1. Consider the linear operator $L: U \to V$. Prove that if L has a non-trivial null space, then $Lu = f$ does not have a unique solution.

2. Consider the linear operator $A : \mathbb{R}^3 \to \mathbb{R}^2$, with singular value decomposition $A = U\Sigma V^T$,

$$U = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \ V = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ \Sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}.$$

(a) What is A?

(b) What is the least-squares pseudo-inverse of A?

(c) Assuming that $\epsilon \ll 1$, consider the rank-two approximation to $Ax = (3, 1)^T$. What is the least squares solution to the approximate problem?

(d) Discuss where that approximate solution lies in relation to the range and/or nullspace of U and V.

3. The equilibrium conditions for deformation of a toroidal membrane (an inner tube) lead to the Poisson equation on a rectangle, $0 < x < a$, $0 < y < b$, with periodic boundary conditions:

$$-u_{xx} - u_{yy} = f(x,y), \; u(x,0) = u(x,b), \; u_y(x,0) = u_y(x,b), \; u(0,y) = u(a,y), \; u_x(0,y) = u_x(a,y).$$

(a) Prove that this toroidal boundary value problem is self-adjoint with respect to the standard L^2 inner product.

(b) Is this boundary value problem positive definite, positive semi-definite, or neither?

(c) What conditions, if any, must be imposed on the forcing function $f(x,y)$ to ensure existence of a solution?

4. Does there exist a solution to the following boundary value problem? If so, write down all solutions.

$$xu''(x) + u'(x) = 1, \ u'(1) = u'(2) = 0.$$

5. Consider the Fourier transform, a linear integral operator $\mathcal{F}$ defined by

$$\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} dx\, e^{ikx}\, f(x).$$

(a) Prove that the Fourier transform is a linear operator.
(b) What is the adjoint of $\mathcal{F}$ with respect to the standard Hermitian L^2 inner product?

6. Let $L = D^2$. Using the L^2 inner products on its domain and target spaces, write down a set of homogenous boundary conditions under which $L^* = D^2$ (the operator is self-adjoint). Then, let $S = L^* \circ L = D^4$. Do your boundary conditions from above lead to a boundary value problem that is 1) positive definite, 2) positive semi-definite, or 3) neither?

7. Consider the heat operator, $L = (\partial_t - \partial_x^2)$, associated with the forced heat equation

$$Lu(t,x) = f(t,x)$$

on $x \in \mathbb{R}$, $t > 0$. Is

$$g(t,x;\tau,\xi) = \frac{\theta(t-\tau)}{2\sqrt{\pi(t-\tau)}} \exp -\frac{(x-\xi)^2}{4(t-\tau)}$$

a Green's function for this problem? Why or why not? (10 points)

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