1. Consider the 2-dimensional Amari-Grossberg neural network model

$$\frac{dv_\alpha}{dt} = -v_\alpha(t) + \sum_\beta J_{\alpha\beta}\, \phi(v_\beta(t)) + \mu_\alpha + \xi_\alpha(t) \tag{1}$$

with an excitatory and inhibitory populations, $\alpha \in \{E, I\}$. μ_α is a constant drive to population α and $\xi_\alpha(t)$ is a zero-mean white Gaussian noise with $\langle \xi_\alpha(t)\xi_\beta(t')\rangle = \delta_{\alpha\beta}\,\delta(t-t')$. Let $\phi(x) = \lfloor x-1 \rfloor_+^2$ and

$$\begin{pmatrix} J_{EE} & E_{EI} \\ J_{IE} & J_{II} \end{pmatrix} = \begin{pmatrix} J & -gJ \\ J & -gJ \end{pmatrix}.$$

(a) What is the action governing the response-variable path integral representation for the probability density functional $p[\boldsymbol{v}]$? (In order to follow the same steps as in class, you could discretize time first and then take a continuous-time limit in the same way we did. Or, you could use the infinite-dimensional delta distribution.)
(b) What is the purely deterministic mean field approximation of Eq. 1?
(c) What are the nullclines of the mean field dynamics?
(d) Expand $\boldsymbol{v}$ around a fixed point: $\boldsymbol{v}(t) = \boldsymbol{v}^* + \delta\boldsymbol{v}(t)$, where $\boldsymbol{v}^*$ is an equilibrium of the deterministic mean field approximation. (Because of this, several terms in the action should cancel.) What is the propagator for this mean-field expansion?
(e) What are the vertices corresponding to the nonlinear terms of that mean field expansion?
(f) What is the tree-level contribution to the second moment $\langle v_\alpha(t)\, v_\beta(t')\rangle$? ("Tree-level" means that it arises from terms in the Wick expansion pairing exactly one factor of $\tilde{\boldsymbol{v}}$ with one factor of $\boldsymbol{v}$; its Feynman diagram is a tree graph.)
(g) What is the tree-level contribution to the third moment $\langle v_\alpha(t)\, v_\beta(t')\, v_\gamma(t'')\rangle$? ("Tree-level" means that it arises from terms in the Wick expansion pairing exactly one factor of $\tilde{\boldsymbol{v}}$ with one factor of $\boldsymbol{v}$; its Feynman diagram is a tree graph.)

2. Simulation. Consider the N-dimensional neural network model

$$\frac{dv_i}{dt} = -v_i + \frac{g}{\sqrt{N}} \sum_{j=1}^{N} J_{ij}\phi(v_j) \tag{2}$$

with $\phi(x) = \tanh(x)$ and $J_{ij} \sim \mathcal{N}(0,1)$. Write a simulation of the dynamics of this network, with the initial conditions $v_i(0) \sim \mathcal{N}(0,1)$, $N = 100$, $dt = .01$ and for total time 100. Plot v against time with $g = 0.5$, $g = 1$, $g = 2$ and $g = 3$. Discuss your results.