1. Determine whether each of the following sets is a vector space or not, and why. (Use the standard element-wise addition and scalar multiplication, with real scalars unless otherwise stated.)

(a) The set of points $(x, 1)$ in $\mathbb{R}^2$.
(b) The set of all continuous functions on the real interval $[a, b]$.
(c) The set of all points on the unit circle in $\mathbb{R}^2$.
(d) The set of complex n-tuples, $\mathbb{C}^n$ (with element-wise addition and complex scalars).
(e) The set of polynomial functions on $\mathbb{R}$.

2. Determine whether each of the following bilinear operators is an inner product for its vector space or not, and why:

(a) $\langle f, g\rangle = \int_a^b f(x)\,\overline{g(x)} + f'(x)\,\overline{g'(x)}\,dx$, for the continuously differentiable complex-valued functions on the real interval $[a, b]$. (The complex derivative is, like the real derivative, a linear operation.)
(b) $\langle f, g\rangle = \int_0^\infty f(x)\,g(x)\,e^{-x}\,dx$, for the polynomials on $\mathbb{R}_+$.

3. Prove that in any inner product space, $\|\boldsymbol{x}+\boldsymbol{y}\|^2 + \|\boldsymbol{x}-\boldsymbol{y}\|^2 = 2\,\|\boldsymbol{x}\|^2 + 2\,\|\boldsymbol{y}\|^2$ under the induced norm. Interpret this statement geometrically in $\mathbb{R}^2$.

4. Prove the triangle inequality under the induced norm for an inner product space, $\|\boldsymbol{x}+\boldsymbol{y}\|^2 \leq \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2$.

5. Consider a forward discretization of the differential operator ∂_t,

$$\boldsymbol{D} = \frac{1}{dt}\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix},$$

a $T \times T$ matrix.

(a) Calculate the left inverse of $\boldsymbol{D}$.
(b) Calculate the adjoint (transpose) of $\boldsymbol{D}$.
(c) Does there exist a solution to the difference system $\boldsymbol{D}\boldsymbol{x} = \boldsymbol{b}$? Why or why not?
(d) Does the difference system $\boldsymbol{D}\boldsymbol{x} = \boldsymbol{b}$ have a unique solution $\boldsymbol{x}$? Why or why not?