Consistent Price Systems and Face-Lifting Pricing under Transaction Costs

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Abstract

In markets with transaction costs, consistent price systems play the same role as martingale measures in frictionless markets. We prove that if a continuous price process has conditional full support, then it admits consistent price systems for arbitrarily small transaction costs. This result applies to a large class of Markovian and non-Markovian models, including geometric fractional Brownian motion.

Using the constructed price systems we show under very general assumptions the following "face-lifting" result: the asymptotic superreplication price of a European contingent claim $g(S_T)$ equals $\hat{g}(S_0)$, where $\hat{g}$ is the concave envelope of g and S_t is the price of the asset at time t. This theorem generalizes similar results obtained for diffusion processes to processes with conditional full support.

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1 Introduction

It is common wisdom in Mathematical Finance that simple and elegant statements in frictionless markets are torn apart and are almost unrecognizable by the slightest market imperfection.

Transaction costs are arguably the most pervasive of frictions, as they affect virtually all financial (and also non-financial) markets, and the presence of bid and ask prices is crucial in the equilibrium between traders and market makers, which ensures market liquidity.

Since bid-ask spreads are usually a tiny fraction of prices, the theory of frictionless markets rests on the tacit assumption that small transaction costs have small effects. It is then unsettling that the basic questions of no-arbitrage and superreplication have radically different solutions with transaction costs, which moreover do *not* converge to the frictionless solution as the bid-ask spread shrinks to a single price (Soner, Shreve & Cvitanić 1995, Levental & Skorohod 1997, Cvitanic, Pham & Touzi 1999, Cherny 2006).

In a frictionless market, the "no free lunch with vanishing risk" (NFLVR) condition for simple strategies dictates that locally bounded asset price processes are semimartingales, and (NFLVR) for general strategies further implies the existence of a local martingale measure (Delbaen & Schachermayer 1994, Theorems 7.1 and 1.1). Then the superreplication price of a contingent claim is obtained as the supremum of its expected value under all such measures (Delbaen & Schachermayer 1994, Theorem 5.7). When this measure is unique, the market is complete, and the price of a claim is uniquely determined.

By contrast, with transaction costs arbitrage disappears for several examples of non-semimartingales, such as Fractional Brownian Motion (Guasoni 2006). Furthermore, market completeness fails except in trivial cases, and even in the Black-Scholes model the superreplication price of a call option is trivial, as it equals the price of the underlying asset (Soner et al. 1995, Levental & Skorohod 1997, Cvitanic et al. 1999).

To overcome this dichotomy between transaction costs and frictionless markets, one needs to better understand the link between the law of asset price processes, and their implications for contingent claim prices. This program, which in frictionless markets involves martingale measures, naturally leads to their transaction costs counterpart, a *consistent price system* (Schachermayer 2004). This concept dates back to the seminal paper of Jouini & Kallal (1995), and was further developed in the subsequent work of Cvitanić & Karatzas (1996), Kabanov (1999), Kabanov & Stricker (2002), Kabanov, Rásonyi & Stricker (2003), and Schachermayer (2004).

A consistent price system (henceforth CPS) is essentially a shadow frictionless asset, which admits a martingale measure, and such that its price is always contained within the bid-ask spread of the original asset. The availability of CPS links the problems of no-arbitrage and superreplication to their frictionless counterparts, where solutions are well-understood. Therefore, it is crucial to find general and easily testable conditions for their existence, and in the previous literature (Levental & Skorohod 1997, Cvitanic et al. 1999), asset prices were assumed semimartingales mainly for the sake of constructing classes of CPS.

In the present paper we study a simple condition on asset prices, namely *conditional full support*, which generates a large class of consistent price systems. In fact, all *natural* examples, we can think of, enjoy this property. We study the problems of no-arbitrage and superreplication for asset prices driven by continuous process with constant proportional transaction costs.

Let $T > 0$ be a fixed time horizon. We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ where the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions of right-continuity and saturatedness with respect to P as well as $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_0$ trivial. Introduce the notation $\mathbb{R}^d_{++} = (0,\infty)^d$. The set of $\mathbb{R}^d$-valued continuous functions on $[u,v]$ is denoted by $C[u,v]$, its subfamily of functions f starting at $f(u) = x$ is $C_x[u,v]$. The class of $\mathbb{R}^d_{++}$-valued continuous functions on $[u,v]$ is $C^+[u,v]$.

When $x \in \mathbb{R}^d_{++}$, we write $C^+_x[u,v]$ for the set of $\mathbb{R}^d_{++}$-valued continuous functions on $[u,v]$ starting at x. The *support* of a $\mathbb{R}^d_{++}$-valued continuous process S on $[u,v]$ is the smallest closed subset A of $C^+[u,v]$ such that $P(S \in A) = 1$.

We say that the $\mathbb{R}^d_{++}$-valued process S *has full support* if its support is the whole $C^+_x[u,v]$, where $S_0 = x$. From now on we shall use regular versions of conditional probabilities without further mention, consult e.g. p. 439 of Billingsley (1995).

We introduce the *conditional full support* condition which prescribes that from any given time on, the asset price path can continue arbitrarily close to any given path with positive conditional probability:

Definition 1.1. *A continuous $\mathbb{R}^d_{++}$-valued process $(S_t)_{t\in[0,T]}$ satisfies the* conditional full support *condition if for all $t \in [0,T)$:*

$$\operatorname{supp} P(S|_{[t,T]}|\mathcal{F}_t) = C^+_{S_t}[t,T] \qquad a.s. \tag{CFS}$$

where $P(S|_{[t,T]}|\mathcal{F}_t)$ denotes the $\mathcal{F}_t$-conditional distribution of the $C^+[t,T]$-valued random variable $S|_{[t,T]}$.

Our first main result shows that this condition implies the existence of consistent price systems, and hence the absence of free lunches:

Theorem 1.2. *Let $(S_t)_{t\in[0,T]}$ be an $\mathbb{R}^d_{++}$-valued, continuous adapted process satisfying* (CFS). *Then S admits an ε-consistent pricing system for all $\varepsilon > 0$ (see Definition 2.1 below).*

The proof of this theorem is quite intuitive, at least in dimension $d = 1$: we show that any continuous price process satisfying the conditional full support condition (CFS) is arbitrarily close to the archetypical model of a "Random Walk with Retirement", where martingale measures are characterized in terms of "retirement probabilities". Exploiting the flexibility of this construction, we discover a large class of consistent price systems, and extend to our setting the "face-lifting" results proved by Cvitanic et al. (1999) and Bouchard & Touzi (2000) for diffusion models. In particular, we may relax the Markov and even the semimartingale properties (see section 4) as well as regularity assumptions on the function g, which were required in previous results:

Theorem 1.3. *Let* $(S_t)_{t\in[0,T]}$ *be an* $\mathbb{R}_{++}$*-valued continuous process satisfying the conditional full support assumption* (CFS)*, and* $g : \mathbb{R}_{++} \mapsto \mathbb{R}$ *a lower semicontinuous function bounded from below.*

Then the asymptotic superreplication price $p_0(g(S_T))$ *(see Definition 2.15 below) of the European claim* $g(S_T)$ *is given by:*

$$p_0(g(S_T)) = \hat{g}(S_0)$$

where $\hat{g}$ *is the concave envelope of* g*.*

The d-dimensional extension of this result is left for future research. The proofs of these results help understanding the difference between transaction costs and frictionless markets as regards option pricing and hedging. Option pricers in the frictionless setting depend critically on the *fine* properties of the paths of the underlying price processes, such as quadratic variation, which are preserved only under equivalent changes of probabilities. Conceptually, such dependence is rather questionable, since these properties are hardly testable, and are not preserved under arbitrarily small measurement errors. By contrast, accounting for transaction costs leads to results that are robust with respect to economically small misspecifications (i.e. within the bid-ask spread), and depend only on the *coarse* properties of the model such as the support of its law in $C^+_x[0,T]$. In this spirit, a consistent price system is the coarse version of a martingale measure.

Since our results are valid under the condition (CFS) of conditional full support, the crucial question is how to verify this assumption in concrete cases. Fortunately, this task can be accomplished exploiting the literature on topological supports of stochastic processes, such as the classical results of Kallianpur (1971) for Gaussian processes and Stroock & Varadhan (1972) for diffusions.

As an application of our result, we show the existence of consistent price systems for Fractional Brownian Motion, which is a continuous process, but neither a Markov process, nor a semimartingale for $H \neq 1/2$. This improves on a result in Guasoni (2006), where the absence of arbitrage was obtained under analogous assumptions, and paves the way to the study of optimization problems. In fact, with Fractional Brownian Motion stochastic control methods are infeasible for lack of the Markov property, while the duality approach hinges on the availability of CPS.

The rest of this paper is organized as follows: in section 2 we describe the model in detail and discuss some preliminaries on markets with transaction costs. The following section contains the proofs of Theorems 1.2 and 1.3 for a market with one risky asset, where simple proofs based on the Random Walk with Retirement are available. For the face-lifting result, we limit ourselves to this setting, where the fact that a one-dimensional random walk leaves almost surely any interval plays a crucial role to achieve a simple proof. Section 3 contains the proof of Theorem 1.2 in the general case of several assets, with stochastic transaction costs, and for possibly negative prices. Here we develop a similar idea as in the one-dimensional case, but need considerably more technique, since no simple analogy with Retired Random Walks is available. Instead, we construct a conditional version of the classical Esscher transform.

As an application of our results, in section 4 we show how to check the conditional full support assumption in the cases of Markov diffusions and Fractional Brownian Motion models. The appendix contains some technical lemmas on conditional supports.

2 One asset with proportional transaction costs

In this section we present the proof of Theorems 1.2 and 1.3 for a market with one asset, and with proportional transaction costs.

We assume that the bid and ask prices are given by $(1+\varepsilon)^{-1}S_t$ and $(1+\varepsilon)S_t$ respectively, where $(S_t)_{t\in[0,T]}$ is a continuous adapted process and $\varepsilon > 0$ is fixed.

We begin with the definition of a CPS, given here in the spirit of Jouini & Kallal (1995):

Definition 2.1. *Let $\varepsilon > 0$. An* ε-consistent price system *is a pair $(\tilde{S}, \tilde{P})$ of a probability $\tilde{P}$ equivalent to P and a $\tilde{P}$-martingale $\tilde{S}$ (adapted to $\mathcal{F}_t$) such that:*

$$\frac{1}{1+\varepsilon} \leq \frac{\tilde{S}_t^i}{S_t^i} \leq 1+\varepsilon \quad \textit{a.s. for all } 1 \leq i \leq d \textit{ and } t \in [0,T] \tag{1}$$

Remark 2.2. We think of a consistent price system as a frictionless price $\tilde{S}$, offering better terms both for buying ($\tilde{S} \leq S(1+\varepsilon)$) and selling ($\tilde{S} \geq S/(1+\varepsilon)$), at all times than the process S with ε transaction costs.

Then it is intuitively obvious (cf. subsection 2.3) that any trading strategy executed at the price $\tilde{S}$ (and without transaction costs) yields a higher payoff than at the bid and ask of S (paying transaction costs).

This observation has two immediate consequences. Firstly, any no-arbitrage condition for the frictionless price $\tilde{S}$ extends to the bid-ask of S. In particular, if S admits an ε-CPS, for some $\varepsilon > 0$,then S does not allow for free lunches (after paying ε-transaction costs) since $\tilde{S}$ has a martingale measure by Definition 2.1.

Secondly, if some capital x hedges a claim X by trading on S with transaction costs, it also hedges X by trading on $\tilde{S}$ without frictions. Although this is also obvious at an intuitive level, see the proof of Theorem 1.3 for a formal argument. Then, as in the usual frictionless case, we obtain an inequality of the form:

$$x \geq E_{\tilde{P}}\left[X\right],$$

where $\tilde{P} \sim P$ is a measure under which $\tilde{S}$ is a martingale.

Thus, the supremum of the right-hand side over the set of all consistent price systems yields a lower bound for the superreplication price. If this bound is large enough, one obtains a static superhedging strategy, and trivial superreplication prices for options.

The previous discussions show that constructing consistent price systems is key to solving the no-arbitrage and superreplication problems under transaction costs by duality methods, and in this section we carry out this program in the one-dimensional case with constant proportional transaction costs. We begin by introducing the basic model of Random Walk with Retirement, which allows to produce a large class of consistent price systems. In the following subsections we employ this construction first to show the existence of consistent price systems (Theorem 1.2), and then to prove the face-lifting result for superreplication prices (Theorem 1.3).

2.1 Random Walk with Retirement

Consider a discrete-time filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_n)_{n\geq 0}, P)$ such that $\mathcal{G}_0$ is trivial and $\vee_n \mathcal{G}_n = \mathcal{G}$.

Definition 2.3. *A* Random Walk with Retirement *is a process* $(X_n)_{n\geq 0}$*, adapted to* $(\mathcal{G}_n)_{n\geq 0}$*, of the form:*

$$X_n = X_0(1+\varepsilon)^{\sum_{i=1}^n R_i}, \qquad n \geq 1 \tag{2}$$

where $\varepsilon > 0$*,* $X_0 \in \mathbb{R}_{++}$*, and the process* $(R_n)_{n\geq 1}$ *has values in* $\{-1, 0, +1\}$ *and satisfies:*

i) $P(R_m = 0 \text{ for all } m \geq n | R_n = 0) = 1$ *for* $n \geq 1$*;*

ii) $P(R_n = x|\mathcal{G}_{n-1}) > 0$ *on* $\{R_{n-1} \neq 0\}$*, for all* $x \in \{-1, 0, +1\}$ *and* $n \geq 1$ *(we set* $\{R_0 \neq 0\} := \Omega$ *as a convention);*

iii) $P(R_n \neq 0 \text{ for all } n \geq 1) = 0$.

In plain English, a Random Walk with Retirement is just a random walk on the geometric grid $(X_0(1+\varepsilon)^k)_{k\in\mathbb{Z}}$, starting at X_0 and "retiring" at the a.s. finite stopping time $\rho = \min\{n \geq 1 : R_n = 0\}$. Note that the filtration $(\mathcal{G}_n)_{n\geq 0}$ is, in general, larger than the one generated by X.

In the following lemma we describe the general form of a probability measure $Q \ll P$ such that X is a Q-martingale. The martingale condition determines the relative weights of probabilities of upward and downward movements. By contrast, at each time we may choose arbitrarily the conditional probability of retirement, denoted by α.

Lemma 2.4. *Let* $(X_n)_{n\geq 0}$ *be a Random Walk with Retirement, and* $(\alpha_n)_{n\geq 1}$ *a predictable (i.e.* α_n *is* $\mathcal{G}_{n-1}$*-measurable) process with values in* $[0,1]$*. Then there exists a unique probability* Q^α *on* $\mathcal{G}$*, such that:*

i) Q^α *is absolutely continuous with respect to* P*;*

ii) X *is a* Q^α*-martingale;*

iii) $Q^\alpha(R_n = 0|\mathcal{G}_{n-1}) = \alpha_n$ *a.s. on* $\{R_{n-1} \neq 0\}$*;*

if and only if α *satisfies:*

$$\lim_{n\to\infty} E\left[1_{\{R_n\neq 0\}}\Pi_{i=1}^n(1-\alpha_i)\right]=0 \tag{3}$$

Vice versa, if $Q\ll P$ *and* X *is a* Q*-martingale, then* $Q=Q^\alpha$ *for some predictable* α *satisfying* (3)*.*

We have $Q\sim P$ *iff* $\alpha_n\in(0,1)$ *a.s. for* $n\geq 1$*.*

Proof. Let α be given, satisfying (3). We construct explicitly the density dQ^α/dP, and then check that it has the desired properties. We define the sequence of sets $(A_n)_{n\geq 0}$ by $A_0=\emptyset$ and $A_n=\{R_n=0\}$ for $n\geq 1$. By *i*) and *iii*) in Definition 2.3, this sequence a.s. increases to Ω, and for $n\geq 1$ we define the $\mathcal{G}_n$-measurable random variable Z_n as:

$$Z_n=\begin{cases}\dfrac{\alpha_n 1_{\{R_n=0\}}}{P(R_n=0|\mathcal{G}_{n-1})}+\dfrac{\lambda_n 1_{\{R_n=-1\}}}{P(R_n=-1|\mathcal{G}_{n-1})}+\dfrac{\mu_n 1_{\{R_n=+1\}}}{P(R_n=+1|\mathcal{G}_{n-1})} & \text{on } \Omega\setminus A_{n-1}\\ 1 & \text{on } A_{n-1}\end{cases} \tag{4}$$

where, on the set $\Omega\setminus A_{n-1}$, the nonnegative $\mathcal{G}_{n-1}$-measurable random variables λ_n,μ_n are uniquely chosen to satisfy the two conditions:

$$E[Z_n|\mathcal{G}_{n-1}]=\alpha_n+\lambda_n+\mu_n=1 \tag{5}$$

$$E[(1+\varepsilon)^{R_n}|\mathcal{G}_{n-1}]=\alpha_n+(1+\varepsilon)^{-1}\lambda_n+(1+\varepsilon)\mu_n=1 \tag{6}$$

We denote by $L_n=\Pi_{i=1}^n Z_i$ and $L=\lim_{n\to\infty}L_n$. Since A_n increases to Ω and $Z_m=1$ on A_m for $m>n$, it follows that $L=L_n$ on A_n and $L>0$ a.s.

L defines a probability density if and only if $E[L]=1$. We have that:

$$E\left[L\right]=E\left[\lim_{n\to\infty}(L1_{A_n})\right]=\lim_{n\to\infty}E\left[L1_{A_n}\right]=\lim_{n\to\infty}E\left[L_n1_{A_n}\right]=1-\lim_{n\to\infty}E\left[L_n1_{\Omega\setminus A_n}\right]$$

where the second equality follows by monotone convergence. Since A_n increases to Ω, and Z_n by definition satisfies $E\left[Z_n1_{\Omega\setminus A_n}|\mathcal{G}_{n-1}\right]=\lambda_n+\mu_n=1-\alpha_n$ on $\Omega\setminus A_{n-1}$, iterating conditional expectations we obtain:

$$E\left[L_n1_{\Omega\setminus A_n}\right]=E\left[\Pi_{i=1}^nZ_i\Pi_{i=1}^n1_{\Omega\setminus A_i}\right]=E\left[\Pi_{i=1}^n(Z_i1_{\Omega\setminus A_i})\right]=E\left[1_{\Omega\setminus A_n}\Pi_{i=1}^n(1-\alpha_i)\right] \tag{7}$$

Thus, $E[L]=1$ if and only if:

$$0=\lim_{n\to\infty}E\left[1_{\Omega\setminus A_n}\Pi_{i=1}^n(1-\alpha_i)\right] \tag{8}$$

which is (3). Then the probability measure $Q^\alpha(A)=E\left[L1_A\right]$ satisfies *ii*) and *iii*) in view of (5) and (6).

Conversely, for any $Q\ll P$ such that X is a Q-martingale define $L_n:=E[dQ/dP|\mathcal{G}_n]$ and $Z_n:=1_{\{L_{n-1}\neq 0\}}L_n/L_{n-1}$. The martingale property of X implies that (5) and (6) hold for some $\alpha_n,\lambda_n,\mu_n,\ n\geq 1$ and the above calculations show that necessarily (3) holds true.

To have $Q^\alpha\sim P$, it is necessary and sufficient that for all $n\geq 1$, $\alpha_n,\lambda_n,\mu_n>0$ a.s. (because of *i*), *ii*) and *iii*) of Definition 2.3), which is equivalent to $\alpha_n\in(0,1)$. $\square$

Corollary 2.5. *Any Random Walk with Retirement admits an equivalent martingale measure.*

Proof. Apply the previous Lemma to $\alpha_n = 1/2$, for example. □

The next lemma shows that, by choosing high probabilities of early retirement, one obtains an equivalent martingale measure with arbitrary integrability conditions. In particular, this implies the existence of equivalent martingale measures for which X is uniformly integrable.

Lemma 2.6. *Let $(X_n)_{n\geq 0}$ be a Random Walk with Retirement. Then for any function $f : \mathbb{R}_{++} \mapsto \mathbb{R}$ and any $\varepsilon > 0$ there exists some $Q^\alpha \sim P$ as in Lemma 2.4 such that $E_{Q^\alpha}[\sup_{n\geq 0} f(X_n)] < \infty$.*

Proof. Observe that the random variables $f(X_n)$ are supported on the double-sided sequence $(s_m)_{m\in\mathbb{Z}}$, where $s_m = f(x_m)$ and $x_m = X_0(1+\varepsilon)^m$. Observe also that we can limit ourselves to the case where:

$$s_m = s_{-m} \quad \text{and} \quad (s_m)_{m\geq 0} \text{ is increasing} \tag{9}$$

up to replacing the function f with $\bar{f}(x_m) = \max_{|l|\leq|m|} f(x_l)$.

We set $M := \sup_{n\geq 0} f(X_n)$ and $Z_n = \sum_{i=1}^n R_i$, so that $X_n = X_0(1+\varepsilon)^{Z_n}$, and by (9) we have that $\{M \geq s_m\} = \{\tau_m < \infty\}$, where $\tau_m = \min\{n : |Z_n| \geq m\}$.

Recalling $i)$ and $iii)$ of Definition 2.3 we know that actually $M = \max_{n\geq 0} f(X_n)$. Using summation by parts, we have that:

$$E_{Q^\alpha}[M] = \sum_{m=0}^{\infty} s_m Q^\alpha(M = s_m) = s_0 + \sum_{m=1}^{\infty} (s_m - s_{m-1}) Q^\alpha(M \geq s_m)$$

Also, the event that M has reached the value s_m is included in the event that the value s_{m-1} has been reached by M and retirement did not occur immediately after. Formally, we have that:

$$Q^\alpha(\tau_m < \infty | \tau_{m-1} < \infty) \leq Q^\alpha(R_{\tau_{m-1}+1} \neq 0 | \tau_{m-1} < \infty) = 1 - \alpha_{\tau_{m-1}} =: \delta_m$$

whence the formula:

$$Q^\alpha(\tau_m < \infty) \leq \Pi_{k=1}^m \delta_k =: \eta_m \tag{10}$$

Thus, we obtain that:

$$\sum_{m=1}^{\infty} (s_m - s_{m-1}) Q^\alpha(M \geq s_m) \leq \sum_{m=1}^{\infty} (s_m - s_{m-1}) \eta_m \tag{11}$$

To have a convergent series, it is thus sufficient to choose η_m small enough, so that $(s_m - s_{m-1})\eta_n < 2^{-m}$. Since $\eta_m = \Pi_{k=1}^m \delta_k$, this is equivalent to recursively choosing δ_m small, or equivalently $\alpha_{\tau_{m-1}}$ close to one. To complete the definition of α outside the sequence of stopping times $(\tau_m)_{m\geq 1}$, we can choose an arbitrary value such as $\alpha = 1/2$ outside $\cup_{n=1}^{\infty} [\![\tau_n]\!]$. □

The next lemma will be needed in section 2.3 to characterize superreplication prices. It shows that, by stopping, we can obtain an $Q^\alpha \ll P$ such that X_∞ is Q^α-a.s. concentrated on any prescribed values u and v below and above X_0.

Lemma 2.7. *Let the filtered probability space and $(R_n)_{n\geq 1}$ be as in Definition 2.3. Let $0 < u < S_0 < v$ be given. For all ε_0 there is $\varepsilon < \varepsilon_0$ and $Q^\alpha \ll P$ such that the corresponding ε-Random Walk with Retirement starting at some $X_0 \in (S_0/(1+\varepsilon), S_0(1+\varepsilon))$ is such that X is a Q^α-martingale, and*

$$Q^\alpha(X_\infty = v) = \frac{X_0 - u}{v - u}, \qquad Q^\alpha(X_\infty = u) = \frac{v - X_0}{v - u} \tag{12}$$

Proof. For $\varepsilon > 0$ small enough, we obtain two integers $j < 0 < k$ such that $u = X_0(1+\varepsilon)^j$, $v = X_0(1+\varepsilon)^k$ and $X_0 \in (S_0/(1+\varepsilon), S_0(1+\varepsilon))$. We define the stopping time:

$$\tau = \min\{n : X_n = u \text{ or } X_n = v\}$$

and set $\alpha_i = 0$ if $i \leq \tau$ and $\alpha_i = 1$ otherwise, which trivially satisfies (3). Apply the construction of Lemma 2.4. We get that X is a bounded martingale under Q^α, and the optional sampling theorem yields:

$$X_0 = E_{Q^\alpha}[X_\tau] = E_{Q^\alpha}[X_\infty] = Q^\alpha(X_\infty = u)u + Q^\alpha(X_\infty = v)v$$

This, combined with $Q^\alpha(X_\infty = u) + Q^\alpha(X_\infty = v) = 1$, implies the claim. □

2.2 Consistent Price Systems

We now employ the previous construction to prove the existence of consistent price systems. We construct an increasing sequence of stopping times at which the process S behaves like a "Random Walk with Retirement"; the conditional full support assumption is key to make this construction possible. Martingale measures for a Random Walk with Retirement are obtained by specifying arbitrarily the probability of retirement as in Lemma 2.4 above. Then the Q-martingale $\tilde{S}$ is defined as the continuous-time martingale determined by the terminal value of the random walk with retirement.

For technical reasons, we state a formally stronger version of the conditional full support condition in terms of stopping times.

Definition 2.8. *Let τ be a stopping time of the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Let us define $S_t := S_T$ for $t > T$ and let $\mu^\tau(\cdot,\omega)$ be (a regular version of) the $\mathcal{F}_\tau$-conditional law of the $C[0,T]$-valued random variable $(S_{\tau+t})_{t\in[0,T]}$.*

We say that the strong conditional full support condition *(SCFS) holds true if, for each $[0,T]$-valued stopping time τ and for almost all $\omega \in \{\tau < T\}$, we have*

$$\mu^\tau(\{g \in C^+_{S_{\tau(\omega)}}[0,T] : \sup_{s\in[0,T-\tau(\omega)]} |f(s) - g(s)| < \eta\}, \omega) > 0,$$

for all $f \in C^+_{S_{\tau(\omega)}}[0, T-\tau(\omega)]$ and for all $\eta > 0$.

Translated into prose, this property means that for all τ,

$$\operatorname{supp} P(S|_{[\tau,T]}|\mathcal{F}_\tau) = C^+_{S_\tau}[\tau, T] \text{ a.s.,} \tag{SCFS}$$

i.e. the conditional full support condition (CFS) holds also with respect to stopping times, while it was formulated in terms of deterministic times in Definition 1.1 above.

The conditions (SCFS) and (CFS) are, in fact, equivalent. The precise formulation of this idea is somewhat technical, thus the proof of the next Lemma is postponed to the Appendix.

Lemma 2.9. *The conditional full support condition* (CFS) *implies the strong conditional full support condition* (SCFS)*, hence they are equivalent.* □

We now present the proof of Theorem 1.2 in dimension one and under the above (SCFS) hypothesis. In this case the arguments are – hopefully – transparent and intuitive.

Proof of Theorem 1.2. We may suppose $\varepsilon \in (0, 1)$ For any such ε, we associate to the process $(S_t)_{t\in[0,T]}$ a "random walk with retirement" as follows. We define the increasing sequence of stopping times:

$$\tau_0 = 0, \qquad \tau_{n+1} = \inf\left\{ t \geq \tau_n : \frac{S_t}{S_{\tau_n}} \notin \left((1+\varepsilon)^{-1}, 1+\varepsilon\right) \right\} \wedge T \tag{13}$$

For $n \geq 1$ we set:

$$R_n = \begin{cases} \operatorname{sign}(S_{\tau_n} - S_{\tau_{n-1}}) & \text{if } \tau_n < T \\ 0 & \text{if } \tau_n = T \end{cases}$$

Remember from the previous section the Random Walk with Retirement $(X_n)_{n\geq 0}$:

$$X_n = X_0(1+\varepsilon)^{\sum_{i=1}^n R_i}$$

adapted to the filtration $(\mathcal{G}_n)_{n\geq 0}$, where $\mathcal{G}_n = \mathcal{F}_{\tau_n}$. To check the properties in Definition 2.3, observe that *i)* is trivial, while *iii)* follows from the continuity of paths. Furthermore, the (CFS) condition implies *ii)* by Lemma 5.1. Note that $X_n = S_{\tau_n}$ on $\{\tau_n < T\}$.

By Lemma 2.6, there exists some $Q^\alpha \sim P$ on $\mathcal{G} = \vee_n \mathcal{G}_n$, such that $E_{Q^\alpha}[\sup_{n\geq 0} X_n] < \infty$. Thus, X is a uniformly integrable $(Q^\alpha, (\mathcal{G}_n)_{n\geq 0})$-martingale and is closed by its terminal value X_∞. Define:

$$\tilde{S}_t := E_{Q^\alpha}[X_\infty|\mathcal{F}_t], \quad t \in [0, T].$$

Fix $0 \leq t \leq T$ and define the random times $\sigma = \max\{\tau_n : \tau_n \leq t\}$ and $\tau = \min\{\tau_n : \tau_n > t\}$ and observe that τ is a stopping time. We have, by definition:

$$(1+\varepsilon)^{-1} \leq \frac{S_t}{S_\sigma}, \frac{S_\tau}{S_\sigma} \leq 1+\varepsilon \quad \text{ a.s. for all } t \in [0, T]$$

and therefore we obtain:

$$(1+\varepsilon)^{-2} \le \frac{S_\tau}{S_t} \le (1+\varepsilon)^2 \quad \text{a.s. for all } t \in [0,T].$$

By construction, $\tilde{S}_{\tau_n} = X_n$ and $S_{\tau_n} = X_n$ on $\{\tau_n < T\}$, for all $n \ge 0$. On $\{\tau_n = T\}$ we have the estimate:

$$(1+\varepsilon)^{-1} \le \frac{\tilde{S}_{\tau_n}}{S_{\tau_n}} \le (1+\varepsilon),$$

for all $n \ge 0$.

Then the optional sampling theorem implies that

$$\frac{\tilde{S}_t}{S_t} = \frac{E_{Q^\alpha}[\tilde{S}_\tau | \mathcal{F}_t]}{S_t} = E_{Q^\alpha}\left[\left. \frac{\tilde{S}_\tau}{S_\tau} \frac{S_\tau}{S_t} \right| \mathcal{F}_t \right],$$

and therefore:

$$(1+\varepsilon)^{-3} \le \frac{\tilde{S}_t}{S_t} \le (1+\varepsilon)^3 \quad \text{a.s. for all } t \in [0,T],$$

which completes the proof, up to the passage to a smaller ε. ☐

2.3 Superreplication

We now prove the “face-lifting” Theorem 1.3, which characterizes superreplication prices of European options in terms of the concave envelopes of their payoffs evaluated at the current price of the underlying asset.

We first need some definitions and notation on trading strategies.

Definition 2.10. *A* trading strategy *is a predictable* $\mathbb{R}$*-valued process* $\theta = (\theta_t)_{t\in[0,T]}$ *such that* $\theta_0 = \theta_T = 0$ *and*

$$\sup_{0\le t_0 \le \ldots \le t_n = T} \sum_{i=1}^{n} |\theta_{t_i} - \theta_{t_{i-1}}| < \infty \text{ a.s.} \tag{14}$$

We may then define the predictable increasing process $\mathrm{Var}_s(\theta)$ by

$$\mathrm{Var}_s(\theta) = \sup_{0\le t_0 \le \ldots \le t_n = s} \sum_{i=1}^{n} |\theta_{t_i} - \theta_{t_{i-1}}|, \text{ for all } 0 \le s \le T.$$

We call $(\mathrm{Var}_s(\theta))_{0\le s\le T}$ the total variation process of θ.

Definition 2.11. *Given a trading strategy* θ*, let us define*

$$V(\theta) := \int_0^T \theta_t dS_t - \varepsilon \int_0^T S_t d\mathrm{Var}_t(\theta), \tag{15}$$

and, for $t \in [0,T]$, we introduce the random variables $V_t(\theta)$ as

$$V_t(\theta) = V(\theta 1_{(0,t)}), \tag{16}$$

so that $V(\theta) = V_T(\theta)$.

*We call the process θ M-*admissible *for some $M \in \mathbb{R}$ if for all $t \in [0,T]$*

$$V_t(\theta) \geq -M$$

almost surely. The set of all such strategies is denoted by $\mathcal{A}_M$. We call $\mathcal{A} = \cup_{M>0}\mathcal{A}_M$, the class of admissible strategies.

Remark 2.12. First of all, let us note that (14) implies that the integrals in (15) are a.s. well-defined, in a pointwise Riemann-Stieltjes sense. Indeed, for the second integral it suffices to observe that S_t has continuous, and therefore bounded, trajectories. As regards the first integral, it suffices to use partial integration and apply once more the previous argument.

Remark 2.13. The random variable $V(\theta)$ clearly has the interpretation of the final gain or loss when applying the trading strategy θ of holding θ_t units of stock at time t: during the infinitesimal interval $[t, t+dt]$ the value of the position in stock (without considering transaction costs) changes by $\theta_t dS_t$, while one has to pay $\varepsilon S_t d\mathrm{Var}_t(\theta)$ transaction costs. The condition $\theta_0 = \theta_T = 0$ corresponds to the requirement that we start and end without a position in the risky asset. Similarly, the term $1_{(0,t)}$ in the definition of $V_t(\theta)$ corresponds to the requirement that we take into account the liquidation cost at time t to determine the value $V_t(\theta)$ of the portfolio at the given instant.

Remark 2.14. In the spirit of Sin (1996) or Yan (1998) in the frictionless case, and Kabanov (1999) in the case of transaction costs, we can also define a more general notion of admissibility. We say that the process θ is M-admissible in the numeraire-free sense if

$$V_t(\theta) \geq -M(1 + S_t).$$

In other words, a trading strategy θ is M-admissible in the numéraire free sense if, by holding M units of the bond *and* the stock, we make sure that the portfolio formed by the trading strategy θ and these two static positions has non-negative liquidation value at all times $t \in [0,T]$.

Clearly, this definition leads to larger classes $\mathcal{A}_M^{nf}$ and $\mathcal{A}^{nf}$ of admissible trading strategies "in the numeraire-free sense". It can be checked that these strategies coincide with the admissible strategies in the sense of Campi & Schachermayer (2006), when we restrict their framework to the present setting.

The name "numéraire free" stems from the fact that this notion of admissibility is invariant when performing a change of numéraire between bond and stock.

It is shown in Yan (1998) for the frictionless case and in Guasoni, Rásonyi & Schachermayer (2007) for the case of transaction costs that the difference of the above two notions of admissibility corresponds to the difference of requiring the martingale or local martingale condition in Definition 1 above.

In the present paper we formulate our results in terms of the notion of admissibility as given in Definition 2.11. The reader can check, however, that Theorem 1.3 still holds true if we choose in Definition 2.15 below the more general concept of *numéraire-free admissibility.* Our results in the present article provide a wide class of models where such price systems exist and hence the above-mentioned hedging theorems are applicable.

Definition 2.15. *In the setting of Definitions 2.10 and 2.11, consider a claim* $X \in L^0(\mathcal{F}_T, P)$ *and* $\varepsilon > 0$*. We define:*

i) the superreplication price*:*

$$p_\varepsilon(X) = \inf\{x : x + V(\theta) \geq X \text{ for some } \theta \in \mathcal{A}\}$$

ii) the static superreplication price*:*

$$p_\varepsilon^{st}(X) = \inf\{x : x + V(\alpha 1_{(0,T)}) \geq X \text{ for some } \alpha \in \mathbb{R}\}$$

iii) the asymptotic superreplication prices*:*

$$p_0(X) = \lim_{\varepsilon \downarrow 0} p_\varepsilon(X) \qquad p_0^{st}(X) = \lim_{\varepsilon \downarrow 0} p_\varepsilon^{st}(X)$$

Remark 2.16. It is clear that $p_\varepsilon^{st}(X) \geq p_\varepsilon(X)$, and hence $p_0^{st}(X) \geq p_0(X)$.

We can now proceed with the proof of Theorem 1.3. In view of the previous discussion, this theorem can be understood as follows: even considering the large class of admissible trading strategies $\mathcal{A}$, we cannot superreplicate a European contingent claim X at a lower cost than using only static hedges as in *ii*) of Definition 2.15.

Proof of Theorem 1.3. We first introduce the set of "absolutely continuous ε-consistent price systems":

$$\mathcal{Z}_\varepsilon = \{(\tilde{S}, \tilde{P}) : \ \tilde{P} \ll P, \ \tilde{S} \text{ is a } \tilde{P}\text{-martingale}, 1 - \varepsilon \leq \frac{\tilde{S}_t}{S_t} \leq 1 + \varepsilon, \ t \in [0,T]\}.$$

Note the minor deviation from (1): in this section estimations are simpler if we take $1 - \varepsilon$ instead of $1/(1+\varepsilon)$. It will become clear that this causes no problems since we let ε tend to 0.

Consider the European claim $g(S_T)$, the initial capital x and a superreplicating admissible strategy $\theta \in \mathcal{A}$ such that $x + V(\theta) \geq g(S_T)$ a.s.

If $(\tilde{S}, \tilde{P}) \in \mathcal{Z}_\varepsilon$, we obtain

$$\begin{aligned} \int_0^T \theta_t d\tilde{S}_t = - \int_0^T \tilde{S}_t d\theta_t \quad &= \\ V_T(\theta) + \varepsilon \int_0^T S_t d\mathrm{Var}_T(\theta) + \int_0^T (S_t - \tilde{S}_t) d\theta_t \quad &\geq \quad V_T(\theta), \end{aligned}$$

and hence:

$$g(S_T) \le x + V(\theta) \le x + \int_{[0,T]} \theta_t d\tilde{S}_t.$$

Since the right-hand side is a supermartingale by the admissibility of θ (it is uniformly bounded from below), it follows that

$$p_\varepsilon(g(S_T)) \ge \sup_{\tilde{P}\in\mathcal{P}_\varepsilon} E_{\tilde{P}}[g(S_T)], \tag{17}$$

where

$$\mathcal{P}_\varepsilon := \{\tilde{P} \ll P : \text{ there exists } \tilde{S} \text{ such that } (\tilde{S},\tilde{P}) \in \mathcal{Z}_\varepsilon\}.$$

Thus, to prove the theorem it is sufficient to show that:

$$\lim_{\varepsilon\downarrow 0} p_\varepsilon(g(S_T)) \ge \lim_{\varepsilon\downarrow 0} \sup_{\tilde{P}\in\mathcal{P}_\varepsilon} E_{\tilde{P}}[g(S_T)] \ge \hat{g}(S_0) \ge p_0^{st}(g(S_T)) \ge p_0(g(S_T)).$$

The first inequality is clear from (17), the second follows from Proposition 2.18 below, the third is a consequence of Proposition 2.17 below, the last one is trivial. By definition, the first quantity equals the last one thus there are equalities everywhere and the theorem is proved. ☐

Proposition 2.17. *Let $g : \mathbb{R}_{++} \to \mathbb{R}$ be a measurable function. Then*

$$p_\varepsilon^{st}(g(S_T)) \le \hat{g}(S_0) + J\varepsilon$$

for a suitable constant $J > 0$.

Proof. The case of $\hat{g}(S_0) = \infty$ being trivial we may assume that $\hat{g}(S_0)$ is finite. By definition of the concave envelope:

$$\hat{g}(S_0) + \hat{g}'_+(S_0)(S_T - S_0) \ge g(S_T),$$

where $\hat{g}'_+$ is the right-hand derivative. Take $\beta \in \mathbb{R}$ satisfying $\beta - \varepsilon|\beta| = \hat{g}'_+(S_0)$. Then we have:

$$\hat{g}(S_0) + 2\varepsilon|\beta|S_0 + \beta(S_T - S_0) - \varepsilon|\beta|(S_T + S_0) \ge g(S_T).$$

Notice that:

$$p_\varepsilon^{st}(g(S_T)) = \inf\{x : x + \alpha(S_T - S_0) - \varepsilon|\alpha|(S_0 + S_T) \ge g(S_T) \text{ for some } \alpha \in \mathbb{R}\},$$

which implies that $p_\varepsilon^{st}(g(S_T)) \le \hat{g}(S_0) + 2\varepsilon|\beta|S_0$. ☐

Proposition 2.18. *Let $g : \mathbb{R}_{++} \mapsto \mathbb{R}$ be lower semicontinuous and bounded from below, and denote by $\hat{g}$ its concave envelope. Then:*

$$\lim_{\varepsilon\downarrow 0} \sup_{\tilde{P}\in\mathcal{P}_\varepsilon} E_{\tilde{P}}[g(S_T)] \ge \hat{g}(S_0). \tag{18}$$

Proof. Let us again suppose that $\hat{g}$ is finite-valued, the case $\hat{g} = \infty$ can be handled in a completely analogous manner.

It is enough to show that for all $\delta > 0$ and for all sufficiently small $\varepsilon > 0$ there is $(\tilde{S}^\varepsilon, Q^\alpha) \in \mathcal{Z}_\varepsilon$ such that:

$$E_{Q^\alpha}[g(S_T)] \geq \hat{g}(S_0) - \delta. \tag{19}$$

By definition of the concave envelope, there exist $0 < u < S_0 < v$ such that:

$$g(u)\frac{v - S_0}{v - u} + g(v)\frac{S_0 - u}{v - u} > \hat{g}(S_0) - \frac{\delta}{3} \tag{20}$$

and, for $\varepsilon > 0$ small enough we have that:

$$g(u)\frac{v - \tilde{S}_0^\varepsilon}{v - u} + g(v)\frac{\tilde{S}_0^\varepsilon - u}{v - u} > \hat{g}(S_0) - \frac{\delta}{2} \tag{21}$$

for all $\tilde{S}_0^\varepsilon \in (S_0(1-\varepsilon), S_0(1+\varepsilon))$.

Since g is lower semicontinuous, we can find, for ε small enough, neighborhoods $U = (u/(1+\varepsilon), u(1+\varepsilon))$ and $V = (v/(1+\varepsilon), v(1+\varepsilon))$ such that $g(U) \geq g(u) - \delta/2$ and $g(V) \geq g(v) - \delta/2$.

Invoking Lemma 2.7 and recalling the construction in the proof of Theorem 1.2, there exists an ε'-consistent price system $(\tilde{S}^\varepsilon, Q^\alpha)$ (i.e. we only have $Q^\alpha \ll P$) such that:

$$Q^\alpha(\tilde{S}_T^\varepsilon = v) = \frac{\tilde{S}_0^\varepsilon - u}{v - u}, \qquad Q^\alpha(\tilde{S}_T^\varepsilon = u) = \frac{v - \tilde{S}_0^\varepsilon}{v - u}, \tag{22}$$

and $\varepsilon' < \varepsilon$ is small enough to be able to use Lemma 2.7 and to guarantee $(\tilde{S}^\varepsilon, Q^\alpha) \in \mathcal{Z}^\varepsilon$.

Thus we obtain:

$$\begin{aligned}
E_{Q^\alpha}\left[g(S_T)\right] =& E_{Q^\alpha}\left[g(S_T)1_{\{\tilde{S}_T^\varepsilon = u\}}\right] + E_{Q^\alpha}\left[g(S_T)1_{\{\tilde{S}_T^\varepsilon = v\}}\right] \\
\geq& \left(g(u) - \frac{\delta}{2}\right)\frac{v - \tilde{S}_0^\varepsilon}{v - u} + \left(g(v) - \frac{\delta}{2}\right)\frac{\tilde{S}_0^\varepsilon - u}{v - u} \\
=& g(u)\frac{v - \tilde{S}_0^\varepsilon}{v - u} + g(v)\frac{\tilde{S}_0^\varepsilon - u}{v - u} - \frac{\delta}{2} \\
>& \hat{g}(S_0) - \delta.
\end{aligned}$$

which is (19), as needed. □

3 The d-dimensional Case

In this section we prove the existence of consistent price systems in the multidimensional case. We consider a market $S = (S^0, S^1, \ldots, S^d)$ with one riskless asset S^0 and d risky assets, based on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$

satisfying the usual assumptions of right-continuity and saturatedness. We also assume that $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$. The riskless asset S^0 is used as numéraire, and therefore assumed to be constantly equal to one. Each risky asset trades at bid and ask prices $S_t^i/(1+\varepsilon)$ and $S_t^i(1+\varepsilon)$ against the numéraire asset S^0.

For a continuous stochastic process $(S_t)_{t\in[u,v]}$ with values in $\mathbb{R}^d_{++}$ and starting at $S_0 = s$, the support is defined as the smallest closed subset $A \subset C_s([u,v],\mathbb{R}^d_{++})$ such that $P(S \in A) = 1$. Given a σ-algebra $\mathcal{G} \subset \mathcal{F}$, the conditional support $\operatorname{supp}(S|\mathcal{G})$ is the support of the regular conditional law of the map $\omega \mapsto S(\omega)$ with respect to $\mathcal{G}$, and therefore it is a random set defined up to a $\mathcal{G}$-null set.

Assumption 3.1. *Let $(S_t)_{t\in[0,T]}$ be a $\mathbb{R}^d_{++}$-valued, continuous, adapted process satisfying the conditional full support condition:*

$$\operatorname{supp}(S|_{[t,T]}|\mathcal{F}_t) = C^+_{S_t}[t,T] \qquad \textit{a.s. for all } t \in [0,T). \tag{23}$$

In this section we prove the following generalization of Theorem 1.2:

Theorem 3.2. *Let S satisfy Assumption 3.1. Then S admits an ε-consistent price system for all $\varepsilon > 0$.*

In the higher-dimensional case, the random walks arising in our proof of the one-dimensional case are no longer supported on a finite grid, therefore we need a slightly different approach – which is similar in spirit – to construct a consistent price system.

The proof given below is based on a conditional version of the classical Esscher transform, which yields a recursive change of measure, without incurring too many technicalities related to measurable selections. This method was used by Rogers (1994) to prove the main result of Dalang, Morton & Willinger (1990). Compare also *b*) below to Theorem 3 of Jacod & Shiryaev (1998).

In the next lemma, X represents the price increment over a given period, $\mathcal{G}$ and $\mathcal{F}$ the initial and final σ-algebras, respectively, A the event where the process is already retired at the beginning of the period. For a subset $W \subset \mathbb{R}^d$, we denote by $\operatorname{conv} W$ the convex hull of W, by $\operatorname{int} W$ the interior of W, and by $\bar{B}_\delta$ the closed ball in $\mathbb{R}^d$ centered at the origin and with radius δ.

Lemma 3.3. *Let $\mathcal{G} \subset \mathcal{F}$ be two σ-algebras, $A \in \mathcal{G}$, $X \in L^\infty(\Omega,\mathcal{F},P;\mathbb{R}^d)$ and $\eta \in L^0(\Omega,\mathcal{G},P;\mathbb{R}_{++})$ such that:*

a) $X = 0$ *on* A*;*

b) $0 \in \operatorname{int}\operatorname{conv}\operatorname{supp}(X|\mathcal{G})(\omega)$ *for almost all* $\omega \in \Omega \setminus A$*;*

c) $P(X = 0|\mathcal{G}) > 0$ *a.s.*

Then there exists $Z \in L^1(\mathcal{F},\mathbb{R}_{++})$ such that we have almost surely:

i) $E[Z|\mathcal{G}] = 1$*;*

ii) $E[ZX|\mathcal{G}] = 0$*;*

iii) $E[Z|X|^2|\mathcal{G}] \leq \eta$*;*

iv) $E[ZI_{\{X\neq 0\}}|\mathcal{G}] \leq \eta$*.*

Proof. We can restrict our attention to the set $\Omega \setminus A$, since setting $Z = 1$ on A trivially satisfies $i) - iv)$. We denote by $\mu(\omega, \cdot)$ the regular conditional law of X with respect to $\mathcal{G}$. As X is bounded,

$$\theta \mapsto \phi(\theta, \omega) = \int e^{\theta\cdot X} d\mu(\omega, \cdot) < \infty \quad \text{for all } \theta \in \mathbb{R}^d, \text{ and a.e. } \omega$$

Since $0 \in \operatorname{int}\operatorname{conv}\operatorname{supp}(X|\mathcal{G}) = \operatorname{int}\operatorname{conv}\operatorname{supp}\mu(\omega,\cdot)$, we can find a $\mathcal{G}$-measurable function $\delta : \Omega \mapsto \mathbb{R}_{++}$ such that $\operatorname{conv}\operatorname{supp}(X|\mathcal{G}) = \operatorname{conv}\operatorname{supp}\mu(\omega,\cdot) \supset \bar{B}_{\delta(\omega)}$.For instance, we can define δ as:

$$1/\delta(\omega) = \min\{n : \bar{B}_{1/n} \subset \operatorname{conv}\operatorname{supp}\mu(\omega,\cdot)\}.$$

Denoting by $\mathcal{S}$ the set of $\mathcal{G}$-measurable functions with values on the unit sphere of $\mathbb{R}^d$, we have that $\operatorname{conv}\operatorname{supp}(v\cdot X|\mathcal{G}) \supset [-\delta, \delta]$ for any $v \in \mathcal{S}$, so $P(v \cdot X > \delta/2|\mathcal{G}) > 0$ a.s. In addition, we claim that (compare to Lemma 2.6 in Schachermayer (1992)):

$$\nu = \operatorname{essinf}_{v\in\mathcal{S}} P(v\cdot X > \delta/2|\mathcal{G}) > 0 \quad \text{a.s.}$$

By contradiction, if $P(\nu = 0) > 0$, there would be a $\mathcal{G}$-measurable sequence $v_n \in \mathcal{S}$ achieving the essential infimum (as this set of functions is easily seen to be directed downwards), and by the compactness of the unit sphere we could choose a $\mathcal{G}$-measurable random subsequence $n_k(\omega)$ such that v_{n_k} converges almost surely to some $v' \in \mathcal{S}$ (see e.g. Lemma 2 of Kabanov & Stricker (2001)). Then $P(v' \cdot X \geq \delta/2|\mathcal{G}) = 0$ with positive probability, which is absurd. Thus we have, for all nonzero $\theta \in \mathbb{R}^d$:

$$\phi(\theta,\omega) = \int e^{\theta\cdot X} d\mu(\omega,\cdot) \geq e^{|\theta|\delta/2}\mu\left(\omega, \frac{\theta}{|\theta|}\cdot X > \delta/2\right) \geq \nu e^{|\theta|\delta/2}$$

and therefore $\lim_{|\theta|\to\infty}\phi(\theta,\omega) = \infty$ almost surely. In particular, the function $\phi(\cdot,\omega)$ admits a unique (by the strict convexity of $\theta \mapsto \phi(\theta,\omega)$) minimum $\theta^*(\omega)$, which solves $\nabla\phi(\theta^*(\omega),\omega) = 0$ and hence is $\mathcal{G}$-measurable. Then dominated convergence implies that:

$$0 = \nabla\phi(\theta^*(\omega),\omega) = \int X e^{\theta^*(\omega)\cdot X} d\mu(\omega,\cdot) = E[Xe^{\theta^*\cdot X}|\mathcal{G}]$$

and $Z' = e^{\theta^*\cdot X}/E[e^{\theta^*\cdot X}|\mathcal{G}]$ satisfies properties $i) - ii)$. To obtain also $iii)$ and $iv)$, we rescale, conditional on $\mathcal{G}$, the relative weight of the events $\{X = 0\}$ and $\{X \neq 0\}$, which does not affect the expectation of X. Formally, we denote by $Y := P(X = 0|\mathcal{G})$ and set $Z = \lambda Z' + \mu 1_{\{X=0\}}$, where the $\mathcal{G}$-measurable positive functions λ and μ will be chosen to satisfy $i)$:

$$\lambda + \mu Y = 1. \tag{24}$$

It is clear that $E[Z'|X|^2|\mathcal{G}] < \infty$ almost surely. Hence we can choose a $\mathcal{G}$-measurable $\lambda > 0$ so small that not only both $\lambda E[Z'|X|^2|\mathcal{G}] \leq \eta$ and $\lambda \leq \eta$ hold but also (24) is satisfied for an appropriate $\mathcal{G}$-measurable $\mu > 0$. This completes the proof. □

Proof of Theorem 3.2. We define the increasing sequence of stopping times:

$$\tau_0 = 0, \quad \tau_{n+1} = \inf\left\{t \geq \tau_n : \frac{S_t^i}{S_{\tau_n}^i} \notin ((1+\varepsilon)^{-1}, 1+\varepsilon) \text{ for some } 1 \leq i \leq d\right\} \wedge T.$$

Path continuity implies that $\tau_n = T$ a.s. for all $n \geq \bar{n}(\omega)$, hence the events $A_n = \{\tau_n = T\}$ increase to Ω as n grows to infinity. For $n \geq 1$ we set:

$$\Delta_n := (S_{\tau_n} - S_{\tau_{n-1}}) 1_{\Omega \setminus A_n}.$$

Obviously, $\operatorname{supp}(\Delta_n|\mathcal{F}_{\tau_{n-1}}) = \{0\}$ on A_{n-1}. Assumption 3.1 implies that $0 \in \operatorname{int}\operatorname{conv}\operatorname{supp}(\Delta_n|\mathcal{F}_{\tau_{n-1}})$ and $P(\Delta_n = 0|\mathcal{F}_{\tau_{n-1}}) > 0$ on $\Omega \setminus A_{n-1}$, this fact is shown in Lemma 5.2 below.

For $n \geq 1$ we now recursively apply Lemma 3.3 to:

$$A = A_{n-1}, \qquad \mathcal{G} = \mathcal{F}_{\tau_{n-1}}, \qquad X = \Delta_n, \qquad \eta := 2^{-n},$$

obtaining a sequence of strictly positive random variables $(Z_n)_{n\geq 1}$ such that, almost surely:

i) $E[Z_n|\mathcal{F}_{\tau_{n-1}}] = 1$;

ii) $E[\Delta_n Z_n|\mathcal{F}_{\tau_{n-1}}] = 0$;

iii) $E[Z_n|\Delta_n|^2|\mathcal{F}_{\tau_{n-1}}] \leq 2^{-n}$;

iv) $E\left[Z_n 1_{\{\Delta_n \neq 0\}}|\mathcal{F}_{\tau_{n-1}}\right] \leq 2^{-n}$.

From the construction it follows that $Z_n = 1$ on A_m for all $n \geq m+1$. Therefore, denoting by $L_n = \prod_{i=1}^n Z_i$, we have that:

$$L = \lim_{n\to\infty} L_n = \prod_{n=1}^{\infty} Z_n > 0 \quad \text{a.s.}$$

because $A_n \uparrow \Omega$.

We obtain using iv) above that:

$$\begin{aligned} E[L] = E\left[L \lim_{n\to\infty} 1_{A_n}\right] = \lim_{n\to\infty} E[L 1_{A_n}] &= \\ 1 - \lim_{n\to\infty} E\left[L_n 1_{\Omega\setminus A_n}\right] \geq 1 - \lim_{n\to\infty} 2^{-n} &= 1. \end{aligned}$$

Hence the density L induces the probability measure $Q(A) := E[L 1_A]$ on $\vee_{t\in[0,T]}\mathcal{F}_t = \mathcal{F}$. Now we define the discrete time process $(M_n)_{n\geq 0}$ as:

$$M_n = S_0 + \sum_{l=1}^n \Delta_l.$$

By construction, M is a Q-martingale with respect to the filtration $(\mathcal{F}_{\tau_n})_{n\geq 0}$. We have $M_n = S_{\tau_n}$ a.s. on $\{\tau_n < T\}$ while on $\{\tau_n = T\}$:

$$(1+\varepsilon)^{-1} \leq \frac{M_n^i}{S_{\tau_n}^i} = \frac{S_{\tau_{n-1}}^i}{S_T^i} \leq 1+\varepsilon \quad \text{for all } t \in [0,T], 1 \leq i \leq d, n \geq 0.$$

Hence we always have that:

$$(1+\varepsilon)^{-1} \leq \frac{M_n^i}{S_{\tau_n}^i} \leq 1+\varepsilon \quad \text{for all } t \in [0,T], 1 \leq i \leq d, n \geq 0. \tag{25}$$

We claim that M is uniformly integrable, in fact, it is bounded in L^2. Indeed, we have that:

$$\begin{aligned} E_Q[|M_n|^2] = |S_0|^2 + \sum_{l=1}^{n} E_Q|\Delta_l|^2 &= |S_0|^2 + \sum_{l=1}^{n} E[L_{l-1}E[Z_l|\Delta_l|^2|\mathcal{F}_{\tau_{l-1}}]] \\ &\leq |S_0|^2 + \sum_{l=0}^{\infty} 2^{-l} < \infty. \end{aligned}$$

Denoting by $\sigma = \max\{\tau_n : \tau_n \leq t\}$ and $\tau = \min\{\tau_n : \tau_n > t\}$, since $t \in [\sigma, t] \cap [\sigma, \tau]$ we have:

$$(1+\varepsilon)^{-1} \leq \frac{S_t^i}{S_\sigma^i}, \frac{S_\tau^i}{S_\sigma^i} \leq 1+\varepsilon \qquad \text{a.s. for all } t \in [0,T], 1 \leq i \leq d,$$

and therefore:

$$(1+\varepsilon)^{-2} \leq \frac{S_\tau^i}{S_t^i} \leq (1+\varepsilon)^2 \qquad \text{a.s. for all } t \in [0,T], 1 \leq i \leq d, \tag{26}$$

We define the martingale $\tilde{S}_t = E_Q\left[M_\infty|\mathcal{F}_t\right],\ t \in [0,T]$, which satisfies by construction $\tilde{S}_{\tau_n} = M_n$ for all $n \geq 0$. Hence, combining (25) and (26) with:

$$\frac{\tilde{S}_\tau^i}{S_t^i} = \frac{\tilde{S}_\tau^i}{S_\tau^i}\frac{S_\tau^i}{S_t^i}$$

it follows that:

$$(1+\varepsilon)^{-3} \leq \frac{\tilde{S}_\tau^i}{S_t^i} \leq (1+\varepsilon)^3 \qquad \text{a.s. for all } t \in [0,T], 1 \leq i \leq d$$

Since $\tilde{S}_t = E[\tilde{S}_\tau|\mathcal{F}_t]$ by optional sampling, the theorem follows up to the passage to a smaller ε. ☐

4 Applications

To illustrate the scope of our results, we now present some important classes of models where the conditions of Theorem 1.2 can be checked.

4.1 Markov Processes

Example 4.1. Consider a (homogeneous) a.s. continuous Markov process $S_t, t \in [0,T]$ with state space $\mathbb{R}^d_{++}$ such that for all $x \in \mathbb{R}^d_{++}$, starting from $S_0 = x$, the process has full support on $C^+_x[0,T]$. Then the Markov property immediately implies the conditional full support condition:

$$\text{supp} P(S|_{[v,T]}|\mathcal{F}_v) = \text{supp} P(S|_{[v,T]}|S_v) = C^+_{S_v}[v,T], \ 0 \le v \le T.$$

Methods to show that a diffusion process has full support can be found in the seminal work of Stroock & Varadhan (1972) and in Revuz & Yor (1999) (p. 340).

4.2 Fractional Brownian Motion

We now turn to models based on fractional Brownian motion (FBM). Recall that $(X_t)_{t\ge 0}$ is FBM with Hurst parameter $0 < H < 1$ if it is a centered Gaussian process with continuous sample paths and covariance function

$$\Gamma(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

If $H = 1/2$ we recover the standard Brownian motion. For a thorough treatment of FBM, we refer the reader to the monograph of Nualart (2006).

Models of asset prices based on Fractional Brownian Motion have long attracted the interest of researchers for their properties of long range dependence (Lo 1991, Maheswaran & Sims 1993, Cutland, Kopp & Willinger 1995, Willinger, Taqqu & Teverovsky 1999). However, in a frictionless setting it turned out that these models lead to arbitrage opportunities (Delbaen & Schachermayer 1994, Rogers 1997, Salopek 1998, Shiryaev 1998, Dasgupta & Kallianpur 2000, Cheridito 2003), and therefore cannot be meaningfully employed for studying optimal investment and derivatives pricing.

In this paper we show how this situation is completely different as soon as arbitrarily small transaction costs are introduced. Then there exist consistent price systems and hence the duality theory and hedging theorems of Kabanov & Stricker (2002) and Campi & Schachermayer (2006) apply.

The next result improves on Proposition 5.1 of Guasoni (2006), and follows from a similar argument.

Proposition 4.2. *Let* $S_t = \exp\{\sigma X_t + f_t\}$ *where* X_t *is FBM with parameter* $0 < H < 1$ *and* f_t *is a deterministic continuous function. Then* $(S_t)_{t\in[0,T]}$ *satisfies the conditional full support condition* (CFS) *with respect to its (right-continuous and saturated) natural filtration.*

Proof. Let us fix $v \in [0,T]$. It is enough to prove that the conditional law $P(X|_{[v,T]}|\mathcal{F}_v)$ has full support on $C_{X_v}([v,T],\mathbb{R})$ almost surely.

From the representation of Corollary 3.1 in Decreusefond & Üstünel (1999) we know that, for some square-integrable kernel $K_H(t,s)$, one has

$$X_t = \int_0^t K_H(t,s)dW_s, \tag{27}$$

for some Brownian Motion $(W_t)_{t\in[0,T]}$ generating the same filtration as $(X_t)_{t\in[0,T]}$.

It is easily seen, by directly calculating the conditional joint characteristic function of finite-dimensional distributions of X, that for any $v \in [0,T]$ the process $(X_t)_{t\in[v,T]}$ is Gaussian, conditionally on $\mathcal{F}_v$. Its conditional expectation and conditional covariance function are given by

$$\begin{aligned} c_t &:= E[X_t|\mathcal{F}_v] = \int_0^v K_H(t,s)dW_s \quad t \geq v, \\ \tilde{\Gamma}(t,s) &:= \operatorname{cov}_{\mathcal{F}_v}(X_t, X_s) = \int_v^{t\wedge s} K_H(t,u)K_H(s,u)du \quad t,s \geq v. \end{aligned} \tag{28}$$

Observe that $\tilde{\Gamma}(t,s)$ does not depend on ω. Hence, for almost all ω, the law of $(X_t)_{t\in[v,T]}$ conditional on $\mathcal{F}_v$ is equal to the law of $Y_t+c_t(\omega)$ where $(Y_t)_{t\in[v,T]}$ is a centered Gaussian process with continuous paths on $[v,T]$ and with covariance function $\tilde{\Gamma}$. Thus, remembering the kernel representation (27), it suffices to prove that the centered Gaussian process

$$Y_t := \int_v^t K_H(t,s)dW_s, \ t \in [v,T]$$

has full support on $C_0([v,T],\mathbb{R})$.

Theorem 3 in Kallianpur (1971) states that the topological support of a continuous Gaussian process $(Y_t)_{t\in[v,T]}$ is equal to the norm closure of its Reproducing Kernel Hilbert space, defined by:

$$\mathbb{H} := \left\{ f \in C_0([v,T],\mathbb{R}) : \ f(t) = \int_v^t K_H(t,s)g(s)ds, \ \text{for some } g \in L^2[v,T] \right\}.$$

Thus, it is sufficient to show that $\mathbb{H}$ is norm dense in $C_0([v,T],\mathbb{R})$.

To achieve this, we need to recall the Liouville fractional integral operator for any $f \in L^1[a,b]$ and $\gamma > 0$:

$$(I^\gamma_{a+}f)(t) := \frac{1}{\Gamma(\gamma)} \int_a^t f(s)(t-s)^{\gamma-1}ds, \ a \leq t \leq b,$$

and we introduce the kernel operator K_H:

$$(K_H f)(t) := \int_0^t K_H(t,s)f(s)ds, \ f \in L^2[0,T], \ t \in [0,T].$$

We first treat the case $H < 1/2$. In this case we have, by Decreusefond & Üstünel (1999, Theorem 2.1) that:

$$K_H f = I^{2H}_{0+}(s^{1/2-H} I^{1/2-H}_{0+}(s^{H-1/2}f(s))).$$

From now on we assume that all functions on a subset of $\mathbb{R}$ are extended to the whole real line by setting them 0 outside their domain of definition. In the case $v = 0$ we could perform the same calculations as in Guasoni (2006, pages 578–579). For general v the argument needs to be split into two steps.

Lemma 4.3. *If $f \in C_0[v,T]$ then $L_1 f \in C_0[v,T]$, where*

$$(L_1 f)(t) = (I_{0+}^{1/2-H}(s^{H-1/2} f(s)))(t).$$

Moreover, $L_1 : C_0[v,T] \to C_0[v,T]$ is continuous and has a dense range (with respect to the uniform norm).

Proof. Clearly, $L_1 f$ is a continuous function and $(L_1 f)(0) = 0$. The operator is continuous by the estimate

$$\|L_1 f - L_1 g\|_\infty \leq v^{H-1/2} \int_0^T (T-s)^{-H-1/2} ds \|f-g\|_\infty.$$

Recall the identity for $a, b > 0$

$$\int_0^t (t-u)^{a-1} u^{b-1} du = C(a,b) t^{a+b-1},$$

where $C(a,b) \neq 0$ is a constant. Defining for a fixed $\alpha > 0$

$$g(s) := 1_{[v,T]} \frac{(s-v)^\alpha}{s^{H-1/2}},$$

we obtain for $t \in [v,T]$:

$$\begin{aligned}(L_1 g)(t) = \int_v^t (t-s)^{-H-1/2} g(s) s^{H-1/2} ds &= \int_v^t (t-s)^{-H-1/2} (s-v)^\alpha ds = \\ \int_0^{t-v} u^\alpha (t-v-u)^{-H-1/2} du &= C(\alpha+1, 1/2-H)(t-v)^{\alpha-H+1/2}.\end{aligned}$$

Varying α, we find that $(t-v)^n \in \mathrm{Im}(L_1)$ for $n \geq 1$ and the Stone-Weierstrass theorem guarantees that $\mathrm{Im}(L_1)$ is dense in $C_0[v,T]$. □

Lemma 4.4. *If $f \in C_0[v,T]$ then $L_2 f \in C_0[v,T]$, where*

$$(L_2 f)(t) = (I_{0+}^{2H}(s^{1/2-H} f(s)))(t)$$

and $L_2 : C_0[v,T] \to C_0[v,T]$ is continuous and has a dense range.

Proof. The same argument applies, this time we use the estimation

$$\|L_1 f - L_1 g\|_\infty \leq T^{1/2-H} \int_0^T (T-s)^{2H-1} ds \|f-g\|_\infty$$

and the functions

$$g(s) := 1_{[v,T]} \frac{(s-v)^\alpha}{s^{1/2-H}}.$$

□

Since the restriction of K_H to $C_0[v,T]$ is exactly $L_2 \circ L_1$ we may conclude that $K_H : C_0[v,T] \to C_0[v,T]$ has a dense range, and *a fortiori* $\mathbb{H}$ is norm-dense in $C_0[v,T]$.

In the case $H \geq 1/2$ a similar representation holds (Decreusefond & Üstünel 1999, Theorem 2.1):

$$K_H f = I^1_{0+}(s^{H-1/2} I^{H-1/2}_{0+}(s^{1/2-H} f))$$

and the same argument carries over. □

5 Appendix

We gather here a few arguments of rather technical nature. First we show the equivalence of (SCFS) and (CFS) by contradiction: if there is an open set of paths with $\mathcal{F}_\tau$-conditional probability 0 on a set of positive measure then we can find a deterministic time q "close enough" to τ such that the same phenomenon arises, which is absurd by (CFS).

Proof of Lemma 2.9. By contradiction, suppose that for some stopping time τ there exists $A \in \mathcal{F}_\tau$ with $P(A) > 0$ such that for almost all $\omega \in A$ there are $f_\omega \in C^+_1[0,T]$ and $\eta_\omega > 0$ such that $\mu^\tau(B(\omega), \omega) = 0$, where

$$B(\omega) := \{g \in C^+_{S_{\tau(\omega)}(\omega)}[0,T] : \sup_{s \in [0, T-\tau(\omega)]} |g(s) - f_\omega(s) S_{\tau(\omega)}(\omega)| \leq \eta_\omega\}.$$

The measurable selection theorem (see sections III. 44–45 of Dellacherie & Meyer (1978)) enables us to choose $\omega \to (f_\omega, \eta_\omega)$ in an $\mathcal{F}_\tau$-measurable way. Set $K_\omega := \|f_\omega\|_\infty \|1/f_\omega\|_\infty$.

Define now, for each $q \in Q := \mathbb{Q} \cap [0,T)$,

$$A_q := A \cap \{q \geq \tau\} \cap \{\sup_{s \in [\tau, q]} |f(s-\tau) S_\tau - S_s| \leq \frac{\eta}{2K}\} \in \mathcal{F}_q.$$

Clearly, $A = \cup_{q \in Q} A_q$, hence we may fix q such that $P(A_q) > 0$. Define

$$\begin{aligned} C_q &:= \{\sup_{s \in [q,T]} |f(s-\tau) S_\tau - S_s| \leq \eta\}, \\ G_q &:= \left\{ \sup_{s \in [q,T]} \left| \frac{S_q f(s-\tau)}{f(q-\tau)} - S_s \right| \leq \frac{\eta}{2} \right\}, \\ H &:= \{\sup_{s \in [\tau,T]} |f(s-\tau) S_\tau - S_s| \leq \eta\}. \end{aligned}$$

Note that on $A_q \cap G_q$ we have for $q \leq s \leq T$:

$$\begin{aligned} |f(s-\tau) S_\tau - S_s| &\leq \left| f(s-\tau) S_\tau - \frac{S_q f(s-\tau)}{f(q-\tau)} \right| + \left| \frac{S_q f(s-\tau)}{f(q-\tau)} - S_s \right| \leq \\ \frac{f(s-\tau)}{f(q-\tau)} \frac{\eta}{2K} + \frac{\eta}{2} &\leq \eta, \end{aligned}$$

hence $A_q \cap G_q \subset C_q \cap A_q$. As $C_q \cap A_q \subset H \cap A$ and

$$P(H \cap A) = E[1_A E[1_H|\mathcal{F}_\tau]] = \int_\Omega 1_A(\omega)\mu^\tau(B(\omega),\omega)dP(\omega) = 0,$$

we get

$$0 = E[1_{A_q}1_{G_q}] = E[1_{A_q}E[1_{G_q}|\mathcal{F}_q]],$$

which is a contradiction as $P(G_q|\mathcal{F}_q) > 0$ almost surely, by the (CFS) condition. $\square$

Lemma 5.1. *Let S be an $\mathbb{R}_{++}$-valued continuous process satisfying* (CFS) *and let R_n be defined as in the proof of Theorem 1.2. Then $P(R_{n+1} = z|\mathcal{F}_{\tau_n}) > 0$ a.s. on $\{\tau_n < T\}$, for $z = 0, \pm 1$ and $n \geq 0$.*

Proof. We write τ instead of τ_n and set $D := \{\tau < T\}$. Define the (random) function

$$\begin{aligned} f(t) &:= S_\tau \left(1 + \frac{2t\varepsilon}{T-\tau}\right)^z, \quad t \in [0,T] - \tau, \\ f(t) &:= f(T-\tau), \quad t > T - \tau. \end{aligned}$$

Define $\eta := S_\tau \varepsilon/2$. Lemma 2.9 implies that $\mu^\tau(B(\omega),\omega) > 0$ for almost all $\omega \in D$, where $B(\omega)$ is the following set of paths:

$$B(\omega) := \{g \in C^+_{S_{\tau(\omega)}(\omega)}[0,T] : \sup_{s\in[0,T-\tau(\omega)]} |f_\omega(s) - g(s)| \leq \eta(\omega)\}.$$

First take $z = 0$. For $\omega \in D$, paths in $B(\omega)$ hit neither $S_\tau(\omega)(1+\varepsilon)$ nor $S_\tau(\omega)(1+\varepsilon)^{-1}$ on $[0, T-\tau(\omega)]$, so

$$\begin{aligned} P(S_{\tau_{n+1}} = S_{\tau_n}|\mathcal{F}_\tau) = P(\tau_{n+1} = T|\mathcal{F}_\tau) &\geq \\ P(\sup_{s\in[\tau,T]} |f(s-\tau) - S_s| \leq \eta|\mathcal{F}_\tau) = \mu^\tau(B(\omega),\omega) &> 0 \text{ on } D, \end{aligned}$$

and we are done.

If $z = \pm 1$ then for $\omega \in D$, each path in $B(\omega)$ attains $S_\tau(\omega)(1+\varepsilon)^z$ on $[0, \frac{3}{4}(T-\tau(\omega))]$ without attaining $S_\tau(\omega)(1+\varepsilon)^{-z}$, so

$$\begin{aligned} P(S_{\tau_{n+1}} = S_{\tau_n}(1+\varepsilon)^z|\mathcal{F}_\tau) &\geq \\ P(\sup_{s\in[\tau,T]} |f(s-\tau) - S_s| \leq \eta|\mathcal{F}_\tau) = \mu^\tau(B(\omega),\omega) &> 0 \text{ on } D, \end{aligned}$$

as claimed. $\square$

The next lemma is the multidimensional counterpart of Lemma 5.1. It shows that, under Assumption 3.1, there is a positive probability of hitting each face of the surface of the "bid-ask cube" (F in the proof below). In the one-asset case it coincides with two points, while in general it has $d-1$ dimensions. The idea of the proof is to see that paths which move linearly (either up or down) along a single coordinate must hit their respective face with positive probability. Hence there is a neighbourhood on each face which has positive conditional probability, thus the support must contain some point from each of these faces, and therefore the interior of its convex hull contains the origin.

Lemma 5.2. *On the set* $\Omega \setminus A_{n-1}$ *we have, almost surely:*

$$0 \in \operatorname{int}\operatorname{conv}\operatorname{supp}(\Delta_n|\mathcal{F}_{\tau_{n-1}}) \quad \textit{and} \quad P(\Delta_n = 0|\mathcal{F}_{\tau_{n-1}}) > 0. \tag{29}$$

Proof. Take $D := \Omega \setminus A_{n-1}$. We will write τ for τ_{n-1}. Let F be the (random) cube with edges $(S_\tau(1+\varepsilon)^{l(1)}, \ldots, S_\tau(1+\varepsilon)^{l(d)})$, $l \in \{\pm 1\}^d$. Define also its faces $F_{iz} = \mathrm{ri}\{x \in F : \ x^i = (1+\varepsilon)^z\}$, $z \in \{\pm 1\}$, $1 \le i \le d$, where ri stands for relative interior.

Introduce the random functions

$$\begin{aligned} f^i_{iz}(t) &:= S^i_\tau(1 + \frac{2t\varepsilon}{T-\tau})^z, \ t \in [0, T-\tau] \\ f^j_{iz}(t) &:= S^j_\tau, \ j \neq i, \ t \in [0, T-\tau] \\ f_{iz}(t) &:= f^{iz}(T-\tau), \ t > T-\tau \end{aligned}$$

for $z \in \{\pm 1\}$ and $1 \le i \le d$. Set $\eta^{iz} := \varepsilon \min_i S^i_\tau/2$.

We get, just like in the previous proof, that on D, with positive $\mathcal{F}_\tau$-probability, the trajectory S^i_t attains $S^i_\tau(1+\varepsilon)^z$ on $[\tau, \tau + 3/4(T-\tau)]$ while the other coordinates S^j_t, $j \neq i$ remain in the intervals $(S^j_\tau(1+\varepsilon)^{-1}, S^j(1+\varepsilon))$. That is,

$$P(S_{\tau_n} - S_\tau = \Delta_n \in F_{iz}|\mathcal{F}_\tau) > 0 \text{ on } D,$$

for each z, i, which means that the $\mathcal{F}_\tau$-conditional support of Δ_n contains some point of each face of the "bid-ask cube" F, hence its convex hull contains 0 in its interior.

Taking $z = 0$ in the above we get $P(\Delta_n = 0|\mathcal{F}_\tau) > 0$ on D. □

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