1. Verify the Hubbard-Stratonovich transform,

$$e^{\frac{b^2}{2}} = \int Dt\, e^{bt}$$

where $Dx = \exp(-t^2/2)/\sqrt{2\pi}$ is the standard Gaussian measure.

2. Consider the Bernoulli distribution,

$$p(x) = \begin{cases} m, \text{ if } x = 1 \\ 1-m, \text{ if } x = 0 \end{cases} \tag{1}$$

Calculate its moment and cumulant generating functions.

3. Consider an Erdős-Rényi network of N neurons with a connection matrix $\boldsymbol{J}$. Each neuron j projects to each neuron i with probability m. The existence of each connection is independent of the others. Since each neuron receives on average mN inputs, we will scale the connection strengths by N: each nonzero element of $\boldsymbol{J}$ is equal to $\bar{J}/N$. That is,

$$\begin{aligned} p(\boldsymbol{J}) &= \prod_{i,j} p(J_{ij}), \\ p(J_{ij}) &= \begin{cases} m, \text{ if } x = \bar{J}/N \\ 1-m, \text{ if } x = 0 \end{cases} \end{aligned} \tag{2}$$

(a) Calculate the first three cumulants of the distribution of synaptic weights.
(b) Repeat that calculation with synaptic weights scaling like $1/\sqrt{N}$ instead of $1/N$.

4. Complete the Gardner capacity calculation. Recall that the saddle point equation for the replica overlap leads to

$$\alpha \int Dt \left(\int_u^\infty dz e^{-z^2/2} \right)^{-1} e^{-u^2/2} \frac{t + \kappa\sqrt{q}}{2\sqrt{q}\,(1-q)^{3/2}} = \frac{q}{1-q} \tag{3}$$

where $\alpha = P/N$ is the capacity, $Dt = \exp(-t^2/2)/\sqrt{2\pi}$, and $u = \left(\kappa + t\sqrt{q}\right)/\sqrt{1-q}$. Using this, compute the critical capacity

$$\alpha_c \equiv \lim_{q \to 1} \alpha = \left(\int_{-\kappa}^\infty Dt\,(t+\kappa)^2 \right)^{-1}. \tag{4}$$

Hint: write Eq. 3 in the form $\alpha = q/F(q;\kappa)$ and apply L'Hôpital's rule.

5. Verify the Garner calculation for the critical capacity of a binary perceptron by comparing the analytical result for the critical capacity against simulations with varying margins. For simulations, use the Margin Perceptron with output:

$$y = \begin{cases} 1, & \boldsymbol{J}^T\boldsymbol{x}/\left\|\boldsymbol{J}\right\| \geq \kappa/2 \\ 0, & \boldsymbol{J}^T\boldsymbol{x}/\left\|\boldsymbol{J}\right\| \leq -\kappa/2 \end{cases} \tag{5}$$

trained with the standard Perceptron learning algorithm where, if $\boldsymbol{J}^T\boldsymbol{x}/\left\|\boldsymbol{J}\right\| \in (-\kappa/2, \kappa/2)$, the pattern $\boldsymbol{x}$ automatically produces an error.