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Limit theorems for Toeplitz quadratic functionals of continuous-time stationary processes

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Abstract Let $X(t)$, $t \in \mathbb{R}$, be a centered real-valued stationary Gaussian process with spectral density $f(\lambda)$. The paper considers a question concerning asymptotic distribution of Toeplitz type quadratic functional Q_T of the process $X(t)$, generated by an integrable even function $g(\lambda)$. Sufficient conditions in terms of $f(\lambda)$ and $g(\lambda)$ ensuring central limit theorems for standard normalized quadratic functionals Q_T are obtained, extending the results of Fox and Taqqu (Prob. Theory Relat. Fields 74: 213–240, 1987), Avram (Prob. Theory Relat. Fields 79:37–45, 1988), Giraitis and Surgailis (Prob. Theory Relat. Fields 86: 87–104, 1990), Ginovian and Sahakian (Theory Prob. Appl. 49:612–628, 2004) for discrete time processes.

Keywords Stationary Gaussian process · Spectral density · Toeplitz type quadratic functional · Central limit theorems

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1 Introduction

Let $X(t)$, $t \in \mathbb{R}$, be a centered real-valued stationary Gaussian process with spectral density $f(\lambda)$ and covariance function $r(t)$, i.e.

$$r(t) = \int_{-\infty}^{+\infty} e^{i\lambda t} f(\lambda)\, d\lambda. \tag{1.1}$$

We consider a question concerning asymptotic distribution (as $T \to \infty$) of the following Toeplitz type quadratic functional of the process $X(t)$:

$$Q_T = \int_0^T \int_0^T \widehat{g}(t-s) X(t) X(s)\, dt\, ds, \tag{1.2}$$

where

$$\widehat{g}(t) = \int_{-\infty}^{+\infty} e^{i\lambda t} g(\lambda)\, d\lambda, \quad t \in \mathbb{R}, \tag{1.3}$$

is the Fourier transform of some integrable even function $g(\lambda)$, $\lambda \in \mathbb{R}$. We will refer $g(\lambda)$ as a generating function for the functional Q_T.
The limit distribution of the functional (1.2) is completely determined by the spectral density $f(\lambda)$ and the generating function $g(\lambda)$, and depending on their properties it can be either Gaussian (that is, Q_T with an appropriate normalization obey central limit theorem), or non-Gaussian. We naturally arise the following two questions:

(a) Under what conditions on $f(\lambda)$ and $g(\lambda)$ will the limits be Gaussian?
(b) Describe the limit distributions, if they are non-Gaussian.

In this paper we essentially discuss the question (a). This question goes back to the classical monograph by Grenander and Szegö [12], where the problem was considered for discrete time processes, as an application of authors' theory of the asymptotic behavior of the trace of products of truncated Toeplitz matrices.

Later the problem (a) was studied by I. Ibragimov [14] and M. Rosenblatt [17], in connection with statistical estimation of the spectral ($F(\lambda)$) and covariance ($r(t)$) functions, respectively. Since 1986, there has been a renewed interest in both questions (a) and (b), related to the statistical inferences for long-range dependent processes (see, e.g., [1,5,9,10,18,21,22], and references therein). In particular, Avram [1], Fox and Taqqu [5], Giraitis and Surgailis [10], Ginovian and Sahakian [9] have obtained sufficient conditions for quadratic form Q_T to obey the central limit theorem (CLT).

For continuous time processes the above questions are less investigated. The question (a) was partially studied in [6,8,14].

In this paper we obtain sufficient conditions in terms of $f(\lambda)$ and $g(\lambda)$ ensuring central limit theorems for standard normalized quadratic functionals Q_T, extending the results of Avram [1], Fox and Taqqu [5], Giraitis and Surgailis [10], Ginovian and Sahakian [9] for discrete time processes.

We will use the following notation: By $\widetilde{Q}_T$ we denote the standard normalized quadratic functional:

$$\widetilde{Q}_T = \frac{1}{\sqrt{T}}\,(Q_T - EQ_T)\,. \tag{1.4}$$

The notation

$$\widetilde{Q}_T \Longrightarrow N(0,\sigma^2) \quad \text{as } T \to \infty \tag{1.5}$$

will mean that the distribution of the random variable $\widetilde{Q}_T$ tends (as $T \to \infty$) to the centered normal distribution with variance σ^2.

By $B_T(\psi)$ we denote the truncated Toeplitz operator generated by a function $\psi \in L^1(\mathbb{R})$ defined as follows (see [12,14]):

$$[B_T(\psi)u](\lambda) = \int_0^T \hat{\psi}(\lambda - \mu)u(\mu)\mathrm{d}\mu, \quad u(\lambda) \in L^2[0,T], \tag{1.6}$$

where $\hat{\psi}(\cdot)$ is the Fourier transform of $\psi(\cdot)$.

Our study of the asymptotic distribution of the quadratic functional (1.2) is based on the well-known representation of the kth order cumulant $\chi_k(\cdot)$ of $\widetilde{Q}_T$ (see, e.g., [12,14]):

$$\chi_k(\widetilde{Q}_T) = \begin{cases} 0 & \text{for } k = 1, \\ T^{-k/2}2^{k-1}(k-1)!\,\mathrm{tr}\,[B_T(f)B_T(g)]^k & \text{for } k \geq 2, \end{cases} \tag{1.7}$$

where $B_T(f)$ and $B_T(g)$ are truncated Toeplitz operators in $L_2[0,T]$ generated by the functions f and g respectively, and $\mathrm{tr}[A]$ stands for the trace of an operator A.

By C, M, C_k, M_k we will denote constants that can vary from line to line.
The remainder of the paper is organized as follows. In Sect. 2 we state the main results of the paper—Theorems 1–5. In Sect. 3 we prove some auxiliary results. Sect. 4 is devoted to the proofs of results stated in Sect. 2.

2 Results

Below we assume that $f, g \in L^1(\mathbb{R})$, and with no loss of generality, that $g \geq 0$. Also, we set

$$\sigma_0^2 = 16\pi^3 \int_{-\infty}^{\infty} f^2(\lambda)g^2(\lambda)\,\mathrm{d}\lambda. \tag{2.1}$$

The main results of the paper are the following theorems.

Theorem 1 *Assume that* $f \cdot g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *and for* $T \to \infty$

$$\chi_2(\widetilde{Q}_T) = \frac{2}{T}\mathrm{tr}\big[B_T(f)B_T(g)\big]^2 \longrightarrow 16\pi^3 \int_{-\infty}^{+\infty} f^2(\lambda)g^2(\lambda)\,\mathrm{d}\lambda. \tag{2.2}$$

Then $\widetilde{Q}_T \Longleftarrow\!\!\!\Longrightarrow N(0,\sigma_0^2)$ *as* $T \to \infty$.

Theorem 2 *If the function*

$$\varphi(x_1,x_2,x_3) = \int_{-\infty}^{+\infty} f(u)g(u-x_1)f(u-x_2)g(u-x_3)\,\mathrm{d}u \tag{2.3}$$

belongs to $L^2(\mathbb{R}^3)$ *and is continuous at* $(0,0,0)$, *then* $\widetilde{Q}_T \Longleftarrow\!\!\!\Longrightarrow N(0,\sigma_0^2)$ *as* $T \to \infty$.

Theorem 3 *Assume that* $f(\lambda) \in L^p(\mathbb{R})$ $(p \geq 1)$ *and* $g(\lambda) \in L^q(\mathbb{R})$ $(q \geq 1)$ *with* $1/p + 1/q \leq 1/2$. *Then* $\widetilde{Q}_T \Longleftarrow\!\!\!\Longrightarrow N(0,\sigma_0^2)$ *as* $T \to \infty$.

Theorem 4 *Let* $f \in L^2(\mathbb{R})$, $g \in L^2(\mathbb{R})$, $fg \in L^2(\mathbb{R})$ *and*

$$\int_{-\infty}^{+\infty} f^2(\lambda)g^2(\lambda-\mu)\,\mathrm{d}\lambda \longrightarrow \int_{-\infty}^{+\infty} f^2(\lambda)g^2(\lambda)\,\mathrm{d}\lambda \quad \text{as } \mu \to 0. \tag{2.4}$$

Then $\widetilde{Q}_T \Longleftarrow\!\!\!\Longrightarrow N(0,\sigma_0^2)$ *as* $T \to \infty$.

Let $SV(\mathbb{R})$ be the class of slowly varying at zero functions $u(\lambda)$, $\lambda \in \mathbb{R}$ satisfying $u(\lambda) \in L^\infty(\mathbb{R})$, and

$$\lim_{\lambda\to 0} u(\lambda) = 0, \quad u(\lambda) = u(-\lambda),\ 0 < u(\lambda) < u(\mu) \text{ for } 0 < \lambda < \mu.$$

Theorem 5 *Assume that functions* f *and* g *are integrable on* $\mathbb{R}$ *and bounded on* $\mathbb{R} \setminus (-\pi,\pi)$ *and let*

$$f(\lambda) \leq |\lambda|^{-\alpha}L_1(\lambda),\ \ |g(\lambda)| \leq |\lambda|^{-\beta}L_2(\lambda), \quad \lambda \in [-\pi,\pi], \tag{2.5}$$

where $L_1(\lambda)$ *and* $L_2(\lambda)$ *are slowly varying at zero functions and*

$$\begin{aligned} &\alpha < 1,\ \beta < 1,\ \alpha+\beta \leq 1/2;\\ &L_i \in SV(\mathbb{R}),\ \ \lambda^{-(\alpha+\beta)}L_i(\lambda) \in L^2(\mathbb{T}), \quad i = 1,2. \end{aligned} \tag{2.6}$$

Then $\widetilde{Q}_T \Longleftarrow\!\!\!\Longrightarrow N(0,\sigma_0^2)$ *as* $T \to \infty$.

Remark 1 For $p = 2, q = \infty$ Theorem 3 was first established by Ibragimov [14], while the general case was proved by Ginovian in [8]. Here we give a different proof of this theorem by showing that it follows from Theorem 2.

Remark 2 The analogs of Theorems 1 and 4 for discrete time processes were proved by Giraitis and Surgailis in [10], the discrete analogs of Theorems 2 and 5 were established by Ginovian and Sahakian in [9] while the discrete analog of Theorem 3 was proved by Avram in [1].

3 Preliminaries

Lemma 1 *Let $B_T(f)$ and $B_T(g)$ be the truncated Toeplitz operators generated by functions $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ respectively. Then*

$$tr\left[B_T(f)B_T(g)\right]^2 = \int\limits_{\mathbb{R}^4} G_T(x_1-x_3)G_T(x_3-x_2)G_T(x_2-x_4)G_T(x_4-x_1) \times f(x_1)f(x_2)g(x_3)g(x_4)\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4, \tag{3.1}$$

where

$$G_T(u) = \int\limits_0^T \mathrm{e}^{\mathrm{i}tu}\,\mathrm{d}t = \frac{\mathrm{e}^{\mathrm{i}Tu}-1}{\mathrm{i}u} = \mathrm{e}^{\mathrm{i}Tu/2}\cdot D_T(u) \tag{3.2}$$

and D_T is the Dirichlet kernel defined by

$$D_T(u) = \frac{\sin(Tu/2)}{u/2}. \tag{3.3}$$

Proof Using (1.6) it is easy to check that $\left[B_T(f)B_T(g)\right]^2$ is an integral operator with kernel

$$K(t,s) = \int\limits_0^T\int\limits_0^T\int\limits_0^T r(t-u_1)\widehat{g}(u_1-u_2)r(u_2-u_3)\widehat{g}(u_3-s)\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3,$$

where $r(t)$ and $\widehat{g}(t)$ are as in (1.1) and (1.3). By the formula for traces of integral operators (see [11, Sect. 3.10]), we have

$$\begin{aligned}\mathrm{tr}\left[B_T(f)B_T(g)\right]^2 &= \int\limits_0^T K(t,t)\,\mathrm{d}t \\ &= \int\limits_0^T\int\limits_0^T\int\limits_0^T\int\limits_0^T r(t-u_1)\widehat{g}(u_1-u_2)r(u_2-u_3) \\ &\quad\times\widehat{g}(u_3-t)\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3\,\mathrm{d}t.\end{aligned} \tag{3.4}$$

Taking into account (1.1), (1.3) and (3.2) from (3.4) we obtain (3.1). Lemma 1 is proved.

Let $\mathbf{u} = (u_1, u_2, u_3)$. Consider the kernel

$$\Phi_T(\mathbf{u}) = \frac{1}{8\pi^3 T} \cdot D_T(u_1) D_T(u_2) D_T(u_3) D_T(u_1 + u_2 + u_3). \tag{3.5}$$

Lemma 2 *The kernel $\Phi_T(\mathbf{u})$ possesses the following properties*:

$$\text{(a)} \quad \sup_T \int_{\mathbb{R}^3} |\Phi_T(\mathbf{u})| \, d\mathbf{u} = C_1 < \infty;$$

$$\text{(b)} \quad \int_{\mathbb{R}^3} \Phi_T(\mathbf{u}) \, d\mathbf{u} = 1;$$

$$\text{(c)} \quad \lim_{T\to\infty} \int_{\mathbb{E}_\delta^c} |\Phi_T(\mathbf{u})| \, d\mathbf{u} = 0 \quad \textit{for any} \quad \delta > 0;$$

(d) *for any $\delta > 0$ there exists a constant $M_\delta > 0$ such that*

$$\int_{\mathbb{E}_\delta^c} \Phi_T^2(\mathbf{u}) d\mathbf{u} \le M_\delta \quad \textit{for } T > 0. \tag{3.6}$$

where $\mathbb{E}_\delta^c = \mathbb{R}^3 \setminus \mathbb{E}_\delta$ and $\mathbb{E}_\delta = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : |u_i| \le \delta, \, i = 1, 2, 3\}$.

Proof The proof of properties (a)–(c) can be found in [2] (Lemma 3.2). To prove (d) first observe that

$$\int_{\mathbb{R}} D_T^2(u) du \le C\, T \;\text{ and }\; |D_T(u)| \le C_\delta \quad \text{for } |u| > \delta, \; T > 0. \tag{3.7}$$

For $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ we have

$$\begin{aligned} \int_{\mathbb{E}_\delta^c} \Phi_T^2(\mathbf{u}) d\mathbf{u} \le \int_{|u_1|>\delta} \Phi_T^2(\mathbf{u}) d\mathbf{u} + \int_{|u_2|>\delta} \Phi_T^2(\mathbf{u}) d\mathbf{u} \\ + \int_{|u_3|>\delta} \Phi_T^2(\mathbf{u}) d\mathbf{u} =: I_1 + I_2 + I_3. \end{aligned} \tag{3.8}$$

It is enough to estimate I_1:

$$I_1 \le \int\limits_{|u_1|>\delta,\ |u_2|>\delta/3} \Phi_T^2(\mathbf{u})\mathrm{d}\mathbf{u} + \int\limits_{|u_1|>\delta,\ |u_3|>\delta/3} \Phi_T^2(\mathbf{u})\mathrm{d}\mathbf{u} + \int\limits_{|u_1|>\delta,\ |u_2|\le\delta/3,\ |u_3|\le\delta/3} \Phi_T^2(\mathbf{u})\mathrm{d}\mathbf{u} =: I_1^{(1)} + I_1^{(2)} + I_1^{(3)}. \tag{3.9}$$

According to (3.7)

$$I_1^{(1)} \le C_\delta \cdot \frac{1}{T^2} \int\limits_{|u_2|>\delta/3} D_T^2(u_2)D_T^2(u_3)D_T^2(u_1+u_2+u_3)\mathrm{d}u_1\mathrm{d}u_3\mathrm{d}u_2 \le C_\delta \int\limits_{|u_2|>\delta/3} \frac{1}{u_2^2}\,\mathrm{d}u_2 \le M_\delta. \tag{3.10}$$

Likewise,

$$I_1^{(2)} \le M_\delta. \tag{3.11}$$

Note that in the integral $I_1^{(3)}$, we have $|u_1+u_2+u_3| > \delta/3$, hence by (3.7)

$$I_1^{(3)} \le C_\delta \cdot \frac{1}{T^2} \int\limits_{|u_1|>\delta} D_T^2(u_1)D_T^2(u_2)D_T^2(u_3)\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_1 \le C_\delta \int\limits_{|u_1|>\delta} \frac{1}{u_1^2}\,\mathrm{d}u_1 \le M_\delta. \tag{3.12}$$

From (3.8)–(3.12) we obtain (3.6). Lemma 2 is proved.

Lemma 3 *If the function* $\Psi(\mathbf{u}) \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ *is continuous at* $(0,0,0)$. *Then*

$$\lim_{T\to\infty} \int\limits_{\mathbb{R}^3} \Psi(\mathbf{u})\Phi_T(\mathbf{u})\mathrm{d}\mathbf{u} = \Psi(0,0,0), \tag{3.13}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ *and* $\Phi_T(\mathbf{u})$ *is defined by* (3.5).

Proof By Lemma 2 (b) we have

$$R_T := \int\limits_{\mathbb{R}^3} \Psi(\mathbf{u})\Phi_T(\mathbf{u})\mathrm{d}\mathbf{u} - \Psi(\mathbf{0}) = \int\limits_{\mathbb{R}^3} [\Psi(\mathbf{u}) - \Psi(\mathbf{0})]\Phi_T(\mathbf{u})\mathrm{d}\mathbf{u}. \tag{3.14}$$

For any $\varepsilon > 0$ we can find $\delta > 0$ to satisfy

$$|\Psi(\mathbf{u}) - \Psi(\mathbf{0})| < \frac{\varepsilon}{C_1}, \tag{3.15}$$

where C_1 is the constant from Lemma 2 (a). Consider the decomposition $\Psi = \Psi_1 + \Psi_2$ such that

$$\|\Psi_1\|_2 \leq \frac{\varepsilon}{\sqrt{M_\delta}} \quad \text{and} \quad \|\Psi_2\|_\infty < \infty, \tag{3.16}$$

where M_δ is defined in Lemma 2 (d). Applying Lemma 2 and (3.14)–(3.16) for sufficiently large T we obtain

$$\begin{aligned} |R_T| &\leq \int_{\mathbb{E}_\delta} |\Psi(\mathbf{u}) - \Psi(\mathbf{0})||\Phi_T(\mathbf{u})|\mathrm{d}\mathbf{u} + \int_{\mathbb{E}_\delta^c} |\Psi_1(\mathbf{u})||\Phi_T(\mathbf{u})|\mathrm{d}\mathbf{u} \\ &+ \int_{\mathbb{E}_\delta^c} |\Psi_2(\mathbf{u}) - \Psi(\mathbf{0})||\Phi_T(\mathbf{u})|\mathrm{d}\mathbf{u} \leq \frac{\varepsilon}{C_1} \int_{\mathbb{E}_\delta} |\Phi_T(\mathbf{u})|\mathrm{d}\mathbf{u} \\ &+ \|\Psi_1\|_2 \left[\int_{\mathbb{E}_\delta^c} \Phi_T^2(\mathbf{u})\mathrm{d}\mathbf{u} \right]^{1/2} + C_2 \int_{\mathbb{E}_\delta^c} |\Phi_T(\mathbf{u})|\mathrm{d}\mathbf{u} \leq 3\,\varepsilon. \end{aligned}$$

This combined with (3.14) yields (3.13). Lemma 3 is proved.

Denote

$$\mu_T(A) = \frac{1}{T} \int_A G_T(\lambda_1 - \lambda_3) G_T(\lambda_3 - \lambda_2) G_T(\lambda_4 - \lambda_1) G_T(\lambda_2 - \lambda_4) \mathrm{d}\lambda_1 \mathrm{d}\lambda_2 \mathrm{d}\lambda_3 \mathrm{d}\lambda_4 \tag{3.17}$$

and let $C_{loc}(\mathbb{R}^n)$ be the space of continuous functions on $\mathbb{R}^n$ with bounded support.

Lemma 4 *If* $f \in C_{loc}(\mathbb{R}^4)$, *then*

$$\lim_{T\to\infty} \int_{\mathbb{R}^4} f \mathrm{d}\mu_T = 8\pi^3 \int_{\mathbb{R}} f(u,u,u,u)\mathrm{d}u \tag{3.18}$$

Proof Making a change of variables

$$\lambda_1 = u, \quad \lambda_3 - \lambda_2 = u_1, \quad \lambda_4 - \lambda_1 = u_2, \quad \lambda_2 - \lambda_4 = u_3,$$

from (3.2)–(3.5), (3.17) and the equality

$$\begin{aligned} &G_T(u_1)G_T(u_2)G_T(u_3)G_T(-u_1-u_2-u_3) \\ &\quad = D_T(u_1)D_T(u_2)D_T(u_3)D_T(u_1+u_2+u_3), \end{aligned}$$

we get

$$\begin{aligned} \int_{\mathbb{R}^4} f\mathrm{d}\mu_T &= \frac{1}{T}\int_{\mathbb{R}^3}\int_{\mathbb{R}} f(u, u+u_2+u_3, u+u_1+u_2+u_3, u+u_2)\mathrm{d}u \\ &\quad \times G_T(u_1)G_T(u_2)G_T(u_3)G_T(-u_1-u_2-u_3)\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3 \\ &=: 8\pi^3\int_{\mathbb{R}^3}\Psi(\mathbf{u})\Phi_T(\mathbf{u})\mathrm{d}\mathbf{u}, \end{aligned} \tag{3.19}$$

where $\mathbf{u}=(u_1,u_2,u_3)$ and

$$\Psi(\mathbf{u}) = \int_{\mathbb{R}} f(u, u+u_2+u_3, u+u_1+u_2+u_3, u+u_2)\mathrm{d}u.$$

It is not difficult to check that the function Ψ satisfies conditions of Lemma 3 and

$$\lim_{\mathbf{u}\to(0,0,0)}\Psi(\mathbf{u}) = \int_{\mathbb{R}} f(u,u,u,u)\mathrm{d}u. \tag{3.20}$$

Hence applying Lemma 3 from (3.19) and (3.20) we get (3.18). Lemma 4 is proved.

Lemma 5 *If $f\in L^1(\mathbb{R})$, then*

$$(1)\quad \int_{\mathbb{R}^4}|f(x_i)|\mathrm{d}|\mu_T| \le C_1\|f\|_{L^1(\mathbb{R})},\quad i=1,2,3,4, \tag{3.21}$$

$$(2)\quad \int_{\mathbb{R}^4}|f(x_i)f(x_j)|\mathrm{d}|\mu_T| \le C_2\|f\|^2_{L^2(\mathbb{R})},\quad i,j=1,2,3,4,\ i\neq j. \tag{3.22}$$

where C_1 and C_2 are absolute constants.

Proof For $T>0$ we have

$$|G_T(x)| \le 2T\psi_T(x),\quad \text{where } \psi_T(x) = \frac{1}{1+T|x|},\ \ x\in\mathbb{R}. \tag{3.23}$$

We show that for $u, v \in \mathbb{R}$,

$$T \int_{\mathbb{R}} \psi_T(x+u)\psi_T(x+v)\mathrm{d}x \le C_\delta \psi_T^{1-\delta}(u-v), \quad \delta > 0, \tag{3.24}$$

Indeed, (3.24) is equivalent to the following

$$\int_{\mathbb{R}} \frac{1}{(1+|x|)(1+|x+t|)}\,\mathrm{d}x \le C_\delta \frac{1}{(1+|t|)^{1-\delta}}, \qquad t \in \mathbb{R}. \tag{3.25}$$

Denoting by $J(t)$ the left-hand side of (3.25), we can write

$$\begin{aligned} J(t) &= \int_{|x|<2|t|} \frac{1}{(1+|x|)(1+|x+t|)}\,\mathrm{d}x + \int_{|x|\ge 2|t|} \frac{1}{(1+|x|)(1+|x+t|)}\,\mathrm{d}x \\ &\le \frac{C}{1+|t|} \int_{|x|<2|t|} \left[\frac{1}{1+|x|} + \frac{1}{1+|x+t|}\right]\mathrm{d}x + C \int_{|x|\ge 2|t|} \frac{1}{(1+|x|)^2}\,\mathrm{d}x \\ &\le C\frac{\ln(1+|t|)}{1+|t|} + C\frac{1}{1+|t|} \le C_\delta \frac{1}{(1+|t|)^{1-\delta}}, \quad \delta > 0, \end{aligned}$$

yielding (3.25). To prove (3.21) for $i = 1$ (say), we apply the inequality (3.24) with $\delta = 1/4$ to obtain

$$\begin{aligned} \int_{\mathbb{R}^4} |f(x_1)|\mathrm{d}|\mu_T| &\le CT^3 \int_{\mathbb{R}^4} |f(x_1)|\psi_T(x_1-x_3)\psi_T(x_3-x_2) \\ &\quad \times \psi_T(x_4-x_1)\psi_T(x_2-x_4)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 \\ &\le CT \int_{\mathbb{R}} |f(x_1)| \int_{\mathbb{R}} \psi_T^{3/2}(x_1-x_2)\mathrm{d}x_2\mathrm{d}x_1 \le C_1\|f\|_{L^1(\mathbb{R})}. \end{aligned}$$

This proves (3.21). To prove (3.22) for $i = 1$, $j = 2$ (say), we use (3.24) and Cauchy inequality to obtain

$$\begin{aligned} &\int_{\mathbb{R}^4} |f(x_1)f(x_2)|\mathrm{d}|\mu_T| \\ &\le CT^3 \int_{\mathbb{R}^4} |f(x_1)f(x_2)|\psi_T(x_1-x_3)\psi_T(x_3-x_2)\psi_T(x_4-x_1) \\ &\quad \times \psi_T(x_2-x_4)dx_1dx_2dx_3dx_4 \\ &\le CT \int_{\mathbb{R}^2} |f(x_1)f(x_2)|\psi_T^{3/2}(x_1-x_2)\mathrm{d}x_1\mathrm{d}x_2 \end{aligned}$$

$$\leq C\left\{T\int\limits_{\mathbb{R}^2} f^2(x_1)\psi_T^{3/2}(x_1-x_2)\mathrm{d}x_1\mathrm{d}x_2\right\}^{1/2}$$
$$+\left\{T\int\limits_{\mathbb{R}^2} f^2(x_2)\psi_T^{3/2}(x_1-x_2)\mathrm{d}x_1\mathrm{d}x_2\right\}^{1/2} \leq C_2\int\limits_{\mathbb{R}} f^2(x)\mathrm{d}x.$$

Lemma 5 is proved.

Lemma 6 *Let* $f(u_1,u_2,u_3,u_4) = \prod_{i=1}^{4} f_i(u_i)$, *where* $f_i \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $i = 1,2,3,4$. *Then* (3.18) *holds.*

Proof Suppose $\|f_i\|_\infty \leq M < \infty$, $i = 1,2,3,4$. Using Lusin's theorem for any $\varepsilon > 0$ we can find functions $h_i,\ g_i$, $i = 1,2,3,4$ satisfying

$$f_i = g_i + h_i, \quad g_i \in C_{loc}(\mathbb{R}), \quad \|h_i\|_{L^1(\mathbb{R})} \leq \varepsilon, \quad \|g_i\|_C \leq M. \tag{3.26}$$

Therefore

$$\int\limits_{\mathbb{R}^4} f\mathrm{d}\mu_T = \int\limits_{\mathbb{R}^4}\prod_{i=1}^{4}(g_i+h_i)\mathrm{d}\mu_T = \int\limits_{\mathbb{R}^4}\prod_{i=1}^{4} g_i\mathrm{d}\mu_T + I_T, \tag{3.27}$$

where by (3.26) and Lemma 5

$$|I_T| \leq \sum_{j=1}^{4}\int\limits_{\mathbb{R}^4}|h_j|\prod_{i=1,i\neq j}^{4}(|g_i|+|h_i|)\mathrm{d}|\mu_T|$$
$$\leq C_M\sum_{j=1}^{4}\int\limits_{\mathbb{R}^4}|h_j|\mathrm{d}|\mu_T| \leq C_M\sum_{j=1}^{4}\|h_j\|_{L^1(\mathbb{R})} \leq C_M\varepsilon. \tag{3.28}$$

By Lemma 4

$$\lim_{T\to\infty}\int\limits_{\mathbb{R}^4}\prod_{i=1}^{4} g_i(u_i)\mathrm{d}\mu_T = \int\limits_{\mathbb{R}}\prod_{i=1}^{4} g_i(u)\mathrm{d}u$$
$$= \int\limits_{\mathbb{R}}\prod_{i=1}^{4}\left[f_i(u)-h_i(u)\right]\mathrm{d}u = \int\limits_{\mathbb{R}} f(u,u,u,u)\mathrm{d}u + J, \tag{3.29}$$

where

$$|J| \le \sum_{j=1}^{4} \int_{\mathbb{R}} |h_j(u)| \prod_{i=1, i\neq j}^{4} (|f_i(u)| + |g_i(u)|)\mathrm{d}u \le C_M \varepsilon. \tag{3.30}$$

From (3.27) – (3.30) we get (3.18). Lemma 6 is proved.

Lemma 7 *Let* $\psi(u) \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$, *with* $1 < p < \infty$. *Then*

$$\mu_T := ||B_T(\psi)||_\infty = o\big(T^{1/p}\big) \quad \textit{as} \quad T \to \infty. \tag{3.31}$$

Proof Let N_T be a positive function of T, which we will specify later. We set

$$M_T = \{\lambda \in \mathbb{R};\ |\psi(\lambda)| > N_T\}. \tag{3.32}$$

We have

$$\begin{aligned} \mu_T = ||B_T(\psi)||_\infty &= \sup_{u \in L^2(\mathbb{R}),\ ||u||_2 = 1} |(B_T(\psi)u, u)| \\ &= \sup_{u \in L^2(\mathbb{R}),\ ||u||_2 = 1} \left| \int_0^T \int_0^T \widehat{\psi}(t-s) u(t) u(s)\, \mathrm{d}t\, \mathrm{d}s \right| \\ &= \sup_{u \in L^2(\mathbb{R}),\ ||u||_2 = 1} \left| \int_0^T \int_0^T \left[\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\lambda(t-s)} \psi(\lambda)\, d\lambda \right] u(t) u(s)\, \mathrm{d}t\, \mathrm{d}s \right|. \end{aligned} \tag{3.33}$$

A square integrable function $u(t)$ is also integrable on $[0, T]$. Hence, switching the order of integration in (3.33), we get

$$\begin{aligned} \mu_T &= \sup_{u \in L^2(\mathbb{R}),\ ||u||_2 = 1} \left| \int_{-\infty}^{+\infty} \psi(\lambda) \left[\int_0^T u(t) \mathrm{e}^{\mathrm{i}t\lambda}\, \mathrm{d}t \int_0^T u(s) e^{-\mathrm{i}s\lambda}\, \mathrm{d}s \right] \mathrm{d}\lambda \right| \\ &\le \sup_{u \in L^2(\mathbb{R}),\ ||u||_2 = 1} \int_{-\infty}^{+\infty} |\psi(\lambda)| \left| \int_0^T u(t)\, \mathrm{e}^{\mathrm{i}\lambda t} \mathrm{d}t \right|^2 \mathrm{d}\lambda. \end{aligned} \tag{3.34}$$

Since for $u(t) \in L^2[0, T]$ with $||u||_2 = 1$ we have $\left| \int_0^T u(t)\, e^{\mathrm{i}\lambda t} \mathrm{d}t \right|^2 \le T$, by Plancherel's theorem from (3.34) we obtain

$$\mu_T \le 2\pi N_T + T \int_{M_T} |\psi(\lambda)|\, \mathrm{d}\lambda, \tag{3.35}$$

where M_T is as in (3.32). We show that for every $p (1 < p < \infty)$

$$\int_{M_T} |\psi(\lambda)|\, \mathrm{d}\lambda \le \gamma_T^p N_T^{(1-p)}, \tag{3.36}$$

where

$$\gamma_T = \left(\int_{M_T} |\psi(\lambda)|^p\, \mathrm{d}\lambda \right)^{1/p}. \tag{3.37}$$

Indeed, by Hölder inequality

$$\int_{M_T} |\psi(\lambda)|\, \mathrm{d}\lambda \le \gamma_T \left[m(M_T)\right]^{(p-1)/p}, \tag{3.38}$$

where $m(M_T)$ is the Lebesgue measure of the set M_T. Next, by Chebyshev inequality

$$m(M_T) \le \gamma_T^p N_T^{-p}. \tag{3.39}$$

A combination of (3.38) and (3.39) yields (3.36). Now from (3.35) and (3.36) we have

$$\mu_T \le 2\pi N_T + T\gamma_T^p N_T^{(1-p)}. \tag{3.40}$$

If $\psi \in L^\infty(\mathbb{R})$, then putting $N_T = \|\psi\|_\infty$ for all $T > 0$, we will have $\gamma_T = 0$ and (3.40) implies $\mu_T = O(1)$.

Now suppose $\psi \notin L^\infty(\mathbb{R})$ and for fixed $T > 0$ consider the function

$$F(N) = N - T^{1/p} \left(\int_{\{\lambda : |\psi(\lambda)| > N\}} |\psi(\lambda)|^p\, \mathrm{d}\lambda \right)^{1/p}, \quad N \in [0, \infty).$$

Since $F(0) < 0$ and $\lim_{N\to\infty} F(N) = +\infty$ there exists a positive number $N = N_T$ with $F(N_T) = 0$, i.e.

$$N_T = T^{1/p} \left(\int_{\{\lambda : |\psi(\lambda)| > N_T\}} |\psi(\lambda)|^p \right) = T^{1/p}\gamma_T. \tag{3.41}$$

It is easy to see that for $\psi \notin L^\infty(\mathbb{R})$ the equality (3.41) implies $\lim_{T\to\infty} N_T = \infty$. Hence $\gamma_T = o(1)$ and from (3.40) and (3.41) we obtain $\mu_T < 8T^{1/p}\gamma_T = o(T^{1/p})$ as $T \to \infty$. Lemma 7 is proved.

Lemma 8 *Let $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then*

$$\frac{1}{T}||B_T(\psi)||_2^2 \longrightarrow 2\pi||\psi||_2^2 \quad as\ T \to \infty. \tag{3.42}$$

Proof Using the formula for Hilbert–Schmidt norm of integral operators (see [11]), by (1.6) we have

$$\begin{aligned}\frac{1}{T}||B_T(\psi)||_2^2 &= \frac{1}{T}\int_0^T\int_0^T |\widehat{\psi}(t-s)|^2\,\mathrm{d}t\,\mathrm{d}s \\ &= \frac{1}{T}\int_{-T}^{T}\big(T-|t|\big)|\widehat{\psi}(t)|^2\,\mathrm{d}t = \int_{-T}^{T}\left(1-\frac{|t|}{T}\right)|\widehat{\psi}(t)|^2\,\mathrm{d}t.\end{aligned} \tag{3.43}$$

Passing to the limit as $T \to \infty$ and using Plancherel's equality, from (3.43) we obtain (3.42). Lemma 8 is proved.

The next lemma is known (see [4], Proposition 1(i)), we deduce it from Lemmas 7 and 8.

Lemma 9 *Let $Y(t), t \in \mathbb{R}$, be a real-valued, centered, separable stationary Gaussian process with the spectral density $f_Y(\lambda) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then the distribution of the normalized quadratic form*

$$L_T := \frac{1}{\sqrt{T}}\left(\int_0^T Y^2(t)\,\mathrm{d}t - \mathbb{E}\int_0^T Y^2(t)\,\mathrm{d}t\right) \tag{3.44}$$

tends (as $T \to \infty$) to the normal distribution $N(0, \sigma_Y^2)$ with variance

$$\sigma_Y^2 = 4\pi\int_{-\infty}^{+\infty} f_Y^2(\lambda)\,\mathrm{d}\lambda. \tag{3.45}$$

Proof Let $R(t)$ be the covariance function of $Y(t)$. For $T > 0$ denote by $\mu_j = \mu_j(T), j \geq 1$ the eigenvalues of the operator $B_T(f_Y)$ (see (1.6)), and let $e_j(t) = e_j(t, T) \in L_2[0.T], j \geq 1$, be the corresponding orthonormal eigenfunctions, i.e.

$$\int_0^T R(t-s)e_j(s)\,\mathrm{d}s = \mu_j e_j(t), \quad j \geq 1. \tag{3.46}$$

Since by Mercer's theorem (see, e.g., [11], Sect. 3.10)

$$R(t-s)=\sum_{j=1}^{\infty}\mu_j e_j(t)e_j(s) \tag{3.47}$$

with positive and summable eigenvalues $\{\mu_j\}$, we have the Karhunen–Loéve expansion (see, e.g., [16], Sect. 34.5)

$$Y(t)=\sum_{j=1}^{\infty}\sqrt{\mu_j}\xi_j e_j(t), \tag{3.48}$$

where $\{\xi_j\}$ are independent $N(0,1)$ random variables. Therefore by (3.44) and (3.48)

$$L_T=\frac{1}{\sqrt{T}}\sum_{j=1}^{\infty}\mu_j(\xi_j^2-1). \tag{3.49}$$

Denote by $\chi_k(L_T)$ the kth order cumulant of L_T. By (3.49) (cf. (1.7)) we have

$$\chi_k(L_T)=\begin{cases}0 & \text{for } k=1,\\ (k-1)!\,2^{k-1}T^{-k/2}\mathrm{tr}[B_T(f_Y)]^k & \text{for } k\geq 2.\end{cases} \tag{3.50}$$

By (3.50) and Lemma 8

$$\chi_2(L_T)=\frac{2}{T}||B_T(f_Y)||_2^2\longrightarrow 4\pi\int_{-\infty}^{+\infty}f_Y^2(\lambda)\,\mathrm{d}\lambda \quad \text{as } T\to\infty. \tag{3.51}$$

Next, by (3.50) for $k\geq 3$

$$|\chi_k(L_T)|\leq C\,\frac{1}{T}||B_T(f_Y)||_2^2\,T^{1-k/2}\mu_T^{k-2}, \tag{3.52}$$

where $\mu_T=||B_T(f_Y)||_\infty$. By Lemmas 7 and 8 the right-hand side of (3.52) tends to zero as $T\to\infty$. Lemma 9 is proved.

4 Proofs

Proof of Theorem 1 By Theorem 16.7.2 from [15] the underlying process $X(t)$ admits the moving average representation

$$X(t) = \int_{-\infty}^{+\infty} \widehat{a}(t-s)\, d\xi(s), \tag{4.1}$$

where

$$\int_{-\infty}^{+\infty} |\widehat{a}(t)|^2\, dt < \infty, \tag{4.2}$$

while $\xi(s)$ is a process with orthogonal increments such that $\mathbb{E}[d\xi(s)] = 0$ and $\mathbb{E}|d\xi(s)|^2 = ds$. Moreover the spectral density $f(\lambda)$ can be represented as

$$f(\lambda) = 2\pi\, |a(\lambda)|^2, \tag{4.3}$$

where $a(\lambda)$ is the inverse Fourier transform of $\widehat{a}(t)$. We set

$$a_1(\lambda) = (2\pi)^{1/2}\, a(\lambda) \cdot [g(\lambda)]^{1/2}, \tag{4.4}$$

where $g(\lambda)$ is the generating function, and consider a process $Y(t)$ ($t \in \mathbb{R}$) defined by

$$Y(t) = \int_{-\infty}^{+\infty} \widehat{a}_1(t-s)\, d\xi(s), \tag{4.5}$$

where $\widehat{a}_1(t)$ is the Fourier transform of $a_1(\lambda)$ and $\xi(s)$ is as in (4.1). Since $fg \in L^1(\mathbb{R})$ by Parseval's identity we have

$$\int_{-\infty}^{+\infty} |\widehat{a}_1(t)|^2\, dt = 2\pi \int_{-\infty}^{+\infty} |a_1(\lambda)|^2\, d\lambda = 4\pi^2 \int_{-\infty}^{+\infty} f(\lambda)\, g(\lambda)\, d\lambda < \infty. \tag{4.6}$$

So, $Y(t)$ is well-defined stationary process with spectral density

$$\varphi(\lambda) := |a_1(\lambda)|^2 = 2\pi\, f(\lambda)\, g(\lambda). \tag{4.7}$$

Since by assumption $f(\lambda)g(\lambda) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, the process $Y(t)$ defined by (4.5) satisfies the conditions of Lemma 9. Hence Lemma 9 and Lemma 10 that follows imply Theorem 1.

Lemma 10 *Under the assumptions of Theorem* 1

$$\operatorname{Var}(Q_T - L_T) = o(T) \quad \text{as } T \to \infty, \tag{4.8}$$

where Q_T *and* L_T *are as in* (1.2) *and* (3.44), *respectively.*

Proof By (1.2) and (4.1) we have

$$Q_T = \int_{\mathbb{R}^2} \left[\int_0^T \int_0^T \widehat{g}(t-s)\widehat{a}(t-u_1)\widehat{a}(s-u_2)\,\mathrm{d}t\,\mathrm{d}s \right] \mathrm{d}\xi(u_1)\,\mathrm{d}\xi(u_2). \tag{4.9}$$

Similarly, by (3.44) and (4.5)

$$L_T = \int_{\mathbb{R}^2} \left[\int_0^T \widehat{a}_1(t-u_1)\widehat{a}_1(t-u_2)\,\mathrm{d}t \right] \mathrm{d}\xi(u_1)\,\mathrm{d}\xi(u_2). \tag{4.10}$$

Setting

$$d_{1T}(u_1,u_2) = \int_0^T \int_0^T \widehat{g}(t-s)\widehat{a}(t-u_1)\widehat{a}(s-u_2)\,\mathrm{d}t\,\mathrm{d}s \tag{4.11}$$

and

$$\begin{aligned} d_{2T}(u_1,u_2) &= \int_0^T \int_0^T \delta(t-s)\widehat{a}_1(t-u_1)\widehat{a}_1(s-u_2)\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_0^T \widehat{a}_1(t-u_1)\widehat{a}_1(t-u_2)\,\mathrm{d}t, \end{aligned} \tag{4.12}$$

from (4.9)–(4.12) we get

$$Q_T - L_T = \int_{\mathbb{R}^2} \left[d_{1T}(u_1,u_2) - d_{2T}(u_1,u_2) \right] \mathrm{d}\xi(u_1)\,\mathrm{d}\xi(u_2). \tag{4.13}$$

Since the underlying process $X(t)$ is Gaussian, we obtain

$$\operatorname{Var}(Q_T - L_T) = 2 \int_{\mathbb{R}^2} \left[d_{1T}(u_1,u_2) - d_{2T}(u_1,u_2) \right]^2 \mathrm{d}u_1\,\mathrm{d}u_2. \tag{4.14}$$

We set

$$p_1(\lambda_1,\lambda_2,\mu) = a(\lambda_1)a(\lambda_2)g(\mu), \tag{4.15}$$
$$p_2(\lambda_1,\lambda_2,\mu) = a_1(\lambda_1)a_1(\lambda_2)\delta(\mu) = a(\lambda_1)a(\lambda_2)[g(\lambda_1)]^{1/2}[g(\lambda_2)]^{1/2}. \tag{4.16}$$

By Plancherel's identity we have

$$\begin{aligned}\int_{\mathbb{R}^2} \mathrm{d}_{iT}^2(u_1,u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2 &= (2\pi)^2 \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}} G_T(\lambda_1-\mu)G_T(\mu-\lambda_2)p_i(\lambda_1,\lambda_2,\mu)\,\mathrm{d}\mu \right| \mathrm{d}\lambda_1\,\mathrm{d}\lambda_2 \\ &= (2\pi)^2 T||p_i||_T^2, \quad i=1,2, \end{aligned}\tag{4.17}$$

where $G_T(u)$ is given by (3.2), $||p||_T^2 = (p,p)_T$,

$$(p,p')_T = \int_{\mathbb{R}^4} p(\lambda_1,\lambda_2,\lambda_3)\overline{p'(\lambda_1,\lambda_2,\lambda_4)}\,\mathrm{d}\mu_T, \tag{4.18}$$

and the measure μ_T is defined by (3.17).

As in (4.17) (see also (4.14))

$$\operatorname{Var}(Q_T - L_T) = 8\pi^2 T \|p_1 - p_2\|_T^2 \tag{4.19}$$

For any $K > 0$ we consider the sets

$$E_1^K = \{u \in \mathbb{R} : |a(u)| < K\}, \quad E_2^K = \{u \in \mathbb{R} : g(u) < K\}, \tag{4.20}$$

and denote

$$\begin{aligned} p_1^K(\mathbf{u}) &= p_1(\mathbf{u})\chi_1^K(u_1)\chi_1^K(u_2)\chi_2^K(u_3), \\ p_2^K(\mathbf{u}) &= p_2(\mathbf{u})\chi_1^K(u_1)\chi_1^K(u_2)\chi_2^K(u_1)\chi_2^K(u_2), \end{aligned}\tag{4.21}$$

where $\mathbf{u} = (u_1,u_2,u_3) \in \mathbb{R}^3$ and $\chi_j^K(u)$ is the characteristic function of the set E_j^K, $j=1,2$. Then

$$\|p_1 - p_2\|_T^2 \le 3\left(\|p_1^K - p_2^K\|_T^2 + \|p_1 - p_1^K\|_T^2 + \|p_2 - p_2^K\|_T^2\right). \tag{4.22}$$

Now, (4.15), (4.16) and (4.21) imply that $\|p_1^K - p_2^K\|_T^2 = \int_{\mathbb{R}^4} \Gamma \mathrm{d}\mu_T$, where $\Gamma = \Gamma(u_1, u_2, u_3, u_4)$ is a sum of four functions satisfying the conditions of Lemma 6. Since $\Gamma(u, u, u, u) = 0$ for $u \in \mathbb{R}$, applying Lemma 6 we get

$$\lim_{T\to\infty} \|p_1^K - p_2^K\|_T = \int_{\mathbb{R}} \Gamma(u, u, u, u)\mathrm{d}u = 0. \tag{4.23}$$

Next, according to (4.18) we have

$$\|p_1\|_T^2 = \|p_1^K + (p_1 - p_1^K)\|_T^2 = \|p_1\|_T^2 + 2(p_1^K, p_1 - p_1^K)_T + \|p_1 - p_1^K\|_T^2.$$

Therefore

$$\|p_1 - p_1^K\|_T^2 \le \left|\|p_1\|_T^2 - \|p_1^K\|_T^2\right| + 2\left|(p_1^K, p_1 - p_1^K)_T\right|. \tag{4.24}$$

By (2.2), (4.17) and Lemma 1

$$\|p_1\|_T^2 = (2\pi)^{-2}\frac{1}{T}\mathrm{tr}\left[B_T(f)B_T(g)\right]^2 \to 2\pi \int_{\mathbb{R}} f^2(u)g^2(u)\mathrm{d}u, \tag{4.25}$$

while according to Lemma 6 and (4.17)

$$\|p_1^K\|_T^2 \to 2\pi \int_{H_K} f^2(u)g^2(u)\mathrm{d}u, \tag{4.26}$$

where $H_K := \{u \in \mathbb{R} : f(u) < K,\ g(u) < K\}$. From (4.25) and (4.26) we get

$$\lim_{T\to\infty} \left(\|p_1\|_T^2 - \|p_1^K\|_T^2\right) = \int_{\mathbb{R}\setminus H_K} f^2(u)g^2(u)\mathrm{d}u = o(1) \quad \text{as } K \to \infty. \tag{4.27}$$

To estimate the last term on the right-hind side of (4.24) we denote

$$\Gamma_K(u_1, u_2, u_3, u_4) = p_1^K(u_1, u_2, u_3)\left[p_1(u_1, u_2, u_4) - p_1^K(u_1, u_2, u_4)\right].$$

From (4.20) and (4.21) for $\Gamma_K(u_1, u_2, u_3, u_4) \neq 0$ we have

$$|a(u_1)| < K, \quad |a(u_2)| < K, \quad g(u_3) < K, \quad g(u_4) > K, \tag{4.28}$$

Next, for any $L > K$ and $\mathbf{u} = (u_1, u_2, u_3, u_4)$ we have

$$(p_1^K, p_1 - p_1^K)_T = \int_{\mathbb{R}^4} \Gamma_K(\mathbf{u}) \mathrm{d}\mu_T = \int_{\mathbb{R}^4} \Gamma_K(\mathbf{u}) \chi_2^L(u_4) \mathrm{d}\mu_T + I, \tag{4.29}$$

where with some constant C_K depending on K,

$$|I| \le C_K \int_{\mathbb{R}^4} g(u_4) \left(1 - \chi_2^L(u_4)\right) \mathrm{d}|\mu_T|. \tag{4.30}$$

It follows from (4.15), (4.16) and (4.21) that $\Gamma_K(\mathbf{u})\chi_2^L(u_4)$ is a linear combination of functions satisfying the conditions of Lemma 6. Applying Lemma 6 and taking into account that $\Gamma_K(u, u, u, u) = 0$ for $u \in \mathbb{R}$ (see (4.28)), we obtain

$$\lim_{T\to\infty} \int_{\mathbb{R}^4} \Gamma_K(\mathbf{u}) \chi_2^L(u_4) \mathrm{d}\mu_T = \int_{\mathbb{R}} \Gamma_K(u, u, u, u) \chi_2^L(u) \mathrm{d}u = 0. \tag{4.31}$$

For given $\varepsilon > 0$ and sufficiently large L by (3.21)

$$C_K \int_{\mathbb{R}^4} g(u) \left(1 - \chi_2^L(u)\right) \mathrm{d}|\mu_T| \le C_K \int_{\{u:\, g(u) > L\}} g(u) \mathrm{d}u \le \varepsilon. \tag{4.32}$$

From (4.29)–(4.32) we obtain

$$\lim_{T\to\infty} (p_1^K, p_1 - p_1^K)_T = 0.$$

This combined with (4.24) and (4.27) yields

$$\lim_{T\to\infty} \|p_1 - p_1^K\|_T = 0. \tag{4.33}$$

Finally, we prove that

$$\lim_{T\to\infty} \|p_2 - p_2^K\|_T = 0. \tag{4.34}$$

Indeed, according to (4.16), (4.21) and (3.22)

$$\|p_2 - p_2^K\|_T \le \int_{\mathbb{R}^4} [1-\chi_1^K(u_1)]f(u_1)g(u_1)f(u_2)g(u_2)\mathrm{d}|\mu_T|$$
$$+ \int_{\mathbb{R}^4} [1-\chi_1^K(u_2)]f(u_1)g(u_1)f(u_2)g(u_2)\mathrm{d}|\mu_T|$$
$$\le \int_{\{u:|f(u)|>\sqrt{K}\}} f^2(u)g^2(u)\mathrm{d}u + \int_{\{u:|g(u)|>K\}} f^2(u)g^2(u)\mathrm{d}u = o(1),$$

when $K \to \infty$ (uniformly on T). A combination of (4.19), (4.23), (4.33) and (4.34) yields (4.8). This completes the proof of Lemma 10. Theorem 1 is proved.

Proof of Theorem 2 By a change of variables $x_1 = u, x_1 - x_3 = u_1, x_3 - x_2 = u_2$ and $x_2 - x_4 = u_3$ from (3.1) we obtain

$$\operatorname{tr}\left[B_T(f)B_T(g)\right]^2 = \int_{\mathbb{R}^4} G_T(u_1)G_T(u_2)G_T(u_3)G_T(-u_1-u_2-u_3) \times f(u)g(u-u_1)f(u-u_1-u_2) \times g(u-u_1-u_2-u_3)\,\mathrm{d}u\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3, \tag{4.35}$$

where $G_T(u)$ is as in (3.2). Taking into account the equality

$$\mathrm{e}^{\mathrm{i}u_1(T+1)/2} \cdot \mathrm{e}^{\mathrm{i}u_2(T+1)/2} \cdot \mathrm{e}^{\mathrm{i}u_3(T+1)/2} \cdot \mathrm{e}^{-\mathrm{i}(u_1+u_2+u_3)(T+1)/2} = 1$$

and that $D_T(u)$ is even function, from (4.35) we obtain

$$\operatorname{tr}\left[B_T(f)B_T(g)\right]^2 = 8\pi^3 \int_{\mathbb{R}^3} \Psi(u_1,u_2,u_3)\Phi_T(u_1,u_2,u_3)\,\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3, \tag{4.36}$$

where $\Phi_T(u_1,u_2,u_3)$ is defined by (5), $\Psi(u_1,u_2,u_3) = \varphi(u_1,u_1+u_2,u_1+u_2+u_3)$ and $\varphi(u_1,u_2,u_3)$ is defined by (2.3). By Theorem 1 and (4.36) we need to prove that

$$\lim_{T\to\infty} \int_{\mathbb{R}^3} \Psi(\mathbf{u})\Phi_T(\mathbf{u})\mathrm{d}\mathbf{u} = \int_{\mathbb{R}} f^2(x)g^2(x)\mathrm{d}x. \tag{4.37}$$

Now, since both functions $\varphi(u_1,u_2,u_3)$ and $\Psi(u_1,u_2,u_3) = \varphi(u_1,u_1+u_2,u_1+u_2+u_3)$ are square integrable and continuous at $(0,0,0)$, and

$$\Psi(0,0,0) = \int_{\mathbb{R}} f^2(x)g^2(x)\mathrm{d}x,$$

from Lemma 3 we obtain (4.37). Theorem 2 is proved.

Proof of Theorem 3 According to Theorem 2 it is enough to prove that the function

$$\varphi(\mathbf{t}) := \int_{\mathbb{R}} f_0(u)f_1(u-t_1)f_2(u-t_2)f_3(u-t_3)\mathrm{d}u, \quad \mathbf{t} = (t_1,t_2,t_3) \in \mathbb{R}^3 \tag{4.38}$$

belongs to $L^2(\mathbb{R}^3)$ and is continuous at $(0,0,0)$, provided that

$$f_i \in L^1(\mathbb{R}) \cap L^{p_i}(\mathbb{R}),\ \ 1 \le p_i \le \infty,\ \ i = 0,1,2,3, \quad \sum_{i=0}^{3} \frac{1}{p_i} \le 1. \tag{4.39}$$

It follows from Hölder inequality and (4.39) that

$$|\varphi(\mathbf{t})| \le \prod_{i=0}^{3} ||f_i||_{L^{p_i}(\mathbb{R})} < \infty, \quad \mathbf{t} = (t_1,t_2,t_3) \in \mathbb{R}^3.$$

Hence, $\varphi \in L^\infty(\mathbb{R}^3)$. On the other hand, the condition $f_i \in L^1(\mathbb{R})$ and (4.38) imply $\varphi \in L^1(\mathbb{R}^3)$. Therefore $\varphi \in L^2(\mathbb{R}^3)$.

To prove the continuity of $\varphi(\mathbf{t})$ at the point $(0,0,0)$ we consider three cases.
Case 1. $p_i < \infty,\ i = 0,1,2,3$
For an arbitrary $\varepsilon > 0$ we can find $\delta > 0$ satisfying (see (4.39))

$$\|f_i(u-t) - f_i(u)\|_{L^{p_i}(\mathbb{R})} \le \varepsilon,\ \ i = 1,2,3, \quad \text{if } |t| \le \delta. \tag{4.40}$$

We fix $\mathbf{t} = (t_1,t_2,t_3)$ with $|\mathbf{t}| < \delta$ and denote

$$\overline{f}_i(u) = f_i(u+t_i) - f_i(u), \quad i = 1,2,3.$$

Then (4.40) implies $\|\overline{f}_i\|_{p_i} \le \varepsilon$, $i = 1,2,3$. So in view of (4.38) we have

$$\varphi(\mathbf{t}) = \int_{\mathbb{R}} f_0(u)\prod_{i=1}^{3}\left(\overline{f}_i(u) + f_i(u)\right)\mathrm{d}u = \varphi(0,0,0) + W.$$

Each of the five integrals comprising W contains at least one function $\bar{f}_i$ and can be estimated as follows

$$\left| \int_{\mathbb{R}} f_o(u)\bar{f}_1(u)f_2(u)f_3(u)\mathrm{d}u \right| \leq \|f_o\|_{L^{p_0}} \|\bar{f}_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \|f_3\|_{L^{p_3}} \leq A\varepsilon.$$

Case 2. $p_i \leq \infty,\ i = 0,1,2,3,\ \sum_{i=0}^{3} \frac{1}{p_i} < 1$
There exist finite numbers $p_i' < p_i,\ i = 0,1,2,3$, such that $\sum_{i=0}^{3} 1/p_i' \leq 1$. Hence according to (4.39) we have $f_i \in L^{p_i'}\ i = 0,1,2,3$ and φ is continuous at $(0,0,0)$ as in the case 1.
Case 3. $p_i \leq \infty,\ i = 0,1,2,3,\ \sum_{i=0}^{3} \frac{1}{p_i} = 1$
First observe that at least one of the numbers p_i is finite. Suppose, without loss of generality, that $p_0 < \infty$. For any $\varepsilon > 0$ we can find functions f_0', f_0'' such that

$$f_0 = f_0' + f_0'', \quad f_0' \in L^{\infty}(\mathbb{R}), \quad \|f_0''\|_{L^{p_0}} < \varepsilon. \tag{4.41}$$

Therefore

$$\varphi(\mathbf{t}) = \varphi'(\mathbf{t}) + \varphi''(\mathbf{t}), \tag{4.42}$$

where the functions φ' and φ'' are defined as φ in (4.38) with f_0 replaced by f_0' and f_0'' respectively. It follows from (4.41) that φ' is continuous at $(0,0,0)$ (see Case 2), while by Hölder inequality $|\varphi''(\mathbf{t})| \leq A\varepsilon$. Hence, for sufficiently small $|\mathbf{t}|$

$$|\varphi(\mathbf{t}) - \varphi(0,0,0)| \leq |\varphi'(\mathbf{t}) - \varphi'(0,0,0)| + |\varphi''(\mathbf{t}) - \varphi''(0,0,0)| \leq (A+1)\varepsilon,$$

and the result follows. Theorem 3 is proved.

Remark 3 Observe that in fact, here we have proved that under (4.39)

$$\lim_{T\to\infty} \int_{\mathbb{R}^4} f_0(u_1)f_1(u_2)f_2(u_3)f_3(u_4)\mathrm{d}\mu_T = 8\pi^3 \int_{\mathbb{R}} f_0(u)f_1(u)f_2(u)f_3(u)\mathrm{d}u.$$

Proof of Theorem 4 By Theorem 2 it is enough to show that the function

$$\varphi(\mathbf{t}) = \int_{\mathbb{R}} f(u)g(u-t_1)f(u-t_2)g(u-t_3)\mathrm{d}u, \quad \mathbf{t} = (t_1,t_2,t_3) \in \mathbb{R}^3$$

belongs to $L^2(\mathbb{R}^3)$ and is continuous at $(0,0,0)$, provided that f and g satisfy conditions of Theorem 4, i. e. $f \in L_2(\mathbb{R}), g \in L_2(\mathbb{R}), fg \in L_2(\mathbb{R})$ and (2.4) holds.
Since

$$\varphi^2(\mathbf{t}) = \int_{\mathbb{R}^2} f(u)f(v)g(u-t_1)g(v-t_1)f(u-t_2)f(v-t_2)g(u-t_3)g(v-t_3)\mathrm{d}u\mathrm{d}v,$$

we have

$$\int\limits_{\mathbb{R}^3} \varphi^2(\mathbf{t})\,\mathrm{d}\mathbf{t} = \int\limits_{\mathbb{R}^2} \Bigg[\int\limits_{\mathbb{R}} g(u-t_1)g(v-t_1)\,\mathrm{d}t_1 \int\limits_{\mathbb{R}} f(u-t_2)f(v-t_2)\,\mathrm{d}t_2$$
$$\times \int\limits_{\mathbb{R}} g(u-t_3)g(v-t_3)\,\mathrm{d}t_3 \Bigg] f(u)f(v)\mathrm{d}u\mathrm{d}v$$
$$\le \|f\|^2_{L^2(\mathbb{R})}\|g\|^4_{L^2(\mathbb{R})} \int\limits_{\mathbb{R}^2} f(u)f(v)\mathrm{d}u\mathrm{d}v < \infty.$$

Now we prove the continuity of $\varphi(\mathbf{t})$ at the point $(0,0,0)$. Let ε be an arbitrary positive number. We denote

$$E_K = \{u \in \mathbb{R} : |f(u)| \le K\}, \quad f_1(u) = \chi_{E_K}(u)f(u), \quad f_2(u) = f(u) - f_1(u),$$

where $K > 0$ is chosen to satisfy $\int_{\mathbb{R}\setminus E_K} f^2(u)g^2(u)\mathrm{d}u \le \varepsilon$. Then

$$f = f_1 + f_2, \quad \|f_1\|_\infty \le K, \quad \int\limits_{\mathbb{R}} f_2^2(u)g^2(u)\mathrm{d}u \le \varepsilon. \tag{4.43}$$

We have

$$\varphi(\mathbf{t}) = \int\limits_{\mathbb{R}} f_1(u)g(u-t_1)f_1(u-t_2)g(u-t_3)\mathrm{d}u$$
$$+ \int\limits_{\mathbb{R}} f_2(u)g(u-t_1)f(u-t_2)g(u-t_3)\mathrm{d}u$$
$$+ \int\limits_{\mathbb{R}} f_1(u)g(u-t_1)f_2(u-t_2)g(u-t_3)\mathrm{d}u$$
$$=: \varphi_1(\mathbf{t}) + \varphi_2(\mathbf{t}) + \varphi_3(\mathbf{t}). \tag{4.44}$$

We estimate the functions $\varphi_k(\mathbf{t})$, $k = 1,2,3$ separately. First,

$$\varphi_1(\mathbf{t}) = \int\limits_{\mathbb{R}} f_1(u)g(u-t_1)f_1(u-t_2)\left[g(u-t_3) - g(u)\right]\mathrm{d}u$$
$$+ \int\limits_{\mathbb{R}} f_1(u)g(u)f_1(u-t_2)\left[g(u-t_1) - g(u)\right]\mathrm{d}u$$
$$+ \int\limits_{\mathbb{R}} f_1(u)g^2(u)f_1(u-t_2)\mathrm{d}u =: I_1 + I_2 + I_3. \tag{4.45}$$

Using Hölder inequality, from (4.43) we get

$$|I_1| \le K^2 \|g\|_2 \cdot \|g(u+t_3) - g(u)\|_2 = o(1) \quad \text{as } t_3 \to 0. \tag{4.46}$$

Similarly

$$|I_2| = o(1) \quad \text{as } t_1 \to 0. \tag{4.47}$$

From (4.43) we have

$$\begin{aligned} |I_3 - \varphi(0,0,0)| &= \left| \int_{\mathbb{R}} f_1(u+t_2) g^2(u+t_2) f_1(u) \mathrm{d}u - \int_{\mathbb{R}} f_1^2(u) g^2(u) \mathrm{d}u \right| \\ &\quad + \left| \int_{\mathbb{R}} f_2^2(u) g^2(u) \mathrm{d}u \right| \\ &\le K \left\| f_1(u+t_2) g^2(u+t_2) - f_1(u) g_1^2(u) \right\|_1 + \varepsilon = o(1) + \varepsilon \end{aligned} \tag{4.48}$$

as $t_2 \to 0$. From (4.45)–(4.48) for sufficiently small $|\mathbf{t}|$ we obtain

$$|\varphi_1(\mathbf{t}) - \varphi(0,0,0)| \le 2\varepsilon. \tag{4.49}$$

Next, for $\varphi_2(\mathbf{t})$ we have

$$\begin{aligned} |\varphi_2(\mathbf{t})|^2 &\le \int_{\mathbb{R}} f_2^2(u) g^2(u-t_1) \mathrm{d}u \int_{\mathbb{R}} f^2(u-t_2) g^2(u-t_3) \mathrm{d}u \\ &= \left| \int_{\mathbb{R}} f^2(u) g^2(u-t_1) \mathrm{d}u - \int_{\mathbb{R}} f_1^2(u) g^2(u-t_1) \mathrm{d}u \right| \\ &\quad \times \int_{\mathbb{R}} f^2(u) g^2(u+t_2-t_3) \mathrm{d}u \\ &\to \left| \int_{\mathbb{R}} f^2(u) g^2(u) \mathrm{d}u - \int_{\mathbb{R}} f_1^2(u) g^2(u) \mathrm{d}u \right| \int_{\mathbb{R}} f^2(u) g^2(u) \mathrm{d}u \end{aligned}$$

as $|\mathbf{t}| \to 0$. Therefore, in view of (4.43) for sufficiently small $|\mathbf{t}|$

$$|\varphi_2(\mathbf{t})|^2 \le \varepsilon \int_{\mathbb{R}} f^2(u) g^2(u) \mathrm{d}u. \tag{4.50}$$

Similarly we can prove that for $|\mathbf{t}|$ small enough

$$|\varphi_3(\mathbf{t})| \leq \varepsilon \int_{\mathbb{R}} f^2(u)g^2(u)\mathrm{d}u. \tag{4.51}$$

A combination of (4.44) and (4.49)–(4.51) yields

$$\lim_{\mathbf{t}\to 0} \varphi(\mathbf{t}) = \varphi(0,0,0).$$

This completes the proof of Theorem 4.

Proof of Theorem 5 In view of (3.1) and (3.17), we need to prove that (2.5) and (2.6) imply

$$\lim_{T\to\infty} \frac{1}{T} \int_{\mathbb{R}^4} f(x_1)f(x_2)g(x_3)g(x_4)\mathrm{d}\mu_T = 8\pi^3 \int_{\mathbb{R}} f^2(x)g^2(x)\mathrm{d}x. \tag{4.52}$$

If $\alpha, \beta \geq 0$, then (2.5), (2.6) imply $f \in L^{1/\alpha}(\mathbb{R})$, $g \in L^{1/\beta}(\mathbb{R})$, and the result follows from Theorem 3. Assuming $\beta < 0$, from (2.5) we have $g \in L^{\infty}(\mathbb{R})$.

Denote

$$\overline{f}(x) = \begin{cases} 0 & \text{if} \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ f(x) & \text{otherwise} \end{cases}, \quad \overline{g}(x) = \begin{cases} 0 & \text{if} \quad x \in [-\pi, \pi] \\ g(x) & \text{otherwise} \end{cases},$$

and let $\underline{f} = f - \overline{f}$, $\underline{g} = g - \overline{g}$. Then

$$\begin{aligned} \frac{1}{T} \int_{\mathbb{R}^4} f(x_1)f(x_2)g(x_3)g(x_4)\mathrm{d}\mu_T &= \frac{1}{T} \int_{\mathbb{R}^4} \overline{f}(x_1)f(x_2)g(x_3)g(x_4)\mathrm{d}\mu_T \\ &\quad + \frac{1}{T} \int_{\mathbb{R}^4} \underline{f}(x_1)\overline{f}(x_2)g(x_3)g(x_4)\mathrm{d}\mu_T \\ &\quad + \frac{1}{T} \int_{\mathbb{R}^4} \underline{f}(x_1)\underline{f}(x_2)g(x_3)g(x_4)\mathrm{d}\mu_T \\ &=: I_T^1 + I_T^2 + I_T^3. \end{aligned} \tag{4.53}$$

Since $\overline{f}$, $g \in L^{\infty}(\mathbb{R})$ and $f \in L^1(\mathbb{R})$ (see also Remark 3) we obtain

$$\begin{aligned} \lim_{T\to\infty} I_T^1 &= 8\pi^3 \int_{\mathbb{R}} \overline{f}(x)f(x)g^2(x)\mathrm{d}x = 8\pi^3 \int_{|x|>\frac{\pi}{2}} f^2(x)g^2(x)\mathrm{d}x, \\ \lim_{T\to\infty} I_T^2 &= 8\pi^3 \int_{\mathbb{R}} \underline{f}(x)\overline{f}(x)g^2(x)\mathrm{d}x = 0. \end{aligned} \tag{4.54}$$

Next,

$$I_T^3 = \frac{1}{T}\int\limits_{\mathbb{R}^4} \underline{f}(x_1)\underline{f}(x_2)\underline{g}(x_3)\underline{g}(x_4)\mathrm{d}\mu_T + \frac{1}{T}\int\limits_{\mathbb{R}^4} \underline{f}(x_1)\underline{f}(x_2)\underline{g}(x_3)\overline{g}(x_4)\mathrm{d}\mu_T$$
$$+\frac{1}{T}\int\limits_{\mathbb{R}^4} \underline{f}(x_1)\underline{f}(x_2)\overline{g}(x_3)\underline{g}(x_4)\mathrm{d}\mu_T$$
$$+\frac{1}{T}\int\limits_{\mathbb{R}^4} \underline{f}(x_1)\underline{f}(x_2)\overline{g}(x_3)\overline{g}(x_4)\mathrm{d}\mu_T =: \sum_{i=1}^{4} J_T^i. \tag{4.55}$$

We have

$$J_T^1 = \frac{1}{T}\int\limits_{[-\pi,\pi]^4} \underline{f}(x_1)\underline{f}(x_2)\underline{g}(x_3)\underline{g}(x_4)\mathrm{d}\mu_T.$$

Arguments similar to those leading to equality (4.3) from [9] may be used to prove that

$$\lim_{T\to\infty} J_T^1 = 8\pi^3\int\limits_{-\pi}^{\pi} \underline{f}^2(x)g^2(x)\mathrm{d}x = 8\pi^3\int\limits_{-\pi/2}^{\pi/2} f^2(x)g^2(x)\mathrm{d}x. \tag{4.56}$$

Since $f(x_1)f(x_2) \in L^1(\mathbb{R}^2)$ for any $\varepsilon > 0$ we can find $\delta > 0$ satisfying

$$\int\limits_{|x_1-x_2|<\delta} |f(x_1)f(x_2)|\mathrm{d}x_1\mathrm{d}x_2 < \varepsilon.$$

Because $g \in L^\infty(\mathbb{R})$, in view of (3.23) and (3.24) for sufficiently large T we obtain

$$|J_T^2| \le C\cdot T\int\limits_{-\pi/2}^{\pi/2}\int\limits_{-\pi/2}^{\pi/2} |f(x_1)f(x_2)|\int\limits_{-\pi/2}^{\pi/2} \psi_T(x_1-x_3)\psi_T(x_2-x_3)$$
$$\times\int\limits_{|x_4|>\pi} \frac{1}{x_4^2}\mathrm{d}x_4\mathrm{d}x_3\mathrm{d}x_1\mathrm{d}x_2$$
$$\le C\int\limits_{-\pi/2}^{\pi/2}\int\limits_{-\pi/2}^{\pi/2} |f(x_1)f(x_2)|(1+T|x_1-x_2|)^{-1/2}\mathrm{d}x_1\mathrm{d}x_2$$

$$\leq C \int_{|x_1-x_2|<\delta} |f(x_1)f(x_2)|\mathrm{d}x_1\mathrm{d}x_2$$
$$+(1+T\delta)^{-1/2} \int_{|x_1-x_2|\geq\delta} |f(x_1)f(x_2)|\mathrm{d}x_1\mathrm{d}x_2 \leq 2\varepsilon.$$

This means that

$$\lim_{T\to\infty} J_T^2 = 0. \tag{4.57}$$

Likewise

$$\lim_{T\to\infty} J_T^3 = 0. \tag{4.58}$$

To estimate the integral J_T^4 in (4.55) note that in this case $|x_i - x_j| > \frac{\pi}{2}$, $i = 1,2$, $j = 3,4$. Therefore

$$|J_T^4| \leq \frac{C}{T}\int_{\mathbb{R}^4} \underline{f}(x_1)\underline{f}(x_2)\overline{g}(x_3)\overline{g}(x_4)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4$$
$$\leq \frac{C}{T}\|f\|^2_{L^1(\mathbb{R})}\|g\|^2_{L^1(\mathbb{R})} \to 0 \quad \text{as } T\to\infty. \tag{4.59}$$

From (4.55)–(4.59) we obtain $\lim_{T\to\infty} I_T^3 = 0$, which combined with (4.53) and (4.54) yields (4.52). Theorem 5 is proved.

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