1. Prove that for $\boldsymbol{x} \in \mathbb{C}^m$, $\boldsymbol{y} \in \mathbb{C}^n$, $A \in \mathbb{C}^{m\times n}$ and the standard complex inner (dot) products, $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \sum_{i=1}^m a_i \bar{b}_i$ and $\langle\!\langle \boldsymbol{a}, \boldsymbol{b} \rangle\!\rangle = \sum_{i=1}^n a_i \bar{b}_i$, $\langle \boldsymbol{x}, \boldsymbol{A}\boldsymbol{y} \rangle = \langle\!\langle \boldsymbol{A}^* \boldsymbol{x}, \boldsymbol{y} \rangle\!\rangle$ where $\boldsymbol{A}^*$ is the conjugate transpose of $\boldsymbol{A}$. That is, prove that the adjoint of a complex matrix is its conjugate transpose.

2. Prove the Fredhold Alternative Theorem for complex matrices:
<u>Theorem</u> For any complex matrix $\boldsymbol{A} \in \mathbb{C}^{m\times n}$, $\boldsymbol{A} : \mathbb{C}^n \to \mathbb{C}^m$, the range of $\boldsymbol{A}$ is the orthogonal complement of the null space of $\boldsymbol{A}^*$.

3. Prove that if the matrix $\boldsymbol{A}$ has linearly independent columns, the Moore-Penrose pseudoinverse of $\boldsymbol{A}$ is its least-squares pseudo-inverse.

4. Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 1 + \epsilon/\sqrt{10}, & 1 - 2\epsilon/\sqrt{10} \\ 2 - \epsilon/\sqrt{10}, & 2 + \epsilon/\sqrt{10} \end{pmatrix}$, with $\epsilon \ll 1$.

(a) Compute the least-squares pseudo-inverse of $\boldsymbol{A}$.
(b) Compute $\tilde{\boldsymbol{A}}$, the matrix given by discarding the smaller singular value of $\boldsymbol{A}$.
(c) What is the least-squares pseudo-inverse of $\tilde{\boldsymbol{A}}$? Compare the two pseudo-inverses.

5. Prove that the adjoint of a linear operator is also a linear operator.

6. Let $L : U \to V$ and $M : V \to W$. Show that the composite operator $M \circ L$ has adjoint $(M \circ L)^* = L^* \circ M^*$.

7. True or false: the adjoint of the divergence operator (with respect to the standard L^2 inner products) is the negative gradient: $(\nabla\cdot)^* = -\nabla$? If true, what boundary conditions do you need to assume? If false, what is the adjoint?