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1. Let $Du = u'(x)$ so D is the derivative operator, acting on $C^2[0,1]$, the vector space of twice continuously differential functions defined on the closed interval $0 \le x \le 1$, with homogenous Dirichlet boundary conditions.

(a) Compute D^*, the adjoint of D with respect to the weighted inner product $\langle u, \tilde{u} \rangle = \int_0^1 u(x)\, \tilde{u}(x)\, e^x\, dx$. (5 points)

(b) Let $S = D^* \circ D$. Write down and solve the boundary value problem $Su = 2e^x$. (5 points)

(a)

$$\begin{aligned}
\langle Du, \tilde{u} \rangle =& \int_0^1 u'(x)\tilde{u}(x)e^x\, dx \\
=& \tilde{u}(x)\, e^x\, u(x)\big|_0^1 - \int_0^1 u(x)\left(\tilde{u}'(x)e^x + \tilde{u}(x)e^x\right) dx \\
=& -\int_0^1 dx\, e^x u(x)\,(D+1)\,\tilde{u}(x) \\
=& \langle u, -(D+1)\tilde{u} \rangle
\end{aligned}$$

The boundary terms vanish due to the homogenous Dirichlet boundary conditions. The adjoint of D with respect to this weighted inner product is $-(D+1)$.

(b) $S = -D(D+1) = -D^2 - D$. As an aside, we know that $S = D \circ D^*$ must be self-adjoint and its nullspace is the same as that of D. With homogenous Dirichlet boundary conditions the nullspace of D is $\{0\}$, so $Su = f$ has a unique solution. Now back to the problem: what BVP does that solution satisfy, and what is the solution?

We have the boundary value problem:

$$-u''(x) - u'(x) = 2e^x$$

with homogenous Dirichlet boundary conditions, $u(0) = u(1) = 0$. Integrating on both sides,

$$\begin{aligned}
u'(x) - u'(0) + u(x) =& -2e^x - A \\
u'(x)e^x + u(x)e^x =& -2e^{2x} - Ae^x + u'(0)e^x \\
\frac{d}{dx}\left(u(x)e^x\right) =& -2e^{2x} - Ae^x + u'(0)e^x \\
u(x)e^x =& -e^{2x} - (A - u'(0))e^x + B \\
u(x) =& -e^x - (A - u'(0)) + Be^{-x}
\end{aligned}$$

And the boundary conditions require

$$\begin{aligned}
u(0) =& -1 - (A - u'(0)) + B = 0 \\
u(1) =& -e - (A - u'(0)) + B/e = 0
\end{aligned}$$

Taking the derivative of $u(x)$ and then plugging in $x = 0$, we see that

$$u'(0) = -1 - B$$

Together with the first boundary condition, this tells us that

$$\begin{aligned}
0 =& -1 - (A + 1 + B) + B \\
2 =& A
\end{aligned}$$

and the second boundary condition gives us

$$B = \left(\frac{e+1}{e}\right)(e+3).$$

The solution is thus

$$u(x) = -e^x - (2 - u'(0)) + \frac{(e+1)(e+3)}{e^2}$$

We resolve the derivative at 0 by again differentiating, to find that $u'(0) = -1$.

2. Consider the Poisson equation on a one-dimensional domain with Dirichlet boundary conditions:

$$-\partial_x^2 u(x) = f(x), \;\; u(0) = u(1) = 1$$

(a) Is this boundary value problem positive definite, positive semi-definite, or neither? Why? (5 points)

(b) What conditions, if any, are required on f to guarantee existence of a solution? (5 points)

(a) With these boundary conditions, the adjoint of D with respect to the L^2 inner product is

$$\begin{aligned}\langle Du, \tilde{u}\rangle &= \int_0^1 u'(x)\tilde{u}(x)\,dx \\ &= u(x)\tilde{u}(x)\big|_0^1 - \int_0^1 u(x)\tilde{u}'(x)\,dx \\ &= \langle u, -D\tilde{u}\rangle\end{aligned}$$

where the boundary terms vanish because of the equal Dirichlet boundary conditions. So, $-D^2 = D \circ D^*$ (where the adjoint is with respect to the standard L^2 inner product). The boundary value problem is thus at least positive semi-definite. Is is positive definite only if the nullspace of $-D^2$ is trivial. The nullspace of $-D^2$ is the same as that of D, and with these boundary conditions the nullspace of D is $\{1\}$. So, this boundary value problem is positive semi-definite.

(b) The Fredholm alternative tells us that a solution exists if $f(x)$ is orthogonal to the nullspace of $(-D^2)^*$. As shown above, $-D^2 = D \circ D^*$. So, $-D^2$ is self-adjoint with respect to the L^2 inner product under these boundary conditions. Since the nullspace of D is $\{1\}$, we must have

$$0 = \langle f, 1\rangle = \int_0^1 f(x)\,dx.$$

3. Consider the differential equation

$$x'(t) + x(t) = f(t),$$

with $x(t) : [0, \infty) \to \mathbb{R}$, $x(t) \in C^1[0, \infty)$. Is

$$\exp\left(-\left(t-s\right)\right)\theta(t-s),$$

where $\theta(x)$ is the Heaviside step function, a fundamental solution (Green's function) for this problem? Why or why not? (10 points)

A fundamental solution $G(t, s)$ to this initial value problem obeys

$$\left(\partial_t + 1\right) G(t, s) = \delta(t-s).$$

Applying $\partial_t + 1$ to the proposed Green's function directly,

$$\begin{aligned}(\partial_t + 1)\exp\left(-\left(t-s\right)\right)\theta(t-s) &= -\exp\left(-\left(t-s\right)\right)\theta(t-s) + \exp\left(-\left(t-s\right)\right)\delta(t-s) + \exp\left(-\left(t-s\right)\right)\theta(t-s \\ &= \exp\left(-\left(t-s\right)\right)\delta(t-s).\end{aligned}$$

Is this this same as just $\delta(t-s)$? If we view it as a function,

$$\exp\left(-\left(t-s\right)\right)\delta(t-s) = \begin{cases} \exp\left(-\left(t-s\right)\right) \times 0 = 0, & t \neq s \\ \exp\left(-\left(s-s\right)\right) \times \delta(0) = \delta(0), & t = s \end{cases}$$

so it matches the definition of $\delta(t-s)$. It also behaves the same way under integration:

$$\int ds \, \exp\left(-\left(t-s\right)\right)\, \delta(t-s)\, f(s) = \int ds \, \delta(t-s)\, \left(\exp\left(-\left(t-s\right)\right)\, f(s)\right) = f(t)$$

So, $\exp\left(-\left(t-s\right)\right)\theta(t-s)$ is the fundamental solution for $\partial_t + 1$.

4. Consider the matrix A with singular value decomposition $A = U\Sigma V^T$, where

$$U = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

What is the least-squares solution, $\hat{x}$, to $Ax = (1, -1)^T$? (10 points)

The least squares solution is given by

$$\begin{aligned} x =& V\Sigma' U^T b \\ =& \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ =& (0, 0)^T. \end{aligned}$$