = \infty$,

$$(4.34) \qquad \frac{1}{n}\, C^2\, \mathrm{tr}\big(T_n^2(f_0)\big) \longrightarrow 8\pi^3 C^2 \int_{-\pi}^{\pi} f_0^2(\lambda)\, d\lambda < \infty.$$

On the other hand, we have

$$\mathrm{tr}\big(T_n(f_0)\, T_n(g_\pm)\big)^2 = \mathrm{tr}\big(T_n(f_0)\, T_n(g_0 \pm C)\big)^2$$
$$= \mathrm{tr}\big(T_n(f_0)\, T_n(g_0)\big)^2 \pm 2C\, \mathrm{tr}\big(T_n^2(f_0)\, T_n(g_0)\big) + C^2 \mathrm{tr}\big(T_n^2(f_0)\big),$$

which combined with (4.33) and (4.34) implies

$$\frac{1}{n}\, \mathrm{tr}\big(T_n(f_0)\, T_n(g_+)\big)^2 + \frac{1}{n}\, \mathrm{tr}\big(T_n(f_0)\, T_n(g_-)\big)^2$$
$$= \frac{2}{n}\, \mathrm{tr}\big(T_n(f_0)\, T_n(g_0)\big)^2 + \frac{2}{n}\, C^2 \mathrm{tr}\big(T_n^2(f_0)\big) \longrightarrow \infty \qquad \text{as} \quad n \to \infty.$$

Therefore, either

$$\limsup_{n \to \infty} \frac{1}{n}\, \mathrm{tr}\big(T_n(f_0)\, T_n(g_+)\big)^2 = \infty$$

or

$$\limsup_{n \to \infty} \frac{1}{n}\, \mathrm{tr}\big(T_n(f_0)\, T_n(g_-)\big)^2 = \infty.$$

Thus, we obtain

$$\limsup_{n \to \infty} \chi_2(\widetilde{Q}_n) = \limsup_{n \to \infty} \frac{2}{n}\, \mathrm{tr}\big(T_n(f)\, T_n(g)\big)^2 = \infty$$

with $f = f_0$ and $g = g_+$ or $g = g_-$. This completes the proof of Proposition 2.2.

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