1. Consider the linear operator $L : U \to V$. Prove that if L has a non-trivial null space, then $Lu = f$ does not have a unique solution.

If L has a non-trivial nullspace $\mathcal{N}(L)$, there exists some $u_0 \neq 0$ such that $Lu_0 = 0$. Let $Lu_1 = f$. Then $L(u_1 + u_0) = Lu_1 + Lu_0 = f + 0 = f$ so $u_1 + u_0$ is also a solution to $Lu = f$.

2. Consider the linear operator $A : \mathbb{R}^3 \to \mathbb{R}^2$, with singular value decomposition $A = U\Sigma V^T$,

$$U = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \; V = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; \Sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}.$$

(a) What is A?
(b) What is the least-squares pseudo-inverse of A?
(c) Assuming that $\epsilon \ll 1$, consider the rank-two approximation to $Ax = (3, 1)^T$. What is the least squares solution to the approximate problem?
(d) Discuss where that approximate solution lies in relation to the range and/or nullspace of U and V.

For (a, b, c), it is convenient to use the SVD formulation of the least-squares solution as in the homework. For (d), recall that the columns of U and V are, respectively, the eigenvectors of AA^T and A^TA.
An eigenvector v of a matrix C with a zero eigenvalue is in the nullspace of C.
The least squares solution is $\hat{x} = V\Sigma' U^T b$. $\hat{x}$ is thus in the range of V, which is the orthogonal complement of the nullspace of V^T. $\hat{x}$ is formed by first projecting b first into the range of U^T (the orthogonal complement of the nullspace of U). However, components in the range of U^T are then scaled (and those in the nullspace of AA^T discarded) before projecting into the range of V.

3. The equilibrium conditions for deformation of a toroidal membrane (an inner tube) lead to the Poisson equation on a rectangle, $0 < x < a$, $0 < y < b$, with periodic boundary conditions:

$-u_{xx}-u_{yy} = f(x,y),\ u(x,0) = u(x,b),\ u_y(x,0) = u_y(x,b),\ u(0,y) = u(a,y),\ u_x(0,y) = u_x(a,y).$

(a) Prove that this toroidal boundary value problem is self-adjoint with respect to the standard L^2 inner product.

(b) Is this boundary value problem positive definite, positive semi-definite, or neither?

(c) What conditions, if any, must be imposed on the forcing function $f(x,y)$ to ensure existence of a solution?

(a) This follows a direct computation.

(b) Since the adjoint of the gradient with respect to the L^2 inner product is the negative divergence, and $\Delta u = \nabla \cdot \nabla u$, the negative Laplacian is a composite-adjoint operator. It is thus guaranteed to be positive semidefinite. And, it is positive definite only if its nullspace is trivial. That nullspace is the set of solutions to the homogenous BVP

$$-u_{xx} - u_{yy} = 0,\ u(x,0) = u(x,b),\ u_y(x,0) = u_y(x,b),\ u(0,y) = u(a,y),\ u_x(0,y) = u_x(a,y).$$

and is composed of the constant functions on the rectangle, so the nullspace is not trivial.

(c) The Fredholm alternative tells us that f must be orthogonal to the nullspace of the adjoint operator. Since the negative Laplacian is self-adjoint, the forcing function must have zero integral on the rectangle.

6. Let $L = D^2$. Using the L^2 inner products on its domain and target spaces, write down a set of homogenous boundary conditions under which $L^* = D^2$ (the operator is self-adjoint). Then, let $S = L^* \circ L = D^4$. Do your boundary conditions lead to a boundary value problem that is 1) positive definite, 2) positive semi-definite, or 3) neither?

We proved that operators of the form $L^* \circ L$ are always positive semi-definite, and positive definite if and only if they have a trivial nullspace.
For example, you could take homogenous Dirichlet boundary conditions as in class. The null space of S is then just the zero function, so S is positive definite. If you took homogenous Neumann boundary conditions the nullspace would not be trivial (all constant functions) so the operator would be just positive semi-definite.

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