1. True or false, and why: for any $x \in \mathbb{R}$, $\delta(2x) = \frac{1}{2}\delta(x)$?

2. True or false, and why: for any $\vec{x} \in \mathbb{R}^n$, $\delta(\vec{x}) = \delta(||\vec{x}||)$?

3. What is $\delta'(x)$? (Consider $\int dx\, f(x)\delta'(x)$.)

4. Consider a bar of length 1, with stiffness $c(x) = 1/\left(1+x^2\right)$, subject to an external force f. Its displacement obeys the ODE

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x) \tag{1}$$

on $0 < x < 1$, and its ends are fixed: $u(0) = u(1) = 0$.

(a) Find the displacement when then bar is subject to a constant force, $f = 1$.
(b) Find the Green's function for the boundary value problem.
(c) Use the Green's function to check your answer from part (a).
(d) Which point $0 < \xi < 1$ is the weakest (has the greatest displacement in response to a unit impulse at ξ)?

5. Prove the Fourier symmetry principle: if the Fourier transform of $f(x)$ is $\hat{f}(k)$, then the Fourier transform of $\hat{f}(x)$ is $f(-k)$.

6. Prove the Fourier shift and dilation theorem: if the Fourier transform of $f(x)$ is $\hat{f}(k)$, then

(a) The Fourier transform of $f(x-\xi)$ is $e^{-ik\xi}\hat{f}(k)$.
(b) The Fourier transform of $e^{ilx}f(x)$ is $\hat{f}(k-l)$.
(c) The Fourier transform of $f(cx)$ is $\frac{1}{|c|}\hat{f}(\frac{k}{x})$.

Hint: write the Fourier transform directly and multiply by 1 (appropriately written) inside the integral.

7. Prove that if the Fourier transform of $f(x)$ is $\hat{f}(k)$, then the Fourier transform of $xf(x)$ is $i\frac{d}{dk}\hat{f}(k)$.

8. Prove the Fourier integral theorem: if $f(x)$ has Fourier transform $\hat{f}(k)$, then the Fourier transform of $g(x) = \int_{-\infty}^{x} f(y)\,dy$ is given by

$$\hat{g}(k) = -\frac{i}{k}\hat{f}(k) + \pi\hat{f}(0)\delta(k)$$

Hint: First compute the Fourier transform of the Heaviside step function.

9. Solve

$$-u''(x) + 4u(x) = \delta(x)$$

on the real line using Fourier transforms.

10. Calculate the following:

(a) The Fourier transform of $e^{-a|x|}$.
(b) The convolution of $e^{-a|x|}$ with itself.
(c) The Fourier transform of $e^{i\omega x}$, and thus the Fourier transforms of $\cos \omega x$ and $\sin \omega x$.