

# The Fundamental Theorem of Asset Pricing under Transaction Costs

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**Abstract** This paper proves the Fundamental Theorem of Asset Pricing with transaction costs, when bid and ask prices follow locally bounded càdlàg (right-continuous, left-limited) processes.

The Robust No Free Lunch with Vanishing Risk (RNFLVR) condition for simple strategies is equivalent to the existence of a strictly consistent price system (SCPS). This result relies on a new notion of admissibility, which reflects future liquidation opportunities. The (RNFLVR) condition implies that admissible strategies are predictable processes of finite variation.

The appendix develops an extension of the familiar Stieltjes integral for càdlàg integrands and finite-variation integrators, which is central to modeling transaction costs with discontinuous prices.

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## 1 Introduction

The equivalence between absence of arbitrage and the existence of a positive linear pricing rule is central to Finance, and was pioneered by Ross (1977), in the wake of the Black and Scholes (1973) option pricing model. Dybvig and Ross (1987) referred to this class of results as “The Fundamental Theorem of Asset Pricing” (FTAP), setting the tone for today’s terminology.

Thirty years later, the Fundamental Theorem remains an active research topic for markets with frictions, and a practically oriented reader may regard with skepticism another paper on this subject. Yet, there is a reason for this ongoing interest: the path to the FTAP leads to the natural notions of *admissibility* and *arbitrage*, which are central to all practical applications. This is one of the lessons of the work of Delbaen and Schachermayer (1994) in frictionless markets. The Fundamental Theorem appears to be inevitable.

This paper proves a version of the FTAP in continuous time, for a risky asset driven by a càdlàg and locally bounded bid-ask process. The main result is the equivalence of no-arbitrage type condition, the Robust No Free Lunch with Vanishing Risk (RNFLVR), and the existence of a Strictly Consistent Price System (SCPS), representing a positive linear pricing rule:

**Theorem 1.1** *Let  $(S, \kappa)$  be a pair of càdlàg adapted, locally bounded processes, with  $\kappa$  nonnegative. Then  $(\text{RNFLVR}) \iff (\text{SCPS})$ .*

The (RNFLVR) condition is a robust version of the (NFLVR) condition of Delbaen and Schachermayer (1994), and requires that (NFLVR) holds under an arbitrarily small perturbation of the bid-ask spread. Schachermayer (2004) shows that robustness arises naturally with transaction costs.

An important feature of this paper is the formulation of (RNFLVR) only in terms of *simple* trading strategies. This is in sharp contrast to the frictionless case, where (NFLVR) for *simple* strategies implies the semimartingale property for the asset price  $S$  (Delbaen and Schachermayer 1994, Theorem 7.1). Thus, for any *predictable* strategy  $H$  the gain process  $(H \cdot S)$  becomes a well-defined stochastic integral, and the (NFLVR) for *predictable* strategies implies the existence of an equivalent local martingale measure.

Such a major difference arises, intuitively, by the impact of trading on wealth dynamics. In a frictionless market, only price shocks affect wealth, while trading merely regulates their magnitude. This allows wide flexibility in trading patterns, leading to the large class of predictable strategies. With transaction costs, trading has always a direct negative impact on wealth, and this restricts the class of allowable strategies.

The departure point of this paper is the concept of *simple* strategy, an almost surely finite sequence of left transactions, occurring immediately before a stopping time, and right transactions, occurring immediately afterwards (Definition 3.5). Rásonyi (2003) first discovered that right-continuous strategies are not sufficient if prices may jump. Campi and Schachermayer (2006) employed both types of transactions to prove a superreplication theorem.

A simple strategy is *x-admissible* if it has *future* liquidation value (Definition 4.2) greater than  $-x$ . This definition embodies the idea that the current value of a position should reflect *sure* future improvements in liquidity, and reduces to the usual definition in frictionless arbitrage-free markets. From the technical viewpoint, this notion ensures that an admissible strategy with positive terminal value retains a positive value at all times (Proposition 4.6). This is in turn crucial for the closedness of the set admissible payoffs, which allows the classic separation arguments.

A Free Lunch with Vanishing Risk (Definition 5.3) is a sequence of *simple admissible* strategies which uniformly approximates an arbitrage payoff, and the (RN-FLVR) property requires that no such sequences exist up to small model misspecifications. Then, any *general x-admissible strategy*, defined as the pointwise limit of simple  $(x + 1/n)$ -admissible strategies (Definition 5.1), must be of finite variation (Proposition 5.6). This justifies with a no-arbitrage argument the long-accepted intuition (see for example Davis and Norman (1990)) that only finite-variation strategies are admissible under transaction costs.

Central to wealth dynamics is the concept of *cost process*, which tracks the cumulative cash flows generated by trading. In frictionless markets, the cost process  $\int Sd\theta$  is linked to the familiar gain process  $\int \theta dS$  by the integration by parts formula:

$$\int_{[0,T]} S_t d\theta_t = S_T \theta_T - S_0 \theta_0 - \int_{[0,T]} \theta_t dS_t \quad (1.1)$$

which holds for a left-continuous, finite-variation strategy  $\theta$  and a semimartingale price  $S$ . Then the right-hand side is understood as a stochastic integral, while the left-hand side is a usual Stieltjes integral. In the language of Dellacherie and Meyer (1982), this is the *elementary stochastic integral*.

In the presence of bid and ask prices, the cost is the basic concept, since it avoids the appraisal of the risky position, which may be liquidated in the future at more favorable terms. The definition of cost is elementary for simple strategies (Definition 3.5), but its consistent extension to general (l\`adl\`ag) finite-variation strategies requires a careful study, which seems to be of independent interest and is developed in the appendix. From the mathematical standpoint, it is clear that the class of left-continuous strategies lacks any closedness property, hence it is unfit as domain of the integral operator. The economic intuition is also straightforward. When asset prices move in response to unpredictable (i.e. *totally inaccessible*) events, such as earthquakes, trading takes place only immediately after the event, i.e. the strategy  $\theta$  is left-continuous. However, if market-sensitive information is announced at a *predictable* time  $\tau$ , as for quarterly earnings announcements and monetary policy meetings, it is perfectly feasible to trade both immediately before and immediately after the announcement. Then all the values  $\theta_{\tau-}$ ,  $\theta_{\tau}$ ,  $\theta_{\tau+}$  can be distinct, and yet economically relevant.

Building on this intuition, the appendix provides a consistent extension of the Stieltjes integral for a c\`adl\`ag integrand  $S$ , and a predictable finite variation, integrator  $\theta$ . This integral is compatible with the usual stochastic integral: in the special case where  $S$  is a semimartingale, the integration by parts formula (1.1) continues to hold for any predictable, finite-variation  $\theta$ , when the left-hand side is understood in the sense defined in the Appendix.

The rest of this paper is organized as follows: after a brief literature review, section 3 introduces the main model. The next section discusses the new concept of admissibility, and compares it to the frictionless case. The main result of this section is Proposition 4.6, which characterizes admissible strategies by their terminal payoffs. Section 5 introduces the (RNFLVR) condition, shows that it implies the finite variation property of admissible strategies, and establishes the Fatou closedness of the payoff space. Section 6 proves the main theorem using standard arguments: the Kreps-Yan theorem to separate the payoff space, and the argument of Jouini and Kallal (1995) to obtain a martingale. The Appendix, which is readable without reference to the main text, constructs the predictable Stieltjes integral. The pathwise definition of the integral is probability free, and only requires a measurable space. The probability structure is needed only for the construction of simple approximations.

## 2 Literature Review

The simple notion of arbitrage, as a positive nonzero payoff with zero price, proved effective in finite discrete time, leading to the FTAP of Dalang, Morton and Willinger (1990), but becomes problematic in models with infinite horizons or continuous time. The seminal papers of Harrison and Kreps (1979) and Kreps (1981) derived the existence of pricing rules assuming a stronger condition, the absence of Free Lunches, defined as nets, or generalized sequences, of payoffs converging to an arbitrage in a weak sense. Harrison and Pliska (1981) highlighted the need for restrictions on trading strategies, to avoid arbitrage through doubling schemes.

The FTAP in continuous time for general càdlàg processes was established by Delbaen and Schachermayer (1994, 1998), featuring the prominent rôle of semimartingale theory in Mathematical Finance. The notion of a Free Lunch with Vanishing Risk, a very strong approximation to an arbitrage payoffs, proved key to establish first the semimartingale property of asset prices, then the existence of equivalent local martingale measures (in general,  $\sigma$ -martingale measures), which are formally equivalent to positive linear pricing rules.

Arbitrage theory with transaction costs was pioneered by Jouini and Kallal (1995) in a continuous time model. They proved that the absence of free lunches is equivalent to the existence of a “shadow price”, evolving within the bid-ask spread, which is a martingale under an equivalent probability measure. Subsequent work of Kabanov and Stricker (2001a), Kabanov, Rásonyi and Stricker (2002), and Schachermayer (2004) proved discrete time versions of the FTAP in increasing degrees of generality, in a numéraire-free framework where all assets are freely exchangeable. Guasoni, Rásonyi and Schachermayer (2008) characterize the absence of arbitrage with arbitrarily small transaction costs for continuous processes.

## 3 The Model

Consider a market model with one risky and one risk-free asset, based on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$  satisfying the usual conditions, setting for

convenience  $\mathcal{F}_t := \mathcal{F}_T$  for  $t \geq T$ . The risk-free asset plays the role of numéraire, hence its price is constantly equal to one.  $S_t - \kappa_t$  and  $S_t + \kappa_t$  denote respectively the bid (selling) and ask (buying) prices of the risky asset. Equivalently, each share trades at price  $S_t$ , and incurs a transaction fee of  $\kappa_t$ . The price-spread pair  $(S, \kappa)$  satisfies the following:

**Assumption 3.1**  $(S, \kappa)$  is a pair of càdlàg adapted, locally bounded processes, and  $\kappa$  is nonnegative.

Note that  $S$  may become negative in this model. A nonnegative  $\kappa$  is necessary to rule out static arbitrage by crossing the bid and ask prices.

*Remark 3.2* Alternatively, let  $\underline{S} \leq \bar{S}$  be the bid and ask price processes. The relations  $S_t := (\underline{S}_t + \bar{S}_t)/2$  and  $\kappa_t = \bar{S}_t - S_t = S_t - \underline{S}_t$  link the bid-ask notation  $(\underline{S}, \bar{S})$  to the price-spread notation  $(S, \kappa)$ . Thus, the choice of either notation does not restrict generality.

An investor trades in the risky asset according to the strategy  $(\theta_t)_{t \in [0, T]}$ , which represents the number of shares held at time  $t$ .

**Definition 3.3** Let  $(\sigma_n)_{n \geq 1}$  be a strictly increasing sequence of stopping times, such that  $\sup_{n \geq 1} \sigma_n > T$  a.s., that is  $P(\cup_{n \geq 1} \{\sigma_n > T\}) = 1$ . A *simple strategy* is a predictable process  $\theta$  such that  $\theta_0 = \theta_T = 0$  and:

$$\theta = \sum_{n=1}^{\infty} \left( \theta_{\sigma_n} 1_{\llbracket \sigma_n \rrbracket} + \theta_{\sigma_n^+} 1_{\llbracket \sigma_n, \sigma_{n+1} \rrbracket} \right) \quad (3.1)$$

where we extend  $\theta_s = 0$  for  $s > T$ .

According to this definition, a simple strategy entails a finite number of transactions, but this number may depend on  $\omega$ . Each stopping time  $\sigma_n$  involves in principle two transactions: the *left transaction* from  $\theta_{\sigma_n^-}$  to  $\theta_{\sigma_n}$  takes place at price  $(S \pm \kappa)_{\sigma_n^-}$ , i.e. immediately before a possible jump. The *right transaction* from  $\theta_{\sigma_n}$  to  $\theta_{\sigma_n^+}$  takes place at price  $(S \pm \kappa)_{\sigma_n}$ , i.e. after the jump.

*Remark 3.4* Because  $\theta$  is predictable,  $\theta_{\sigma} = \theta_{\sigma^-}$  for any totally inaccessible stopping time. Thus, left transactions may only take place on the accessible part of  $\sigma$ , while right transactions have no restrictions. This observation leads to an equivalent definition of simple strategy. Let  $(\pi_n)_{n \geq 1}$  and  $(\tau_n)_{n \geq 1}$  be two strictly increasing sequences of predictable times and stopping times respectively, such that  $\sup_{n \geq 1} \pi_n > T$  and  $\sup_{n \geq 1} \tau_n > T$  a.s. A *simple strategy* is a predictable process  $\theta$  such that  $\theta_0 = \theta_T = 0$  and (recalling the notation  $\Delta \theta_t = \theta_t - \theta_{t^-}$  and  $\Delta^+ \theta_t = \theta_{t^+} - \theta_t$ ):

$$\theta = \sum_{n \geq 1} \Delta \theta_{\pi_n} 1_{\llbracket \pi_n, \infty \rrbracket} + \sum_{n \geq 1} \Delta^+ \theta_{\tau_n} 1_{\llbracket \tau_n, \infty \rrbracket} \quad (3.2)$$

The equivalence with the original definition is straightforward: (3.2) implies (3.1) with  $(\sigma_n)_{n \geq 1}$  as the ordered sequence obtained from  $(\pi_n)_{n \geq 1}$  and  $(\tau_n)_{n \geq 1}$ . Viceversa, to obtain (3.2) from (3.1), set  $\tau_n = \sigma_n$ , and choose the sequence  $(\pi_n)_{n \geq 1}$  as the ordered sequence obtained from an exhausting sequence of the accessible parts of  $\sigma_n$  (such ordered sequence exists because  $\sigma_n$  is strictly increasing).

The definition of a simple strategy immediately leads to that of cost:

**Definition 3.5** The *cost* of a simple strategy  $\theta$  is:

$$C(\theta) = \sum_{n=1}^{\infty} \left( S_{\sigma_n^-} (\theta_{\sigma_n} - \theta_{\sigma_{n-1}^+}) + S_{\sigma_n} (\theta_{\sigma_n^+} - \theta_{\sigma_n}) + \kappa_{\sigma_n^-} |\theta_{\sigma_n} - \theta_{\sigma_{n-1}^+}| + \kappa_{\sigma_n} |\theta_{\sigma_n^+} - \theta_{\sigma_n}| \right) \quad (3.3)$$

with the convention  $\sigma_0 = 0$ . Define the *liquidation value* as  $V(\theta) = -C(\theta)$ .

The first two terms in  $C(\theta)$  represent the cash flows generated by trading at price  $S$ , which are positive for buying and negative for selling. The last two terms represent the transaction costs, which are always positive. Equivalently,  $C(\theta)$  reflects purchases and sales respectively at bid and ask prices. To see this, regroup the terms in (3.3) as:

$$C(\theta) = \sum_{n=1}^{\infty} \left[ (S + \kappa)_{\sigma_n^-} (\theta_{\sigma_n} - \theta_{\sigma_{n-1}^+})^+ + (S + \kappa)_{\sigma_n} (\theta_{\sigma_n^+} - \theta_{\sigma_n})^+ \right] \quad (3.4)$$

$$- \sum_{n=1}^{\infty} \left[ (S - \kappa)_{\sigma_n^-} (\theta_{\sigma_n} - \theta_{\sigma_{n-1}^+})^- + (S - \kappa)_{\sigma_n} (\theta_{\sigma_n^+} - \theta_{\sigma_n})^- \right] \quad (3.5)$$

Since a strategy begins and ends with cash only, the liquidation value is  $-C(\theta)$ .

#### 4 Admissibility

To introduce admissibility, it is convenient to recast the notion of simple strategy in a discrete-time notation, where the clock ticks at each transaction. This device overcomes the cumbersome distinction between left and right transactions.

Let  $\theta$  be a simple strategy, and  $(\sigma_n)_{n \geq 1}$  the sequence of transaction times in Definition 3.3. Define the discrete filtration  $(\hat{\mathcal{F}})_{n \geq 0}$  as  $\hat{\mathcal{F}} = (\mathcal{F}_0, \mathcal{F}_{\sigma_1^-}, \mathcal{F}_{\sigma_1}, \mathcal{F}_{\sigma_2^-}, \mathcal{F}_{\sigma_2}, \dots)$ , that is  $\hat{\mathcal{F}}_{2n-1} = \mathcal{F}_{\sigma_n^-}$  and  $\hat{\mathcal{F}}_{2n} = \mathcal{F}_{\sigma_n}$  for  $n \geq 1$ . The corresponding prices  $(\hat{S}, \hat{\kappa})_{n \geq 0}$ , defined analogously, are adapted to  $\hat{\mathcal{F}}$  because  $(S, \kappa)$  are adapted to  $\mathcal{F}$ . Using this notation, any simple strategy induces a  $\hat{\mathcal{F}}$ -adapted process  $(\hat{\theta}_n)_{n \geq 0}$  defined as  $\hat{\theta} = (0, \theta_{\sigma_1}, \theta_{\sigma_1^+}, \theta_{\sigma_2}, \theta_{\sigma_2^+}, \dots)$ , or  $\hat{\theta}_{2n-1} = \theta_{\sigma_n}$  and  $\hat{\theta}_{2n} = \theta_{\sigma_n^+}$  for  $n \geq 1$ .

$\hat{\mathcal{F}}_n$  represents the information available at the  $n$ -th transaction, and  $(\hat{S}_n, \hat{\kappa}_n)$  the prices at which such transaction takes place. In addition, the ‘‘hat’’ operation maps predictable to adapted processes, in the following sense:

**Proposition 4.1** Let  $\tau_n = \sigma_{nA_n} := \sigma_n 1_{A_n} + \infty 1_{\Omega \setminus A_n}$  denote the totally inaccessible part of  $\sigma_n$ , for some  $A_n \in \mathcal{F}_{\sigma_n} = \hat{\mathcal{F}}_{2n}$ .

- i) If  $(\theta_t)_{t \geq 0}$  is  $\mathcal{F}$  is a simple strategy, then  $(\hat{\theta}_n)_{n \geq 0}$  is  $\hat{\mathcal{F}}$ -adapted, and  $\hat{\theta}_{2n-1} = \hat{\theta}_{2(n-1)}$  on  $A_n$ . In addition,  $\hat{\theta}_{2n-1} = \theta_{\sigma_n} = \theta_{\sigma_n^-} = \hat{\theta}_{2(n-1)}$  on  $A_n$ .
- ii) If  $(\eta_n)_{n \geq 0}$  is  $\hat{\mathcal{F}}$ -adapted, and  $\eta_{2n-1} = \eta_{2(n-1)}$  on  $A_n$ , then  $\theta = \sum_{n \geq 0} (\Delta \eta_{2n-1} 1_{\llbracket \sigma_n, \infty \rrbracket} + \Delta \eta_{2n} 1_{\llbracket \sigma_n, \infty \rrbracket})$  is a simple strategy and  $\hat{\theta} = \eta$  (recall that  $\Delta \eta_n = \eta_n - \eta_{n-1}$ ).

*Proof* For  $i$ ), observe that  $\theta_\sigma$  is  $\mathcal{F}_{\sigma^-}$ -measurable because  $\theta$  is predictable, and that  $\theta_{\sigma^+}$  is  $\mathcal{F}_\sigma$ -measurable since the filtration is right-continuous.

To obtain  $ii$ ), it suffices to show that both terms in the definition of  $\theta$  are predictable.  $\Delta\eta_{2n}1_{\llbracket\sigma_n, \infty\rrbracket}$  is left-continuous, hence predictable. For  $\Delta\eta_{2n-1}1_{\llbracket\sigma_n, \infty\rrbracket}$ , observe that  $\Delta\eta_{2n-1} = 0$  on  $A_n$  by assumption. Since  $\sigma_{n\Omega \setminus A_n}$  is the accessible part of  $\sigma_n$ , its graph admits the representation:

$$\llbracket\sigma_{n\Omega \setminus A_n}\rrbracket = \cup_{m \geq 1} \llbracket\pi_m\rrbracket \quad (4.1)$$

where  $(\pi_m)_{m \geq 1}$  is a sequence of predictable times, whose graphs  $(\llbracket\pi_m\rrbracket)_{m \geq 1}$  are pairwise disjoint. Thus,  $\Delta\eta_{2n-1}1_{\llbracket\pi_m, \infty\rrbracket}$  is predictable for each  $m$  by (Jacod and Shiryaev 2003, 1.2.12), hence also  $\Delta\eta_{2n-1}1_{\llbracket\sigma_n, \infty\rrbracket} = \Delta\eta_{2n-1} \sum_{m \geq 1} 1_{\llbracket\pi_m, \infty\rrbracket}$  is predictable.

Define the cumulative cost process  $(\hat{C}_k)_{k \geq 0}$  as:

$$\hat{C}_k = \sum_{n=1}^k (\hat{S}_n(\hat{\theta}_n - \hat{\theta}_{n-1}) + \hat{\kappa}_n |\hat{\theta}_n - \hat{\theta}_{n-1}|) \quad (4.2)$$

which implies that  $V(\theta) = -C(\theta) = -\hat{C}_\infty(\hat{\theta})$ .

This notation streamlines the definition of an admissible strategy by avoiding the distinction between left and right transactions. Intuitively, a strategy is  $x$ -admissible if, after any transaction, the clearing broker can freeze the agent's account, and liquidate it to cash at a later date for a cost less than  $x$ . This condition ensures that a credit line of  $x$  is effectively in force.

**Definition 4.2** For  $x \geq 0$ , a simple strategy  $\theta$  is  $x$ -admissible if, for all  $k \geq 0$ , there exists another simple strategy  ${}^k\theta$ , called *liquidation strategy*, such that:

- i)  $\widehat{{}^k\theta} := {}^k\hat{\theta} := \hat{\theta} \wedge_k 1_{\{\cdot < \lambda_k\}}$  for some  $\mathcal{F}$ -stopping time  $\lambda_k > k$  a.s. (*liquidation time*).
- ii)  $x + V({}^k\theta) \geq 0$ .

$\mathcal{A}_x^s$  denotes the set of simple  $x$ -admissible strategies, and  $\mathcal{A}^s = \cup_{x > 0} \mathcal{A}_x^s$  the set of simple admissible strategies. A *simple arbitrage* is  $\theta \in \mathcal{A}^s$  such that  $P(V(\theta) \geq 0) = 1$  and  $P(V(\theta) > 0) > 0$ . A market satisfies (NA-S) if  $\theta \in \mathcal{A}^s$  and  $P(V(\theta) \geq 0) = 1$  implies that  $V(\theta) = 0$ .

*Remark 4.3* The liquidation strategy  ${}^k\hat{\theta}$  satisfies  ${}^k\hat{\theta}_n = \hat{\theta}_n 1_{\{n < k\}} + \hat{\theta}_k 1_{\{k \leq n < \lambda_k\}}$ , and induces a unique  $\mathcal{F}$ -predictable process  ${}^k\theta$ , which satisfies  $\widehat{{}^k\theta} = {}^k\hat{\theta}$  and is piecewise constant between  $\sigma_k$  and  $\sigma_{k+1}$ .  ${}^k\theta$  takes four possible forms, depending on whether the freeze ( $k$ ) and liquidation ( $\lambda_k$ ) are left or right transactions:

$$\begin{aligned} & \theta 1_{\llbracket 0, \sigma_k \rrbracket} + \theta_{\sigma_k^+} 1_{\llbracket \sigma_k, \sigma_{\lambda_k} \rrbracket} \\ & \theta 1_{\llbracket 0, \sigma_k \rrbracket} + \theta_{\sigma_k^+} 1_{\llbracket \sigma_k, \sigma_{\lambda_k} \llbracket} \\ & \theta 1_{\llbracket 0, \sigma_k \llbracket} + \theta_{\sigma_k} 1_{\llbracket \sigma_k, \sigma_{\lambda_k} \rrbracket} \\ & \theta 1_{\llbracket 0, \sigma_k \llbracket} + \theta_{\sigma_k} 1_{\llbracket \sigma_k, \sigma_{\lambda_k} \llbracket} \end{aligned}$$

*Remark 4.4* In Definition 4.2, the freeze occurs after a fixed transaction, but this may equivalent occur at any  $\mathcal{F}$ -stopping time  $\sigma$ . Indeed, if  $(\lambda_k)_{k \geq 1}$  are the liquidation times in Definition 4.2, then  $\lambda_\sigma = \sum_{k=1}^{\infty} \lambda_k 1_{\{\sigma=k\}}$  is a liquidation time for  $\sigma$ .

Compare definition (4.2) to its frictionless counterpart:

$$x + V(\theta 1_{[0,t]}) \geq 0 \quad \text{for all } t \in [0, T] \quad (4.3)$$

In general, (4.3) is more restrictive, but in a frictionless, arbitrage free market the two definitions are equivalent:

**Proposition 4.5** *Let  $\kappa = 0$ . For any  $\theta \in \mathcal{A}^s$ , either:*

$$\sum_{n=1}^k \hat{\theta}_{n-1} (\hat{S}_n - \hat{S}_{n-1}) \geq -x \quad \text{a.s. for all } k \geq 0 \quad (4.4)$$

*or there exists an arbitrage opportunity.*

*Proof* If  $\kappa = 0$ , observe that:

$$V(\theta) = - \sum_{n=1}^{\infty} \hat{S}_n (\hat{\theta}_n - \hat{\theta}_{n-1}) = \sum_{n=1}^{\infty} \hat{\theta}_{n-1} (\hat{S}_n - \hat{S}_{n-1}) \quad (4.5)$$

and therefore:

$$V(\theta 1_{\{\cdot < k\}}) = \sum_{n=1}^k \hat{\theta}_{n-1} (\hat{S}_n - \hat{S}_{n-1}) \quad (4.6)$$

$$V({}^k \hat{\theta}) = \left( \sum_{n=1}^k \hat{\theta}_{n-1} (\hat{S}_n - \hat{S}_{n-1}) \right) + \theta_k (\hat{S}_{\lambda_k} - \hat{S}_k) \quad (4.7)$$

If  $\lambda_{k-1} = k$  is not a liquidation time, there exists  $A \in \mathcal{F}_k$  on which  $V(\theta 1_{\{\cdot < k\}}) < -x$ . Thus, the buy-and-hold strategy  $\eta_n = \theta_k 1_{A \cap \{k \leq n < \lambda_k\}}$  is an arbitrage opportunity.

The main result of this section characterizes an  $x$ -admissible strategy by its terminal payoff. This property essentially confirms that Definition 4.2 is the relevant notion of admissibility in this model.

**Proposition 4.6** *If (NA-S) holds, then  $\mathcal{A}_x^s = \{\theta \in \mathcal{A}^s : x + V(\theta) \geq 0 \text{ a.s.}\}$ .*

The nontrivial statement is that  $\theta \in \mathcal{A}_x^s$  if  $\theta \in \mathcal{A}^s$  and  $x + V(\theta) \geq 0$  a.s. In a frictionless market, this is a straightforward consequence of absence of arbitrage and the concatenation property  $V(\theta) = V(\theta 1_{[0,t]}) + V(\theta 1_{(t,T]})$ . However, concatenation fails under transaction costs, since splitting a strategy at time  $t$  creates two new transactions, thereby increasing costs.

The next proof exploits the following idea: if a strategy has positive final liquidation value, then after every transaction there is either a liquidation time, or an arbitrage. The trouble is that the arbitrage is not necessarily the remaining part of the strategy (because concatenation fails), but it may be hidden inside. Discovering the arbitrage involves the elimination of “damaging transactions”, i.e. the ones executed at a worse (either bid or ask) price than the average liquidation price required by the current position. This elimination process eventually leads either to a liquidation time, or to an arbitrage.

*Proof* The inclusion  $\mathcal{A}_x^s \subset \{\theta \in \mathcal{A}^s : x + V(\theta) \geq 0 \text{ a.s.}\}$  is straightforward. If  $\theta \in \mathcal{A}_x^s$ , by assumption the strategies  $(\hat{\theta}^k)_{k \geq 0}$  satisfy  $x + V(\hat{\theta}^k) \geq 0$  for all  $k \geq 0$ . Since  $\hat{\theta}_n \rightarrow \theta_n$  a.s. for each  $n$  as  $k \rightarrow \infty$ . Since for each  $\omega$ ,  $V(\hat{\theta}^k), V(\theta)$  are finite sums with the same number of terms,  $V(\hat{\theta}^k) \rightarrow V(\theta)$  a.s., and it follows that  $x + V(\theta) \geq 0$  as required.

To prove the reverse inclusion, the following argument constructs explicitly a liquidation time  $\lambda_k$  from the initial assumption that  $x + V(\theta) = x - \hat{C}_\infty \geq 0$ .

Set  $m = 1$ .

**Step 1:** Find arbitrage or liquidation.

After the  $k$ -th transaction, the portfolio consists of  $\hat{\theta}_k$  shares and a cash position equal to  $x - \hat{C}_k$ . On the set  $A_0 = \{\hat{\theta}_k = 0\}$ , set the liquidation time  $\lambda_k = k + 1$ .

Consider the following  $\mathcal{F}_{k+1}$ -partition of  $\Omega \setminus A_0$ :

$$\begin{aligned} A_+ &= \{\hat{\theta}_k > 0, \hat{\theta}_{k+1} > 0\} && \text{(stay long)} \\ A_- &= \{\hat{\theta}_k < 0, \hat{\theta}_{k+1} < 0\} && \text{(stay short)} \\ A_\pm &= \{\hat{\theta}_k \neq 0, \hat{\theta}_{k+1} \hat{\theta}_k \leq 0\} && \text{(long/short change)} \end{aligned}$$

Define  $\hat{\eta} = \hat{\theta} 1_{\{\cdot \leq k\}}$ , ( $\hat{\eta}_n = \hat{\theta}_n$  for  $n \leq k$  and  $\hat{\eta}_n = 0$  for  $n \geq k + 1$ ), and  $\hat{\zeta} = \hat{\theta} 1_{\{\cdot \geq k+1\}} = \hat{\theta} - \hat{\eta}$ . Since  $|\hat{\theta}_k - \hat{\theta}_{k+1}| = |\hat{\theta}_k| + |\hat{\theta}_{k+1}|$  on  $A_\pm$ , concatenation holds:

$$\hat{C}(\hat{\theta}) = \hat{C}(\hat{\eta}) + \hat{C}(\hat{\zeta}) \quad (4.8)$$

and therefore  $x + V(\eta) \geq 0$  on  $A_\pm$ , otherwise  $\hat{\zeta}$  is an arbitrage on the event  $\{x + \hat{C}_k(\hat{\eta}) < 0\}$ . Thus,  $\lambda_k = k + 1$  is a liquidation time on  $A_\pm$ .

On  $A_+$  (“stay long”), denote by  $l = \frac{\hat{C}_k(\hat{\theta}_k) - x}{\hat{\theta}_k}$  the minimum average liquidation price for solvability. Distinguish now three cases:

$$\begin{aligned} A_+^l &= A_+ \cap \{\hat{\theta}_{k+1} - \hat{\theta}_k > 0, (\hat{S} + \hat{\kappa})_{k+1} < l\} && \text{(buy low)} \\ A_+^h &= A_+ \cap \{\hat{\theta}_{k+1} - \hat{\theta}_k < 0, (\hat{S} - \hat{\kappa})_{k+1} > l\} && \text{(sell high)} \\ A_+^s &= A_+ \setminus (A_+^h \cup A_+^l) \end{aligned}$$

First, observe that on  $A_+^h$  (“sell high”)  $\lambda_k = k + 1$  is a liquidation time by definition. Second, on  $A_+^l$  (“buy low”), an arbitrage opportunity arises, hence  $P(A_+^l) = 0$ . This is intuitively clear, as the knowledge that the current long position  $\hat{\theta}_k$  will be sold at least at price  $l$  guarantees that any purchase below  $l$  price yields an arbitrage. To see it formally, define  $\hat{\zeta} = \hat{\theta} 1_{\{\cdot \geq k+1\}}$  as before, and observe that:

$$\begin{aligned} V(\hat{\zeta}) &= -\hat{C}_\infty(\hat{\theta}) + \hat{C}_k(\hat{\theta}) - (\hat{S} + \hat{\kappa})_{k+1} \hat{\theta}_k \\ &> -\hat{C}_\infty(\hat{\theta}) + \hat{C}_k(\hat{\theta}) - l \hat{\theta}_k \\ &= -\hat{C}_\infty(\hat{\theta}) + \hat{C}_k(\hat{\theta}) + x - \hat{C}_k(\hat{\theta}) \\ &= x + V(\theta) \geq 0 \end{aligned}$$

The appearance of the term  $(\hat{S} + \hat{\kappa})_{k+1} \hat{\theta}_k$  in the first equality reflects the additional purchase of  $\hat{\theta}_k$  shares, necessary to build a position of  $\hat{\theta}_{k+1}$  shares from 0 at the  $k + 1$

transaction. The next inequality follows from the definition of  $A_+^l$ , and the following equality from the definition of  $l$ .

The situation for  $A_-$  (“stay short”) is symmetric, in that liquidation is trivial on the “buy low” case  $A_-^l = A_- \cap \{\hat{\theta}_{k+1} - \hat{\theta}_k > 0, (\hat{S} + \hat{\kappa})_{k+1} < l\}$ , while arbitrage arises in the “sell high” case  $A_-^h = A_- \cap \{\hat{\theta}_{k+1} - \hat{\theta}_k < 0, (\hat{S} - \hat{\kappa})_{k+1} > l\}$ . In summary, liquidation at the  $(k+1)$ -th transaction is feasible on the event  $B_m = A_0 \cup A_\pm \cup A_+^h \cup A_-^l$ .

**Step 2:** Else skip a damaging transaction.

It remains to consider the cases  $A_+^s$  and  $A_-^s$ . Intuitively, any transaction taking place on these events is “damaging”, since it neither buys low, nor it sells high. To skip this transaction, while ensuring positive final liquidation, rescale now the remaining part of the strategy. Formally, define  $\hat{\zeta}$  as  $\hat{\zeta}_n = \hat{\theta}_n$  for  $n \leq k$ , and  $\hat{\zeta}_n = \frac{\hat{\theta}_k}{\hat{\theta}_{k+1}} \hat{\theta}_n$  for  $n \geq k+1$ .

Then note that:

$$\begin{aligned} 0 &\leq x + V(\theta) \\ &= x - \hat{C}_k(\hat{\theta}) - (\hat{S} \pm \hat{\kappa})_{k+1}(\hat{\theta}_{k+1} - \hat{\theta}_k) - (\hat{C}_\infty(\hat{\theta}) - \hat{C}_{k+1}(\hat{\theta})) \\ &\leq x - \hat{C}_k(\hat{\theta}) + \frac{\hat{\theta}_{k+1} - \hat{\theta}_k}{\hat{\theta}_k} (x - \hat{C}_k(\hat{\theta})) - (\hat{C}_\infty(\hat{\theta}) - \hat{C}_{k+1}(\hat{\theta})) \\ &= \frac{\hat{\theta}_{k+1}}{\hat{\theta}_k} \left( x - \hat{C}_k(\hat{\theta}) - \frac{\hat{\theta}_k}{\hat{\theta}_{k+1}} (\hat{C}_\infty(\hat{\theta}) - \hat{C}_{k+1}(\hat{\theta})) \right) \\ &= \frac{\hat{\theta}_{k+1}}{\hat{\theta}_k} (x + V(\zeta)), \end{aligned}$$

where the  $\pm$  in the second line is equal to  $+$  on  $(A_+^s \cup A_-^s) \cap \{\hat{\theta}_{k+1} - \hat{\theta}_k > 0\}$ , and is equal to  $-$  on  $(A_+^s \cup A_-^s) \cap \{\hat{\theta}_{k+1} - \hat{\theta}_k < 0\}$ . Since the strategy  $\hat{\zeta}$  satisfies the same assumptions as  $\hat{\theta}$ , and coincides with  $\hat{\theta}$  up to (and including) the  $k$ -th transaction, a liquidation time for  $\hat{\zeta}$  is also valid for  $\hat{\theta}$ . Thus, on the event  $A_+^s \cup A_-^s$  replace  $\hat{\theta}$  by  $\hat{\zeta}$  and return to Step 1, increasing  $m$  by one.

Since the number of transactions in a simple strategy is a.s. finite, this recursion takes place only a finite number of times, i.e.  $P(\cup_{m \geq 1} B_m) = 1$ . The liquidation time is thus  $\lambda_k = \sum_{m=1}^{\infty} (k+m) \mathbf{1}_{B_m \setminus \cup_{j < m} B_j}$ .

One last check is necessary: the strategy  ${}^k \hat{\theta}$  must induce a  $\hat{\mathcal{F}}$ -predictable strategy  ${}^k \theta$ . By Proposition 4.1, this is the case if  ${}^k \hat{\theta}$  is  $\hat{\mathcal{F}}$ -adapted, and  ${}^k \hat{\theta}_{2n-1} = {}^k \hat{\theta}_{2(n-1)}$  on  $A_n$  (which defines the totally inaccessible part of  $\sigma_n$  in Proposition 4.1).  ${}^k \hat{\theta}$  is  $\hat{\mathcal{F}}$ -adapted because  $\theta$  is  $\hat{\mathcal{F}}$ -adapted and  $\{\lambda_k = k+m\}$  is  $\hat{\mathcal{F}}_{k+m}$ -measurable (the events  $A_\pm, A_+^h, A_-^l$  in the construction depend only on  $\hat{\theta}_{k+i}$  for  $i \leq m$ ). For the predictability part, observe that the original strategy satisfies  $\hat{\theta}_{2n} = \hat{\theta}_{2n-1}$  on  $A_n$  by assumption. Thus it suffices to check that the operations of liquidation and scaling in the above construction preserve this property. Indeed, the events  $A_\pm, A_+^h, A_-^l$  are disjoint from  $\{\hat{\theta}_{k+1} = \hat{\theta}_k\}$ , hence liquidation (i.e. setting  ${}^k \hat{\theta}_{k+1} = 0$ ) takes place outside  $A_n$ . On the event  $A_0$  liquidation is consistent with the property, since  ${}^k \hat{\theta}_{k+1} = {}^k \hat{\theta}_k = 0$ . Finally, the scaling operation  $\hat{\zeta}_n = \frac{\hat{\theta}_k}{\hat{\theta}_{k+1}} \hat{\theta}_n$  for  $n \geq k+1$  entails that, on the event  $\{\hat{\theta}_{k+1} = \hat{\theta}_k\}$ , the property  $\{\hat{\zeta}_{k+1} = \hat{\zeta}_k\}$  still holds.  $\square$

An important consequence of the definition of admissibility is that payoffs attainable with zero wealth have nonpositive value under any consistent price system. In frictionless markets, this is the supermartingale property of the gain process under equivalent martingale measures. Liquidation times allow to carry out a similar argument in the present setting.

First, note the intuitively obvious *domination property*: executing a given strategy at better (bid and ask) prices achieves a better payoff. The discrete time notation streamlines calculations.

**Lemma 4.7** *Let  $(S, \kappa)$  and  $(S', \kappa')$  satisfy Assumption 3.1 and  $\kappa - \tilde{\kappa} - |S - \tilde{S}| \geq 0$ . Then, for all  $\theta \in \mathcal{A}^S$ :*

$$V^{(S, \kappa)}(\theta) \leq V^{(S', \kappa')}(\theta) - \sum_{n=1}^{\infty} (\hat{\kappa}_n - \hat{\kappa}'_n - |\hat{S}_n - \hat{S}'_n|) |\hat{\theta}_n - \hat{\theta}_{n-1}| \quad (4.9)$$

*Proof* Since  $V(\theta) = -C(\theta)$ , the claim follows from:

$$\hat{C}_{\infty}^{(S, \kappa)}(\theta) - \hat{C}_{\infty}^{(S', \kappa')}(\theta) = \sum_{n=1}^{\infty} ((\hat{S}_n - \hat{S}'_n)(\hat{\theta}_n - \hat{\theta}_{n-1}) + (\hat{\kappa}_n - \hat{\kappa}'_n) |\hat{\theta}_n - \hat{\theta}_{n-1}|) \quad (4.10)$$

$$\geq \sum_{n=1}^{\infty} (\hat{\kappa}_n - \hat{\kappa}'_n - |\hat{S}_n - \hat{S}'_n|) |\hat{\theta}_n - \hat{\theta}_{n-1}| \quad (4.11)$$

□

Adapting the definitions of Jouini and Kallal (1995) and Campi and Schachermayer (2006) to the present setting, consistent price systems take the following form:

**Definition 4.8** *Let  $(S, \kappa)$  satisfy Assumption 3.1. A *Strictly Consistent Price System* is a pair  $(M, Q)$  of a probability  $Q$  equivalent to  $P$  and a  $Q$ -local martingale  $M$  lying within the bid-ask spread:*

$$\inf_{t \in [0, T]} (\kappa_t - |S_t - M_t|) > 0 \quad \text{a.s.}$$

$(M, Q)$  is a *Consistent Price System* if the inequality is not necessarily strict.

As announced earlier, the basic property of consistent price systems is to assign non-positive value to simple admissible strategies starting with no initial capital.

**Proposition 4.9** *Let  $(M, Q)$  be a consistent price system. Then  $E_Q[V^{(M, 0)}(\theta)] \leq 0$  for all  $\theta \in \mathcal{A}^S$ .*

The proof of this property requires a lemma:

**Lemma 4.10** *Let  $(M_t)_{0 \leq t \leq T}$  be a martingale,  $\theta \in \mathcal{A}^S$  bounded, and denote by  $N_n = \sum_{k=1}^n \hat{\theta}_k (M_k - M_{k-1})$ . Then  $(N_{2n})_{n \geq 0}$  is a  $\hat{\mathcal{F}}_{2n}$ -martingale.*

*Proof* The claim amounts to  $E[N_{2n} - N_{2(n-1)} | \hat{\mathcal{F}}_{2(n-1)}] = 0$ . Since  $\theta_{\sigma_n^-} = \theta_{\sigma_{n-1}^+}$ , it follows that:

$$N_{2n} - N_{2(n-1)} = \theta_{\sigma_n^-} (M_{\sigma_n^-} - M_{\sigma_{n-1}}) + \theta_{\sigma_n} (M_{\sigma_n} - M_{\sigma_n^-}) \quad (4.12)$$

$$= \theta_{\sigma_{n-1}^+} (M_{\sigma_n} - M_{\sigma_{n-1}}) + \Delta \theta_{\sigma_n} \Delta M_{\sigma_n} \quad (4.13)$$

Thus, the first term has the martingale property:

$$E[\theta_{\sigma_{n-1}^+} (M_{\sigma_n} - M_{\sigma_{n-1}}) | \hat{\mathcal{F}}_{2(n-1)}] = \theta_{\sigma_{n-1}^+} E[M_{\sigma_n} - M_{\sigma_{n-1}} | \mathcal{F}_{\sigma_{n-1}}] = 0$$

Since  $\Delta \theta_{\sigma_n}$  is  $\mathcal{F}_{\sigma_n^-}$ -measurable, the second term satisfies:

$$E[\Delta \theta_{\sigma_n} \Delta M_{\sigma_n} | \mathcal{F}_{\sigma_{n-1}}] = E\left[\Delta \theta_{\sigma_n} E[\Delta M_{\sigma_n} | \mathcal{F}_{\sigma_n^-}] | \mathcal{F}_{\sigma_{n-1}}\right]$$

Consider now  $A_n \in \mathcal{F}_{\sigma_n}$  which decomposes the stopping time  $\sigma_n$  into its totally inaccessible part  $\sigma_{nA_n}$  and accessible part  $\sigma_{n\Omega \setminus A_n}$ .  $\{\Delta \theta_{\sigma_n} \neq 0\}$  is a predictable thin set because  $\Delta \theta_{\sigma_n} = 0$  on  $A_n$ , while  $E[\Delta M_{\sigma_n} | \mathcal{F}_{\sigma_n^-}] = 0$  on  $\Omega \setminus A_n$  by Jacod and Shiryaev (2003, 1.2.27).<sup>1</sup>

*Remark 4.11* Note that  $(N_k)_{k \geq 1}$  may not be  $\hat{\mathcal{F}}$ -martingale, because on the totally inaccessible part of  $\sigma_n$  the equality  $E[\Delta M_{\sigma_n} | \mathcal{F}_{\sigma_n^-}] = 0$  may fail.

*Proof* of Proposition 4.9  $M$  is locally bounded because so are  $S, \kappa$ . Take a localizing sequence  $(\tau_l)_{l \geq 0}$ . Since  $\theta$  is simple admissible, by Definition 4.2 it allows a sequence  $(\hat{\theta}^k)_{k \geq 1}$  of liquidation strategies satisfying  $V^{(S, \kappa)}(k \hat{\theta}) \geq -x$ . Furthermore, by Lemma 4.7:

$$V^{(M, 0)}(k \hat{\theta}) \geq V^{(S, \kappa)}(k \hat{\theta}) \geq -x \quad (4.14)$$

and in particular  $k \hat{\theta}$  is  $M$ -admissible. Since  $(M, 0)$  is a frictionless, arbitrage-free market, Proposition 4.5 implies admissibility in the frictionless sense:

$$\sum_{n=1}^j k \hat{\theta}_{n-1}^{\tau_l} (\hat{M}_n^{\tau_l} - \hat{M}_{n-1}^{\tau_l}) \geq -x \quad \text{a.s. for all } j, k \geq 1 \quad (4.15)$$

Setting  $j = k$ , and using the martingale property of  $(N_{2k})_{k \geq 1}$  (Lemma 4.10):

$$E_Q \left[ \sum_{n=1}^{2k} 2^k \theta_{n-1}^{\tau_l} (\hat{M}_n^{\tau_l} - \hat{M}_{n-1}^{\tau_l}) \right] = 0,$$

so, passing to the limit in  $l$ , Fatou's lemma implies

$$E_Q \left[ \sum_{n=1}^{2k} 2^k \theta_{n-1} (\hat{M}_n - \hat{M}_{n-1}) \right] \leq 0,$$

Since  $2^k \theta$  converges to  $\theta$  a.s.,  $k \rightarrow \infty$ , the thesis follows by one more application of Fatou's Lemma.  $\square$

<sup>1</sup>  $\sigma_{n\Omega \setminus A_n}$  admits a decomposition as a countable union  $\cup_{m \geq 1} \llbracket \pi_m \rrbracket$  of predictable times  $\pi_m$ . On each  $\llbracket \pi_m \rrbracket$  the following holds (cf. Jacod and Shiryaev (2003, 1.2.27))  $E[M_{\sigma_n} - M_{\sigma_n^-} | \mathcal{F}_{\sigma_n^-}] = 0$

The rest of this section shows that both left and right transactions are necessary in Definition 3.3. Two examples illustrate this point, showing that L-simple (simple and left-continuous) admissible strategies cannot approximate R-simple (simple and right-continuous) admissible strategies, and viceversa. Thus, simple strategies must include both types of transactions. Transaction costs are inessential in both examples, hence we assume  $\kappa = 0$ .

*Example 4.12* Consider the parametric family of densities  $f_\theta(x) = 1_{[-1,1]} \frac{\theta}{e^\theta - e^{-\theta}} e^{\theta x}$ , which correspond to Esscher transforms of a uniform distribution on  $[-1, 1]$ . All densities in this family are supported by  $[-1, 1]$ , and the expectations span the interval  $(-1, 1)$  as the parameter  $\theta$  varies in  $(-\infty, \infty)$ .

Now define a sequence of random variables  $(\varepsilon_n)_{n \geq 1}$  by setting the density of  $\varepsilon_1$  equal to  $f_0$  (i.e. uniformly distributed on  $[-1, 1]$ ) and the density of  $\varepsilon_{n+1}$ , conditional on  $\mathcal{G}_n := \sigma((\varepsilon_k)_{k \leq n})$ , equal to  $f_{\theta_n}$ , where  $\theta_n(\varepsilon_n)$  is chosen such that  $E[\varepsilon_{n+1} | \mathcal{G}_n] = \varepsilon_n$ . Since  $(\varepsilon_n)_{n \geq 1}$  is a bounded martingale by construction, it converges a.s. to some random variable  $\varepsilon_\infty$  whose support contains  $\{-1, 1\}$  (even conditionally on  $\mathcal{G}_n$ , for each  $n \in \mathbb{N}$ , a.s.).

Finally, set  $\mathcal{F}_t = \sigma(\varepsilon_n, 1 - \frac{1}{n} \leq t)$  and  $S = \varepsilon_\infty 1_{\llbracket 1, 2 \rrbracket}$ . Since the support of  $\Delta S_1 = \varepsilon_\infty$  contains  $\{-1, 1\}$  conditionally on  $\mathcal{F}_t$  for all  $t \in (0, 1)$ , any L-simple approximation of the R-simple strategy  $\theta = \frac{1}{\varepsilon_\infty} 1_{\llbracket 1, 2 \rrbracket}$  can be at best 1-admissible. But  $\theta$  is an arbitrage, hence 0-admissible. Since the definition of  $x$ -admissibility in section 4 will require that a general  $x$ -admissible strategy should be approximated by  $x + 1/n$ -admissible simple strategies, the present example shows that it is impossible to consider L-simple admissible strategies alone.

*Example 4.13* Set  $S_t = -1_{\llbracket \tau_1 + \tau_2, 2 \rrbracket}$ , where  $\tau_1, \tau_2$  are two independent exponential random variables with parameter  $\lambda$ . The filtration  $\mathcal{F}_t$  is generated by  $\tau_1$  and  $\tau_1 + \tau_2$ :

$$\mathcal{F}_t := \sigma(\tau_1 \wedge t, (\tau_1 + \tau_2) \wedge t).$$

The L-simple strategy  $\theta = 1_{\llbracket 1/2, \tau_1 \rrbracket}$  has zero payoff, hence it is 0-admissible. Consider any R-simple admissible strategy  $\eta$ , and denote by  $\tau'_1 = \inf\{t > \tau_1 : \eta_t \neq \eta_{\tau_1^-}\}$ .  $P(\eta_{\tau_1^-} = \eta_{\tau_1}) = 1$  since  $\tau_1$  is totally inaccessible, and  $P(\tau'_1 > \tau_1) = 1$  by right-continuity. Observe that  $P(\tau_1 + \tau_2 < \tau'_1) > 0$ . Indeed, by contradiction suppose that  $\tau'_1 \leq \tau_1 + \tau_2$  a.s. Then  $\tau'_1$  would be  $\mathcal{F}_{\tau_1}$ -measurable, i.e.  $\tau'_1 = u(\tau_1)$  (cf. Dellacherie and Meyer (1978, Theorem 105)), and this implies that:

$$0 = P(\tau'_1 > \tau_1 + \tau_2 | \tau_1) = P(\tau_2 < u(\tau_1) - \tau_1 | \tau_1) = 1 - e^{-\lambda(u(\tau_1) - \tau_1)} > 0 \quad (4.16)$$

which is absurd. But  $\eta_{\tau_1 + \tau_2} = \eta_{\tau_1^-}$  on the event  $\{\tau_1 + \tau_2 < \tau'_1\}$ . Since  $\theta_{\tau_1^-} = 1$ , it follows that any R-simple approximation is at best 1-admissible.

## 5 (RNFLVR) and Finite Variation Strategies

Financially, it is natural to define a general  $x$ -admissible strategy as the limit of a sequence of simple strategies that are asymptotically  $x$ -admissible. In the same spirit, the cost  $C(\theta)$  of such a strategy should be the best (i.e. lowest) possible limit of the cost of such approximating sequences.

**Definition 5.1** A sequence  $(\theta^n)_{n \geq 1} \subset \mathcal{A}^s$  converges *admissibly* to a process  $(\theta_t)_{t \in [0, T]}$  if  $\theta^n$  converge to  $\theta$  a.s. and  $\theta^n \in \mathcal{A}_{x+\varepsilon_n}^s$  for some  $x > 0$  and  $(\varepsilon_n)_{n \geq 1} \downarrow 0$ . Any such limit is an *x-admissible strategy*. Denote the space of *x-admissible* and *admissible* strategies by  $\mathcal{A}_x$  and  $\mathcal{A} := \cup_{x>0} \mathcal{A}_x$  respectively. For  $\theta \in \mathcal{A}$ , define the *cost*  $C(\theta)$  as:

$$C(\theta) = \text{ess inf} \left\{ \liminf_{n \rightarrow \infty} C(\theta^n) : \theta^n \xrightarrow{\text{adm}} \theta \right\} \quad (5.1)$$

where the limit is in the a.s. sense.

*Remark 5.2* The a.s. convergence of processes is understood as:

$$P \left( \lim_{n \rightarrow \infty} \theta_t^n = \theta_t \text{ for all } t \in [0, T] \right) = 1 \quad (5.2)$$

This condition is stronger than  $\theta_t^n \rightarrow \theta_t$  a.s. for all  $t \in [0, T]$ , because  $(\theta_t^n)_{n \geq 1}$  may not be càdlàg.

Mathematically, the previous definition amounts to a relaxation of the cost map  $\theta \mapsto C(\theta)$  from the space of simple admissible strategies  $\mathcal{A}^s$  to its closure  $\mathcal{A}$  with respect to the admissible convergence. Bringing to life these definition requires an explicit representation for  $C(\theta)$  in terms of  $\theta$  alone. This section achieves such a representation as follows: first, the (RNFLVR) property below implies that all admissible strategies are of finite variation. Second, the limit of  $C(\theta^n)$  becomes a predictable Stieltjes integral defined in the Appendix.

**Definition 5.3**

- i)  $(S, \kappa)$  satisfies (NFLVR) if, for any sequence  $(\theta^n)_{n \geq 1}$  such that  $\theta^n \in \mathcal{A}_{1/n}^s$  and  $V(\theta^n)$  converges a.s. to some limit  $V$ , then  $V = 0$  a.s.
- ii)  $(S, \kappa)$  satisfies (RNFLVR) if there exists a pair  $(S', \kappa')$  satisfying (NFLVR) and such that the bid-ask spread of  $(S', \kappa')$  is a.s. strictly contained within that of  $(S, \kappa)$ , pathwise uniformly, i.e.:

$$\inf_{t \in [0, T]} (\kappa_t - \kappa'_t - |S_t - S'_t|) > 0 \quad \text{a.s.} \quad (5.3)$$

*Remark 5.4* Equation (5.3) is equivalent to:

$$\inf_{t \in [0, T]} ((S + \kappa)_t - (S' + \kappa')_t) > 0 \quad \text{and} \quad \inf_{t \in [0, T]} ((S' - \kappa')_t - (S - \kappa)_t) > 0 \quad \text{a.s.}$$

which means that the inner bid and ask prices never touch their outer counterparts. Observe also that (RNFLVR) implies the *efficient friction* condition:

$$\inf_{t \in [0, T]} \kappa_t > 0 \quad \text{a.s.}$$

so the bid-ask spread is always strictly positive, in pathwise uniform sense.

The next Lemma follows from Definition 5.3 with essentially the same argument as Proposition 3.1 in Delbaen and Schachermayer (1994):

**Lemma 5.5** *If  $(S, \kappa)$  satisfies (NFLVR), then  $\mathcal{C}_x^S := \{V(\theta) : \theta \in \mathcal{A}_x^S\} - L_+^0$  is bounded from above in  $L^0$  for all  $x > 0$ , i.e.  $\lim_{n \rightarrow \infty} \sup_{F \in \mathcal{C}_x^S} P(F > n) = 0$ .*

*Proof* It is sufficient to prove the claim for  $x = 1$ . By contradiction, suppose there exists  $\alpha > 0$  and a sequence  $(\theta^n)_{n \geq 1} \subset \mathcal{A}_1^S$  such that  $P(V(\theta^n) > n) > \alpha$  for all  $n$ . Then the sequence  $(\eta^n)_{n \geq 1}$  defined by  $\eta_t^n = \theta_t^n/n$  satisfies  $\eta^n \in \mathcal{A}_{1/n}^S$  for all  $n \geq 1$ , and  $P(V(\eta^n) > 1) > \alpha$ . Up to a sequence of convex combinations (Lemma B.1),  $\max(V(\eta^n), 1)$  converges a.s. to some  $f \geq 0$ . Then it follows that  $E[f] \geq \alpha > 0$ , which violates the (NFLVR) condition.  $\square$

The previous lemma, combined with the domination property (Lemma 4.7), implies that the total variations of  $x$ -admissible strategies are bounded in probability:

**Lemma 5.6** *Let  $(S, \kappa)$  satisfy (RNFLVR). Then  $\{\|\theta\|_T : \theta \in \mathcal{A}_x^S\}$  is bounded in  $L^0$  for all  $x > 0$ .*

*Proof* Rearranging (4.9), for any  $\theta \in \mathcal{A}_x^S$  the following holds:

$$\|\theta\|_T \inf_{n \geq 1} (\hat{\kappa}_n - \hat{\kappa}'_n - |\hat{S}_n - \hat{S}'_n|) \leq \sum_{n=1}^{\infty} (\hat{\kappa}_n - \hat{\kappa}'_n - |\hat{S}_n - \hat{S}'_n|) |\hat{\theta}_n - \hat{\theta}_{n-1}| \leq x + V^{(S', \kappa')}(\theta)$$

Note that  $\inf_{n \geq 1} (\hat{\kappa}_n - \hat{\kappa}'_n - |\hat{S}_n - \hat{S}'_n|) \geq Y := \inf_{t \in [0, T]} (\kappa_t - \kappa'_t - |S_t - S'_t|) > 0$ . Lemma 5.5 implies that the set  $\{\|\theta\|_T Y : \theta \in \mathcal{A}_x^S\}$  and hence also  $\{\|\theta\|_T : \theta \in \mathcal{A}_x^S\}$  are bounded in  $L^0$ .  $\square$

The consequence of the boundedness of total variations is that all admissible strategies have finite variation. Let  $\mathcal{P}_V$  denote the set of predictable finite variation processes. For each  $\theta \in \mathcal{P}_V$ ,  $\|\theta\|$  denotes the corresponding pathwise total variations (see Appendix A.2 for details).

**Lemma 5.7** *If  $(S, \kappa)$  satisfies (RNFLVR) and  $\theta \in \mathcal{A}_x$ . Then, there exists  $(\theta^n)_{n \geq 1}$  which converges admissibly to  $\theta$ , and  $\sup_{n \geq 1} \|\theta^n\|_T$  is a.s. finite. Hence,  $\theta \in \mathcal{P}_V$ .*

*Proof* Take an approximating sequence  $(\theta^n)_{n \geq 1}$  as in Definition 5.1. Since  $\theta$  is a.s. limit of predictable processes, it is predictable. The set  $(\|\theta^n\|_T)_{n \geq 1}$  is bounded in  $L^0$  by Lemma 5.6. Lemma B.1 yields convex combinations  $(\tilde{\theta}^n)_{n \geq 1}$  such that  $\sup_{n \geq 1} \|\tilde{\theta}^n\|_T$  is a.s. finite and  $\tilde{\theta}^n$  still converges admissibly to  $\theta$ . Proposition A.1 ii) implies that  $\liminf_{n \rightarrow \infty} \|\tilde{\theta}^n\|_T \geq \|\theta\|_T$  a.s., which completes the proof.

The cost of an admissible strategy admits a representation in terms of the predictable Stieltjes integral constructed in the Appendix. The key to this result is the ‘‘dominated convergence’’ Theorem A.9 iii), and the approximation property (Corollary A.12).

**Proposition 5.8** *Let  $(S, \kappa)$  satisfy (RNFLVR) and  $\theta \in \mathcal{A}_x$ . Then:*

i) *If  $\theta^n \xrightarrow{adm} \theta$ , and  $\sup_{n \geq 1} \|\theta^n\|_T < \infty$  a.s., then*

$$\liminf_{n \rightarrow \infty} C(\theta^n) \geq \int_{[0, T]} S d\theta + \int_{[0, T]} \kappa d\|\theta\| \quad \text{a.s.} \quad (5.4)$$

*where the integrals are defined in the predictable Stieltjes sense (see the Appendix).*

ii) There exists  $(\theta^n)_{n \geq 1}$ ,  $\theta_n \in \mathcal{A}_{x+1/n}^s$  such that equality holds:

$$\lim_{n \rightarrow \infty} C(\theta^n) = \int_{[0,T]} Sd\theta + \int_{[0,T]} \kappa d\|\theta\| \quad \text{a.s.}, \quad (5.5)$$

*Proof* The parts iii) and iv) of Theorem A.9 respectively imply that:

$$\lim_{n \rightarrow \infty} \int_{[0,T]} Sd\theta^n = \int_{[0,T]} Sd\theta \quad \text{a.s.} \quad (5.6)$$

$$\liminf_{n \rightarrow \infty} \int_{[0,T]} \kappa d\|\theta^n\| \geq \int_{[0,T]} \kappa d\|\theta\| \quad \text{a.s.} \quad (5.7)$$

which proves i). For ii), Corollary A.12 yields a simple strategy  $\theta^\varepsilon$  which satisfies  $|\theta^\varepsilon - \theta| < \varepsilon$  as well as  $|C(\theta^\varepsilon) - C(\theta)| < \varepsilon$  so  $V(\theta^\varepsilon) \geq -x - \varepsilon$ . This in turn implies that  $\theta^\varepsilon \in \mathcal{A}_{x+\varepsilon}^s$  by Proposition 4.6, and the claim follows since  $\varepsilon$  is arbitrary.  $\square$

As an immediate consequence, the representation of  $C(\theta)$  follows:

**Corollary 5.9** *Let  $\theta \in \mathcal{A}$ . Then  $C(\theta) = \int_{[0,T]} Sd\theta + \int_{[0,T]} \kappa d\|\theta\|$ .*

*Proof* Part i) of Proposition 5.8 implies that  $C(\theta) \geq \int_{[0,T]} Sd\theta + \int_{[0,T]} \kappa d\|\theta\|$ , while part ii) proves that equality holds.  $\square$

Denote by  $\mathcal{C}_x = \{V(\theta) : \theta \in \mathcal{A}_x\} - L_+^0$  and  $\mathcal{C} = \cup_{x>0} \mathcal{C}_x$  the set of claims dominated by general admissible strategies. Recall that  $(X_n)_{n \geq 1} \subset L^0$  Fatou converges to  $X$  if  $X_n$  converges to  $X$  a.s., and  $X_n \geq -c$  a.s. for all  $n$  and some  $c > 0$ .

**Proposition 5.10** *Under (RNFLVR),  $\mathcal{C}$  is Fatou closed.*

*Proof* Let  $(X_n)_{n \geq 1} \subset \mathcal{C}$  Fatou converge to  $X$ . It suffices to find some  $\theta \in \mathcal{A}$  such that  $V(\theta) \geq X$  a.s. By assumption,  $X_n = V(\theta^n) - f_n \geq -x$  for some  $\theta^n \in \mathcal{A}$  and  $f_n \geq 0$  a.s. In particular,  $V(\theta^n) \geq -x$ , and Corollary A.12 yields a sequence  $\tilde{\theta}^n \in \mathcal{A}_{x+1/n}^s$  such that  $|V(\tilde{\theta}^n) - V(\theta^n)| \leq 1/n$ . Since  $(\tilde{\theta}^n)_{n \geq 1} \subset \mathcal{A}_{x+1}^s$ ,  $(\|\tilde{\theta}^n\|_T)_{n \geq 1}$  is bounded in  $L^0$  by Lemma 5.6. Thus, by Lemma B.4, up to a sequence of convex combinations  $\tilde{\theta}^n$  converge a.s. to some  $\theta \in \mathcal{P}_V$ . Then Proposition 5.8 i) implies that:

$$V(\theta) \geq \liminf_{n \rightarrow \infty} V(\tilde{\theta}^n) \geq \liminf_{n \rightarrow \infty} (V(\theta^n) - 1/n) \geq \liminf_{n \rightarrow \infty} (X_n - 1/n) = X$$

which completes the proof.  $\square$

**Proposition 5.11** *Let  $(M, Q)$  be a consistent price system, with  $M$  locally bounded. Then  $E_Q[V^{(M,0)}(\theta)] \leq 0$  for all  $\theta \in \mathcal{A}$ .*

*Proof* As in the previous proof, Corollary A.12 yields a sequence  $\tilde{\theta}^n \in \mathcal{A}_{x+1/n}^s$  such that  $|V(\tilde{\theta}^n) - V(\theta^n)| \leq 1/n$ . Then  $V^{(M,0)}(\theta^n) \rightarrow V^{(M,0)}(\theta)$  a.s. by (5.6), and  $V^{(M,0)}(\theta^n) > V^{(S,\kappa)}(\theta^n) \geq -x - \varepsilon_n$  for some  $\varepsilon_n \downarrow 0$ . Since  $E_Q[V^{(M,0)}(\theta^n)] \leq 0$  by Proposition 4.9, the claim follows by Fatou's lemma:

$$E_Q[V^{(M,0)}(\theta)] \leq \liminf_{n \rightarrow \infty} E_Q[V^{(M,0)}(\theta^n)] \leq 0 \quad (5.8)$$

$\square$

## 6 Proof of Main Result

This section proves Theorem 1.1. The proof employs the usual separation method, combined with the following argument, in the spirit of Jouini and Kallal (1995), which is proved after the main theorem.

Let  $(S, \kappa)$  satisfy (RNFLVR), and let  $(S', \kappa')$  be its corresponding bid-ask pair. Then, the auxiliary pair  $(\hat{S}, \hat{\kappa}) := ([S + \tilde{S}]/2, [\kappa + \tilde{\kappa}]/2)$  satisfies (RNFLVR). Indeed:

$$\inf_{t \in [0, T]} (\tilde{\kappa}_t - \hat{\kappa}_t - |\tilde{S}_t - \hat{S}_t|) = \inf_{t \in [0, T]} (\kappa_t - \hat{\kappa}_t - |S_t - \hat{S}_t|) = \inf_{t \in [0, T]} \frac{(\kappa_t - \tilde{\kappa}_t - |S_t - \tilde{S}_t|)}{2} > 0$$

**Lemma 6.1** *If  $Q$  is equivalent to  $P$  and  $E_Q[V^{(\hat{S}, \hat{\kappa})}(\theta)] \leq 0$  for all  $\theta \in \mathcal{A}^{(\hat{S}, \hat{\kappa})}$ , then there exists a  $Q$ -local martingale  $M$  such that  $(M, Q)$  is a strictly consistent price system for  $(S, \kappa)$ .*

*Proof* of Theorem 1.1 (SCPS)  $\Rightarrow$  (RNFLVR): It suffices to check that  $\tilde{S} = M$  and  $\tilde{\kappa} = 0$  satisfies (NFLVR). First, note that (5.3) follows immediately:

$$\inf_{t \in [0, T]} (\kappa_t - \tilde{\kappa}_t - |S_t - \tilde{S}_t|) = \inf_{t \in [0, T]} (\kappa_t - |S_t - M_t|) > 0 \quad \text{a.s.}$$

To check that (NFLVR) holds, let  $(\theta^n)_{n \geq 1}$  be such that  $\theta^n \in \mathcal{A}_{1/n}^s$  and  $V^{(M, 0)}(\theta^n)$  converge a.s. to some  $V$ . Proposition 5.11 implies that  $E_Q[V^{(M, 0)}(\theta^n)] \leq 0$ . Thus, by Fatous's lemma  $E_Q[V] \leq 0$  and  $V = 0$  a.s.

(RNFLVR)  $\Rightarrow$  (SCPS): The convex set  $\mathcal{C}$  (with respect to  $(\hat{S}, \hat{\kappa})$ ) is Fatou closed by Proposition 5.10. Thus,  $C := \mathcal{C} \cap L^\infty$  is also Fatou closed and convex, and Lemma B.2 implies that  $C$  is also  $\sigma(L^\infty, L^1)$ -closed. By Theorem B.3, there exists a probability  $Q$ , equivalent to  $P$  such that  $E_Q[C] \leq 0$ , and Lemma 6.1 concludes the proof.  $\square$

The rest of this section proves Lemma 6.1. The arguments are standard, and included only for completeness. The next lemma is due to Jouini and Kallal (1995) (cf. also Choulli and Stricker (1998) and Cherny (2007)).

**Lemma 6.2** *Let  $(X_t)_{t \in [0, T]}$  be a submartingale and  $(Y_t)_{t \in [0, T]}$  a supermartingale, such that  $X \leq Y$  a.s. Then there exists a martingale  $(M_t)_{t \in [0, T]}$  such that  $X \leq M \leq Y$  a.s.*

*Proof* For finitely many time instants  $0 \leq t_0 < \dots < t_n \leq T$ , set  $M_{t_0} = Y_{t_0}$  and recursively define  $M_{t_{n+1}} = \alpha_n X_{t_{n+1}} + (1 - \alpha_n) Y_{t_{n+1}}$ , where the  $\mathcal{F}_{t_n}$ -measurable  $\alpha_n$  satisfies:

$$M_{t_n} = \alpha_n E[X_{t_{n+1}} | \mathcal{F}_{t_n}] + (1 - \alpha_n) E[Y_{t_{n+1}} | \mathcal{F}_{t_n}]$$

Then  $M$  is a martingale for the filtration  $(\mathcal{F}_t)_{0 \leq t \leq n}$  and  $X_i \leq M_i \leq Y_i$  for  $i = 0, \dots, n$ . From the discrete case just considered, each dyadic partition  $D_n = \{kT/2^n : 0 \leq k \leq 2^n\}$  yields a martingale  $M^n$  with respect to the discrete filtration  $(\mathcal{F}_t)_{t \in D_n}$ , such that:

$$X_t \leq M_t^n \leq Y_t \quad \text{for all } t \in D_n \quad (6.1)$$

In particular,  $X_T \leq M_T^n \leq Y_T$  for all  $n \geq 1$ , therefore  $(M_T^n)_{n \geq 1}$  is bounded in  $L^1$ , and by Komlós Theorem it converges up to a sequence of convex combinations to some random variable  $M$  almost surely and in  $L^1$  (due to the integrability of  $Y_T$ ). Then define the martingale  $M_t = E[M_T | \mathcal{F}_t]$ , and passing to the limit in (6.1) as  $n \rightarrow \infty$  yields  $X_t \leq M_t \leq Y_t$  a.s. for all  $t \in [0, T]$ .  $\square$

The next Lemma extends Theorem 4.5 in Cherny (2007) to the present setting:

**Lemma 6.3** *Let  $(X_t)_{t \in [0, T]}$  and  $(Y_t)_{t \in [0, T]}$  be two càdlàg processes. The following conditions are equivalent:*

i) *There exists a càdlàg martingale  $(M_t)_{t \in [0, T]}$  such that:*

$$X \leq M \leq Y \quad \text{a.s.} \quad (6.2)$$

ii) *For all stopping times  $\sigma, \tau$  such that  $0 \leq \sigma \leq \tau \leq T$  a.s., the following hold:*

$$E[X_\tau | \mathcal{F}_\sigma] \leq Y_\sigma \quad \text{and} \quad E[Y_\tau | \mathcal{F}_\sigma] \geq X_\sigma \quad \text{a.s.} \quad (6.3)$$

*Proof*

i)  $\Rightarrow$  ii): (6.2) and the optional sampling theorem imply that:

$$E[X_\tau | \mathcal{F}_\sigma] \leq E[M_\tau | \mathcal{F}_\sigma] = M_\sigma \leq Y_\sigma$$

and the second equation in (6.3) follows similarly.

ii)  $\Rightarrow$  i): Denoting by  $\mathcal{O}_t$  the set of stopping times with values in the interval  $[t, T]$ , define the auxiliary processes:

$$X'_t = \text{ess sup}_{\tau \in \mathcal{O}_t} E[X_\tau | \mathcal{F}_t] \quad \text{and} \quad Y'_t = \text{ess inf}_{\tau \in \mathcal{O}_t} E[Y_\tau | \mathcal{F}_t]$$

Note that for  $\sigma, \tau \in \mathcal{O}_t$  and  $A \in \mathcal{F}_t$  we have that  $\sigma 1_A + \tau 1_{\Omega \setminus A} \in \mathcal{O}_t$ . Thus:

$$\text{ess sup}_{\tau \in \mathcal{O}_t} E[X_\tau | \mathcal{F}_t] = E[\text{ess sup}_{\tau \in \mathcal{O}_t} X_\tau | \mathcal{F}_t]$$

and therefore  $X'$  is a supermartingale. Likewise,  $Y'$  is a submartingale, and they both admit càdlàg versions (this follows as in Proposition 4.3 in Kramkov (1996)). ii) implies that, for all  $\sigma, \tau \in \mathcal{O}_t$ :

$$\begin{aligned} E[X_\tau | \mathcal{F}_t] - E[Y_\sigma | \mathcal{F}_t] &= E[E[X_\tau - Y_\sigma | \mathcal{F}_{\tau \wedge \sigma}] | \mathcal{F}_t] = \\ &= E[(X_\tau - E[Y_\sigma | \mathcal{F}_\tau]) 1_{\{\tau \leq \sigma\}} + (E[X_\tau | \mathcal{F}_\sigma] - Y_\sigma) 1_{\{\sigma < \tau\}} | \mathcal{F}_t] \leq 0 \quad \text{a.s.} \end{aligned}$$

and hence  $X'_t \leq Y'_t$  a.s. for all  $t \in [0, T]$ . Lemma 6.2 concludes the proof.  $\square$

*Proof of Lemma 6.1* By localization, assume that  $S, \kappa$  are bounded. Then, for any stopping times  $\sigma < \tau$  and  $A \in \mathcal{F}_\sigma$  the strategy  $\theta = \pm 1_{A \cap ]\sigma, \tau]}$  is admissible for  $(\hat{S}, \hat{\kappa})$ . Therefore:

$$E_Q[(\hat{S} - \hat{\kappa})_\tau - (\hat{S} + \hat{\kappa})_\sigma] 1_A \leq 0, \quad E_Q[(\hat{S} + \hat{\kappa})_\tau - (\hat{S} - \hat{\kappa})_\sigma] 1_A \leq 0$$

Since these equations hold for any  $A \in \mathcal{F}_\sigma$ , it follows that:

$$E_Q[(\hat{S} + \hat{\kappa})_\tau | \mathcal{F}_\sigma] \leq (\hat{S} - \hat{\kappa})_\sigma, \quad E_Q[(\hat{S} - \hat{\kappa})_\tau | \mathcal{F}_\sigma] \leq (\hat{S} + \hat{\kappa})_\sigma$$

and Lemma 6.3 implies the existence of a  $Q$ -martingale  $M$  such that:

$$|\hat{S}_t - M_t| \leq \hat{\kappa}_t \quad \text{a.s. for all } t \in [0, T]$$

Therefore, (6) implies that:

$$\inf_{t \in [0, T]} (\kappa_t - |S_t - M_t|) \geq \inf_{t \in [0, T]} (\kappa_t - \hat{\kappa}_t - |\hat{S}_t - S_t|) > 0$$

as required.  $\square$

## 7 Conclusion

This paper proves the Fundamental Theorem of Asset Pricing under transaction costs in continuous time for potentially discontinuous – but locally bounded – prices.

The treatment focuses on a single risky asset mainly to ease notation. With the exception of Proposition 4.6, these results hold virtually unchanged for models with several assets trading against a common numéraire. The extension of Proposition 4.6 to several risky assets becomes exceedingly complicated, as each risky asset may require a different liquidation time. Such an extension is not treated here.

The main conceptual difference between this model (which follows Jouini and Kallal (1995)) and the pure exchange models initiated by Kabanov (1999) is in the role of the numéraire. Exchange models are by construction numéraire free and – unlike frictionless models – define admissibility symmetrically with respect to all assets. This symmetry is possible because in such models relative prices are always positive.

In the bid-ask model considered here, the numéraire is the only measure of value, since prices can be positive as well as negative. A trading strategy is admissible when its eventual liquidation leads to a bounded loss. Boundedness crucially depends on the choice of numéraire, as in the frictionless theory.

### A Predictable Stieltjes integral

The usual Stieltjes integral  $\int S d\theta$  is well-defined for continuous integrands  $S$  and integrators  $\theta$  of finite variation. This subsection defines an extension of the Stieltjes integral to càdlàg integrands, which makes the integral operator continuous with respect to the pointwise convergence of integrators.

The discussion is divided into three parts: the first subsection recalls some properties of predictable finite variation processes, available in most textbooks under the extra assumption of càdlàg processes, relaxed here to làdlàg. The second subsection defines the integral, and establishes its Lebesgue- and Fatou-type properties. Since the integral is defined pathwise, the first two subsections are probability-free, requiring only a filtered measurable space with a right-continuous filtration. This includes the special case when both the integrator and the integrand are deterministic functions. The third subsection establishes an approximation result for the predictable integral, which requires an underlying probability measure, since it relies on the decomposition of a stopping time into its accessible and totally inaccessible parts. This part requires a filtered probability space satisfying the usual conditions.

### A.1 Predictable finite variation processes

This subsection recalls some properties of finite variation processes, dropping the extra assumption of right-continuity appearing in most textbooks.

Consider a measurable space  $(\Omega, \mathcal{F})$ , endowed with a right-continuous filtration  $(\mathcal{F}_t)_{t \geq 0}$  such that  $\mathcal{F}_\infty = \mathcal{F}$ . On the set  $\Omega \times \mathbb{R}_+$ , the predictable  $\sigma$ -algebra is generated by the class of adapted and left-continuous processes. Denote by  $\mathcal{P}_V$  the class of predictable processes with pathwise finite variation; by  $\mathcal{D}$  the class of adapted càdlàg processes and by  $\mathcal{L}$  the class of adapted làglàd (right-limited and left-limited) processes. Since a function of finite variation has right and left limits at all points, it follows that  $\mathcal{P}_V \subset \mathcal{L}$ . A sequence of processes  $X^n$  converges pointwise to a process  $X$  if  $X^n(\omega, t) \rightarrow X(\omega, t)$  for all  $(\omega, t) \in \Omega \times \mathbb{R}_+$ .

**Proposition A.1** *Let  $\theta \in \mathcal{P}_V$  and let  $\|\theta\|$  denote its pathwise total variation. Then:*

- i)  $\|\theta\| \in \mathcal{P}_V$ ;
- ii) if  $(\theta^n)_{n \geq 1} \subset \mathcal{P}_V$  converge pointwise to  $\theta$ , then:

$$\|\theta\| \leq \liminf_{n \rightarrow \infty} \|\theta^n\| \quad \text{pointwise}$$

The proof requires a representation of the set of jumps of predictable finite variation processes. The finite-variation property avoids the use of section theorems. Recall that a random set  $A \subset \Omega \times [0, \infty)$  is *thin* if  $A \subset \cup_{n \geq 1} \llbracket \tau_n \rrbracket$  for a sequence of stopping times  $(\tau_n)_{n \geq 1}$ .

**Lemma A.2** *Let  $X \in \mathcal{P}_V$ . Then the sets  $\{X \neq X_-\}$  and  $\{X \neq X_+\}$  are thin, and  $\{X \neq X_-\}$  is predictable.*

*Proof* For  $n, p \geq 0$ , define the stopping times  $S_{n,p}$  by induction as  $S_{n,0} = 0$  and:

$$S_{n,p+1} = \begin{cases} \inf\{t > S_{n,p} : |X_t - X_{S_{n,p}^+}| > 2^{-n}\} & \text{for even } p \\ \inf\{t \geq S_{n,p} : |X_t - X_{S_{n,p}}| > 2^{-n}\} & \text{for odd } p \end{cases}$$

By definition,  $S_{n,p} < S_{n,p+2}$ . Define also the random sets:

$$A_{n,p} = \begin{cases} \{X_{S_{n,p}} \neq X_{S_{n,p}^-}, S_{n,p} < \infty\} & \text{for even } p \\ \{X_{S_{n,p}^+} \neq X_{S_{n,p}}, S_{n,p} < \infty\} & \text{for odd } p \end{cases}$$

and observe that  $A_{n,p} \in \mathcal{F}_{S_{n,p}}$ . This is obvious for  $p$  even, while for  $p$  odd the right-continuity of the filtration implies that  $X_{S_{n,p}^+}$  is  $\mathcal{F}_{S_{n,p}}$ -measurable, whence the claim. Then, set  $T_{n,p} = S_{n,p} 1_{A_{n,p}} + \infty 1_{A_{n,p}^c}$ , which is a stopping time since  $A_{n,p} \in \mathcal{F}_{S_{n,p}}$ . As  $X$  has finite variation,  $\lim_{p \rightarrow \infty} S_{n,p} = \infty$  and therefore:

$$\{X \neq X_-\} = \cup_{n,p \text{ even}} \llbracket T_{n,p} \rrbracket \quad \text{and} \quad \{X_+ \neq X\} = \cup_{n,p \text{ odd}} \llbracket T_{n,p} \rrbracket,$$

so the sets  $\{X \neq X_+\}$  and  $\{X_- \neq X\}$  are thin. Also,  $X_-$  is left-continuous, therefore the set  $\{X \neq X_-\}$  is predictable.  $\square$

*Proof* of Proposition A.1. The proof of *i*) is similar to the càdlàg case (Delbaen and Schachermayer 2006, Theorem 12.2.1). By Lemma A.2, there is an exhausting sequence of stopping times  $(\sigma_k)_{k \geq 1}$  for the set of jumps of  $\theta$ . For a fixed  $n \geq 1$ , denote by  $\Pi_n = (\tau_k)_{0 \leq k \leq N_n}$  the finite ordered sequence of stopping times obtained from the set  $(k2^{-n})_{0 \leq k \leq 2^n T} \cup (\sigma_k)_{k \leq n}$ , and define the processes:

$$V_t(\theta, \Pi_n) = \sum_{k=1}^{N_n} |\theta_{\tau_k \wedge t} - \theta_{\tau_{k-1} \wedge t}| \quad (\text{A.1})$$

$V(\theta, \Pi_n)$  is predictable because  $\theta$  is, and since  $V(\theta, \Pi_n)$  converges to  $\|\theta\|$  pointwise, also  $\|\theta\|$  is predictable and *i*) follows.

For *ii*), denote by  $\Pi = (\tau_k)_{0 \leq k \leq n}$  a finite ordered sequence of stopping times, and define  $V(\theta, \Pi)$  as in (A.1). The pointwise convergence of  $(\theta^n)_{n \geq 1}$  to  $\theta$  implies that:

$$\lim_{n \rightarrow \infty} V_t(\theta^n, \Pi) = V_t(\theta, \Pi) \quad \text{for all } t, \Pi$$

By *i*) it follows that  $\|\theta\|_t = \sup_{\Pi} V_t(\theta, \Pi)$ , and therefore:

$$\liminf_{n \rightarrow \infty} \|\theta^n\|_t = \liminf_{n \rightarrow \infty} \sup_{\Pi} V_t(\theta^n, \Pi) \geq \sup_{\Pi} \lim_{n \rightarrow \infty} V_t(\theta^n, \Pi) = \sup_{\Pi} V_t(\theta, \Pi) = \|\theta\|_t$$

which proves *ii*).  $\square$

## A.2 The Predictable Stieltjes Integral

This section defines the integral  $\int_0^t S d\theta$  for each  $\theta \in \mathcal{P}_V$  and  $S \in \mathcal{D}$ . Without additional assumptions on  $\theta$  and  $S$ , such an integral may not exist in the standard Stieltjes sense. A simple example demonstrates the problem:

*Example A.3* Consider the deterministic  $S = \theta = 1_{[1, 2]}$ . Then  $S$  is not Stieltjes integrable with respect to  $\theta$ , because there are arbitrarily fine subdivisions such that the corresponding Riemann sums are either 0 or 1. Indeed, some Riemann sums include the jump  $\Delta S_1(\theta_{1+} - \theta_1)$ , while others do not.

However,  $\int S d\theta$  is a Stieltjes integral if  $\theta$  is left-continuous:

**Lemma A.4** *Let  $S$  be a càdlàg function and  $\theta$  a function of finite variation. Then  $\int_{[0, T]} S d\theta_-$  exists as a Stieltjes integral.*

The proof requires a simple Lemma.

**Lemma A.5** *Let  $S$  be a càdlàg (deterministic) function such that  $|\Delta S_t| \leq s$  for all  $t$ . Then for all  $\eta > s$  there exists  $\delta > 0$  such that  $|S_u - S_t| \leq \eta$  for all  $u, t$  such that  $|t - u| \leq \delta$ .*

*Proof* By the càdlàg assumption, for all  $t \in [0, T]$  there exists some  $\delta_t > 0$  such that  $|S_u - S_t| < (\eta - s)/2$  for  $u \in [t, t + \delta_t)$  and  $|S_u - S_{t-}| < (\eta - s)/2$  for  $u \in (t - \delta_t, t)$ . Therefore  $|S_u - S_t| < \eta$  for all  $u \in U_t = (t - \delta_t, t + \delta_t)$ . By compactness, the open covering of  $(U_t)_{t \in [0, T]}$  admits a finite subcovering  $(U_i)_{i=1}^n$ , and the thesis follows with  $\delta = \min_{1 \leq i \leq n} \delta_i$ .  $\square$

*Proof of Lemma A.4.* Recall two basic facts (cf. Wheeden and Zygmund (1977)). First, the Stieltjes integral  $\int f dg$  exists if and only if for all  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for any pair of partitions  $\Gamma, \Gamma'$  with mesh smaller than  $\delta$ , the corresponding Riemann sums differ by less than  $\varepsilon$ . Second, if two functions  $f$  and  $g$  have a finite number of jump discontinuities, then the integrals  $\int f dg$  and  $\int g df$  are defined if and only if at each (common) discontinuity point one function is right-continuous and the other is left-continuous.

For a given  $\varepsilon > 0$ , set  $J_t^\varepsilon = \sum_{s \leq t} \Delta S_s 1_{\{|\Delta S_s| > \varepsilon\}}$ , and  $S^\varepsilon = S - J^\varepsilon$ . Since  $S$  is càdlàg, it has a finite number of jumps larger than any given size, therefore  $J^\varepsilon$  is piecewise constant and right-continuous. By construction,  $S^\varepsilon$  is càdlàg with jumps bounded by  $\varepsilon$ . Since  $S$  is càdlàg and  $\theta_-$  left-continuous, by the previous observation  $J^\varepsilon$  is Stieltjes integrable with respect to  $\theta_-$ . For  $S^\varepsilon$ , consider two partitions  $\Gamma = (t_i)_{i=0}^n, \Gamma' = (s_i)_{i=0}^m$ , and denote by  $\Gamma \cup \Gamma' = (r_i)_{i=0}^l$ . Let  $u_i \in [t_{i-1}, t_i], v_i \in [s_{i-1}, s_i]$  arbitrary points. The corresponding Riemann sums satisfy:

$$\begin{aligned} I_T(S^\varepsilon, \theta; \Gamma) - I_T(S^\varepsilon, \theta; \Gamma') &= \sum_{t_i \leq T} S_{u_i}(\theta_{t_i^-} - \theta_{t_{i-1}^-}) - \sum_{s_i \leq T} S_{v_i}(\theta_{s_i^-} - \theta_{s_{i-1}^-}) = \\ &= \sum_{r_i \leq T} (S_{t(r_i)} - S_{s(r_{i-1})})(\theta_{r_i^-} - \theta_{r_{i-1}^-}) \end{aligned}$$

where  $t(r_j) = u_i$  with  $i$  such that  $[r_{j-1}, r_j] \subset [t_{i-1}, t_i]$  and  $s(r_j)$  is defined analogously. If the meshes of  $\Gamma$  and  $\Gamma'$  are smaller than  $\delta$ , then  $|t(r) - s(r)| \leq \delta$  for all  $r$ . Applying Lemma A.5 with  $s = \varepsilon$  and  $\eta = 2\varepsilon$  for some small  $\delta$ , it follows that:

$$|I_T(S^\varepsilon, \theta; \Gamma) - I_T(S^\varepsilon, \theta; \Gamma')| \leq \sum_{r_i \leq T} |S_{t(r_i)} - S_{s(r_{i-1})}| |\theta_{r_i^-} - \theta_{r_{i-1}^-}| \leq 2\varepsilon \|\theta\|_T$$

Since  $J^\varepsilon$  is Stieltjes integrable,  $|I_T(J^\varepsilon, \theta; \Gamma) - I_T(J^\varepsilon, \theta; \Gamma')| \leq 2\varepsilon \|\theta\|_T$  up to a smaller  $\delta' < \delta$ . Adding up, for any  $\varepsilon > 0$  there is some  $\delta' > 0$  such that:

$$|I_T(S, \theta; \Gamma) - I_T(S, \theta; \Gamma')| \leq 4\varepsilon \|\theta\|_T$$

for any pair of partitions  $\Gamma, \Gamma'$  with mesh smaller than  $\delta'$ , which amounts to pathwise integrability in the sense of Stieltjes.  $\square$

The predictable Stieltjes integral is defined as follows:

**Definition A.6** For an arbitrary finite variation  $\theta$  and càdlàg  $S$ , define:

$$I_T(S, \theta) := \int_{[0, T]} S_u d\theta_u := \int_{[0, T]} S d\theta_- - \sum_{s \leq T} (\theta_s - \theta_{s^-}) \Delta S_s. \quad (\text{A.2})$$

The first term is a well-defined Stieltjes integral by Lemma A.4, and the second term is absolutely convergent:

$$\sum_{s \leq T} (\theta_s - \theta_{s^-}) \Delta S_s \leq \sum_{s \leq T} |\theta_s - \theta_{s^-}| 2S_T^* \leq 2\|\theta\|_T S_T^* < \infty$$

since  $S_T^* = \sup_{t \in [0, T]} |S_t|$  is finite by the càdlàg assumption.

*Remark A.7* The case for definition (A.2) rests on both mathematical and economic arguments. Proposition A.15 below justifies (A.2) from the viewpoint of stochastic analysis, connecting the predictable Stieltjes integral to the usual stochastic integral via integration by parts.

From the economic viewpoint, consider  $S \in \mathcal{D}$  and  $\theta \in \mathcal{P}_V$ , regarding  $S$  as the asset price, and  $\theta_t$  (resp.  $\theta_{t-}/\theta_{t+}$ ) as the number of shares held at time  $t$  (resp. immediately before/after  $t$ ). Intuitively, a jump of  $S$  at a predictable time  $\tau$  corresponds to a shock whose size is unknown but whose timing is known before its occurrence (e.g. the announcement of macroeconomic data). Viceversa, a totally inaccessible  $\tau$  corresponds to a shock with sudden timing (e.g. a natural catastrophe). The definition of  $\int Sd\theta$  should reflect the different natures of such jumps.

On a predictable jump, the investor may rebalance her portfolio “immediately” before  $\tau$  (and change from position  $\theta_{\tau-}$  to  $\theta_{\tau}$ ) and react after the price has changed from  $S_{\tau-}$  to  $S_{\tau}$  (by moving from  $\theta_{\tau}$  to  $\theta_{\tau+}$ ). On an unpredictable jump (i.e.  $\tau$  totally inaccessible) necessarily  $\theta_{\tau-} = \theta_{\tau}$  as  $\theta$  is predictable (there is no possibility to prepare for  $\tau$ ), but there is still a possibility of rebalancing from  $\theta_{\tau}$  to  $\theta_{\tau+}$ . Thus, the cost of the portfolio  $\theta$  should be:

$$S_{\tau-}(\theta_{\tau} - \theta_{\tau-}) + S_{\tau}(\theta_{\tau+} - \theta_{\tau}) = S_{\tau}(\theta_{\tau+} - \theta_{\tau-}) - \Delta S_{\tau}(\theta_{\tau} - \theta_{\tau-}) \quad (\text{A.3})$$

in the case of predictable  $\tau$  and

$$S_{\tau}(\theta_{\tau+} - \theta_{\tau}) = S_{\tau}(\theta_{\tau+} - \theta_{\tau-}) \quad (\text{A.4})$$

in the unpredictable case. These formulas are indeed consistent with (A.2).<sup>2</sup>

Looking at (A.3) and (A.4) under a different angle, the predictability of  $\theta$  dictates that  $\int Sd\theta$  should include  $\Delta S_{\tau}(\theta_{\tau+} - \theta_{\tau})$  at all predictable stopping times, and should not include it at totally inaccessible times.

The pathwise definition of the integral (A.2) extends to a linear map from processes into processes.

**Proposition A.8** *The function  $(S, \theta) \mapsto I(S, \theta)$*

$$I_T(S, \theta) = \int_{[0, T]} Sd\theta_- - \sum_{s \leq T} (\theta_s - \theta_{s-}) \Delta S_s \quad (\text{A.5})$$

maps  $\mathcal{D} \times \mathcal{P}_V$  into  $\mathcal{L}$ .

*Proof* It is necessary to check that  $\omega \mapsto I_T(S, \theta)(\omega)$  is  $\mathcal{F}_T$ -measurable for all  $T > 0$ , and that the paths  $t \mapsto I_t(S, \theta)$  has have right and left limits. For the latter property, observe that the integral in (A.5) is left-continuous with right limits, while the summation in (A.5) is right-continuous with left limits.

For the measurability part, note that the second term in (A.5) is trivially  $\mathcal{F}_T$ -measurable, hence it suffices to check the first term, which is a pathwise Stieltjes

<sup>2</sup> Note that in the present paper there are also transaction costs represented by the process  $\kappa$  hence the additional cost term  $\int_{[0, T]} \kappa_u d\|\theta\|_u$  appears, see Definition 4.2. Remember also that  $\int_{[0, T]} S_t d\theta_t$  (as well as  $\int_{[0, T]} \kappa_u d\|\theta\|_u$ ) represent the *cost* of trading with strategy  $\theta$  on the interval  $[0, T]$ , cf. Definition 4.2.

integral by Lemma A.4. Thus, it is the limit of Riemann sums along a deterministic grid:

$$\int_{[0,T]} Sd\theta_- = \lim_{n \rightarrow \infty} \sum_{k=0}^{\lfloor nT \rfloor} S_{\frac{k}{n}} (\theta_{\frac{k+1}{n}-} - \theta_{\frac{k}{n}-}) \quad (\text{A.6})$$

and therefore it is  $\mathcal{F}_T$ -measurable.  $\square$

The following Theorem establishes the main properties of the predictable Stieltjes integral. First, if  $\theta$  is simple (i.e. constant on a sequence of intervals, which may collapse to points), then  $\int Sd\theta$  is consistent with the definition of cost process in the main text. Second, the integral satisfies the natural bound (A.8). Finally, the integral satisfies a dominated convergence theorem and a Fatou property with respect to the pointwise convergence of the integrator  $\theta$ .

**Theorem A.9** *The map  $I$  in (A.2) satisfies the following properties:*

i) *If  $(\tau_n)_{n \geq 0}$  is a sequence of stopping times such that  $\theta = \sum_{n=1}^{\infty} (\theta_{\tau_n} 1_{[\tau_n, \tau_{n+1}[} + \theta_{\tau_n^+} 1_{] \tau_n, \tau_{n+1}])}$  is predictable then*

$$I_T(S, \theta) = \sum_{\tau_i \leq T} S_{\tau_i^-} (\theta_{\tau_i} - \theta_{\tau_i^-}) + \sum_{\tau_i < T} S_{\tau_i} (\theta_{\tau_i^+} - \theta_{\tau_i}) \quad (\text{A.7})$$

ii)  *$I_T$  is linear both in  $S$  and in  $\theta$ , and, denoting by  $S_T^* = \sup_{t \in [0, T]} |S_t|$ :*

$$|I_T(S, \theta)| \leq \|\theta\|_T S_T^* \quad (\text{A.8})$$

iii) *If  $\theta^n \rightarrow \theta$  pointwise and  $\sup_{n \geq 1} \|\theta^n\|_T < \infty$ , then  $I(S, \theta^n) \rightarrow I(S, \theta)$  pointwise.*  
 iv) *If  $(\theta^n)_{n \geq 1}$  are as in iii) and  $S \geq 0$ , then  $\liminf_{n \rightarrow \infty} I(S, \|\theta^n\|) \geq I(S, \|\theta\|)$  pointwise.*

*Proof i):* (A.7) follows immediately from (A.2).

For ii), the linearity in  $\theta$  and  $S$  is immediate from (A.2), which also implies the estimate:

$$|I_T(S, \theta)| \leq \left| \int_{[0, T]} Sd\theta_- \right| + \left| \sum_{t \leq T} (\theta_t - \theta_{t-}) \Delta S_t \right| \leq \|\theta\|_T S_T^* + \|\theta\|_T 2S_T^* = 3\|\theta\|_T S_T^* \quad (\text{A.9})$$

Now, define the sequence of stopping times  $(\sigma_n^\varepsilon)_{n \geq 0}$ :

$$\sigma_0^\varepsilon = 0 \quad \text{and} \quad \sigma_{n+1}^\varepsilon = \inf\{t > \sigma_n^\varepsilon : |S_t - S_{\sigma_n^\varepsilon}| > \varepsilon\} \quad (\text{A.10})$$

and set  $S^\varepsilon = \sum_{j=0}^{\infty} S_{\sigma_j^\varepsilon} 1_{[\sigma_j^\varepsilon, \sigma_{j+1}^\varepsilon[}$ , which is piecewise constant and right-continuous, and satisfies  $|S_t - S_t^\varepsilon| \leq \varepsilon$  for all  $t \in [0, T]$ . By linearity in  $S$  and (A.9), for any  $\varepsilon > 0$  the following holds:

$$|I_T(S, \theta)| \leq |I_T(S^\varepsilon, \theta)| + 3\varepsilon \|\theta\|_T$$

Fix  $T > 0$ , we may and will assume  $\theta_0 = \theta_s = 0$  for  $s \geq T$  from now on. To calculate  $I_T(S^\varepsilon, \theta)$ , first observe that (with the convention  $\sigma_{-1} = 0$ ):

$$\begin{aligned} \int_{[0,T]} S^\varepsilon d\theta_- &= \sum_{n \geq 0} S_{\sigma_n}^\varepsilon (\theta_{\sigma_{n+1}^-} - \theta_{\sigma_n^-}) = - \sum_{n \geq 0} \theta_{\sigma_n^-} (S_{\sigma_n} - S_{\sigma_{n-1}}), \\ \sum_{t \in [0,T]} (\theta_t - \theta_{t-}) \Delta S_t^\varepsilon &= \sum_{n \geq 0} (\theta_{\sigma_n} - \theta_{\sigma_n^-}) (S_{\sigma_n}^\varepsilon - S_{\sigma_n^-}) = \sum_{n \geq 0} (\theta_{\sigma_n} - \theta_{\sigma_n^-}) (S_{\sigma_n} - S_{\sigma_{n-1}}). \end{aligned}$$

and therefore:

$$I_T(S^\varepsilon, \theta) = - \sum_{n \geq 0} \left[ \theta_{\sigma_n^-} (S_{\sigma_n} - S_{\sigma_{n-1}}) + (\theta_{\sigma_n} - \theta_{\sigma_n^-}) (S_{\sigma_n} - S_{\sigma_{n-1}}) \right] \quad (\text{A.11})$$

$$= - \sum_{n \geq 0} \theta_{\sigma_n} (S_{\sigma_n} - S_{\sigma_{n-1}}) = \sum_{n \geq 0} S_{\sigma_{n-1}} (\theta_{\sigma_n} - \theta_{\sigma_{n-1}}). \quad (\text{A.12})$$

Consequently,

$$|I_T(S^\varepsilon, \theta)| = \left| \sum_{\sigma_n \leq T} S_{\sigma_{n-1}} (\theta_{\sigma_n} - \theta_{\sigma_{n-1}}) \right| \leq S_T^* \|\theta\|_T$$

and *ii*) follows since  $\varepsilon > 0$  is arbitrary.

*iii*): By the linearity in  $S$ , it suffices to consider the case of  $\theta^n$  converging pointwise to zero. *ii*) implies that:

$$|I_T(S, \theta^n)| \leq |I_T(S^\varepsilon, \theta^n)| + \varepsilon \|\theta^n\|_T$$

The calculations in the proof of *ii*) show that the term  $|I_T(S^\varepsilon, \theta^n)|$  depends only on finitely many values of  $\theta$ , thus for large  $n$  it becomes arbitrarily small. As  $M := \sup_{n \geq 1} \|\theta^n\|_T$  is finite,

$$\limsup_{n \rightarrow \infty} |I_T(S, \theta^n)| \leq \varepsilon M$$

and convergence follows since  $\varepsilon$  is arbitrary.

*iv*): By the same argument as in the proof of Proposition A.1 *ii*), for  $s(\omega) < t(\omega)$  the following holds:

$$\liminf_{n \rightarrow \infty} (\|\theta^n\|_t - \|\theta^n\|_s) \geq \|\theta\|_t - \|\theta\|_s,$$

in the pointwise sense. Then, recalling (A.11),  $S \geq 0$  implies that:

$$\begin{aligned} \liminf_{n \rightarrow \infty} I_T(S^\varepsilon, \|\theta^n\|) &= \liminf_{n \rightarrow \infty} \sum_{k \geq 1} S_{\sigma_{k-1}} (\|\theta_{\sigma_k}^n\| - \|\theta_{\sigma_{k-1}}^n\|) \\ &\geq \sum_{k \geq 1} S_{\sigma_{k-1}} (\|\theta_{\sigma_k}\| - \|\theta_{\sigma_{k-1}}\|) = I_T(S^\varepsilon, \|\theta\|) \end{aligned}$$

Since  $|I_T(S, \|\theta^n\|) - I_T(S^\varepsilon, \|\theta^n\|)| \leq \varepsilon \|\theta^n\|_T$  and  $|I_T(S, \|\theta\|) - I_T(S^\varepsilon, \|\theta\|)| \leq \varepsilon \|\theta\|_T$ , it follows that:

$$\varepsilon \left( \sup_n \|\theta^n\|_T + \|\theta\|_T \right) + \liminf_{n \rightarrow \infty} \int_{[0,T]} S d\|\theta^n\| \geq \int_{[0,T]} S d\|\theta\|.$$

and the thesis follows as  $\varepsilon \downarrow 0$ .  $\square$

### A.3 Approximations of predictable integrals

The previous subsection shows that the integral  $\theta \mapsto I_T(S, \theta)$  is continuous with respect to the pointwise convergence. This result is probability free.

However, a much more precise result holds in a filtered probability space satisfying the usual conditions, when  $S$  is locally bounded: both  $\theta$  and  $I_T(S, \theta)$  admit almost surely uniform approximations. Any underlying probability admits an approximating sequence, but this sequence may depend on the probability itself. In this regard, approximating sequence are analogous to announcing sequences for predictable times.

Let us fix a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ .

**Theorem A.10** *Let  $\theta \in \mathcal{P}_V$  and  $S \in \mathcal{D}$  be locally bounded. Then, for all  $\varepsilon > 0$  there exists a strictly increasing sequence of stopping times  $(\sigma_n)_{n \geq 0}$  such that  $\sup_{n \geq 1} \sigma_n > T$ , and a predictable process  $\theta''$  of the form:*

$$\theta'' = \sum_{n=0}^{\infty} \left( \theta_{\sigma_n} 1_{\llbracket \sigma_n \rrbracket} + \theta_{\sigma_n^+} 1_{\llbracket \sigma_n, \sigma_{n+1} \rrbracket} \right) \quad (\text{A.13})$$

satisfying  $\theta'' \in \mathcal{P}_V$ ,  $|\theta'' - \theta| \leq \varepsilon$ ,  $|\int S d\theta'' - \int S d\theta| \leq \varepsilon$  and  $\|\theta''\| \leq \|\theta\|$  pointwise on  $[0, T]$  (outside a  $P$ -zero set).

The proof of this theorem requires a property of predictable finite variation processes:

**Proposition A.11** *Any  $\theta \in \mathcal{P}_V$  is locally bounded.*

*Proof* Let  $\tau_n = \inf\{t > 0 : |\theta_t| > n\}$ , so that  $\lim_{n \rightarrow \infty} \tau_n = \infty$  a.s. Then  $|\theta| 1_{\llbracket 0, \tau_n \rrbracket}$  is bounded, i.e.  $|\theta|$  is prelocally bounded. A prelocally bounded predictable process is also locally bounded by section 11 of Chapter 8 in Dellacherie and Meyer (1982).  $\square$

*Proof* of Theorem A.10.

Fix  $T > 0$ . Note that  $\theta \in \mathcal{P}_V$  implies that  $\|\theta\|$  is locally bounded by Proposition A.11, while  $S$  is locally bounded by assumption. Thus, there exists a sequence of stopping times  $(v_M)_{M \geq 1}$  such that  $|S_t|, \|\theta\|_t \leq M$  for  $t \in \llbracket 0, v_M \rrbracket$ , and it suffices to prove the claim on  $\llbracket 0, v_M \rrbracket$ . Indeed, if  $\theta^M$  satisfies the claim for  $\varepsilon^M = \varepsilon 2^{-(M+1)}$ , then  $\theta'' = \theta^1 1_{\llbracket 0, v_1 \rrbracket} + \sum_{M=2}^{\infty} \theta^M 1_{\llbracket v_{M-1}, v_M \rrbracket}$  satisfies the claim on  $[0, T]$ . Thus, from now on assume that  $S = S 1_{\llbracket 0, v_M \rrbracket} + S_{v_M} 1_{\llbracket v_M, \infty \rrbracket}$ . Fix  $T > 0$ . Define the sequence of stopping times  $(\rho_m)_{m=0}^{\infty}$ :

$$\rho_0 = 0 \quad \rho_{m+1} = \inf\{t > \rho_m : |S_t - S_{\rho_m}| \geq \delta\} \wedge T.$$

Since  $S$  is càdlàg,  $\sup_m \rho_m > v_M$  a.s. on  $\{v_M < \infty\}$ .

**Step 1:** Construction of the approximation on  $\llbracket \rho_{m-1}, \rho_m \rrbracket$ .

For  $m \geq 1$ , set  $\varepsilon_m = \frac{\varepsilon}{2M} 2^{-(m+1)}$ , and define the stopping times  $(\sigma_n)_{n \geq 0}$  as follows<sup>3</sup>:

$$\sigma_0 = \rho_{m-1} \quad (\text{A.14})$$

$$v_0 = 0 \quad (\text{A.15})$$

$$\sigma_{n+1} = \inf\{t : \|\theta\|_t \geq v_n + \varepsilon_m\} \wedge \rho_m \quad (\text{A.16})$$

$$v_{n+1} = \inf\{\|\theta\|_t : \|\theta\|_t \geq v_n + \varepsilon_m, t \leq \rho_m\} \quad (\text{A.17})$$

<sup>3</sup> The clock  $\sigma_n$  ticks each time that the total variation  $\|\theta\|$  increases by at least  $\varepsilon_m$ , stopping at  $\rho_m$ .

Since  $\theta$  has finite variation, the random set  $\cup_{n \geq 0} \llbracket \sigma_n \rrbracket$  has finite sections. Consider  $A_n \in \mathcal{F}_{\sigma_n}$  such that  $\tau_n := (\sigma_n)_{A_n}$  is the totally inaccessible part of  $\sigma_n$ . The sequence  $(\tau_n)_{n \geq 1}$  is strictly increasing, because  $(\sigma_n)_{G_n}$  is predictable for  $G_n := \{\|\theta_{\sigma_n}\| \geq v_{n-1} + \varepsilon_m\}$ , hence  $A_n \subset \{\|\theta_{\sigma_n}\| < v_{n-1} + \varepsilon_m, \|\theta_{\sigma_n^+}\| \geq v_{n-1} + \varepsilon_m\}$ .

The predictable set  $B = \cup_{n \geq 0} (\llbracket \sigma_n \rrbracket \setminus \llbracket \tau_n \rrbracket)$  admits an exhausting sequence of predictable times  $(\pi_n)_{n \geq 0}$ . Since  $B$  has finite sections, this sequence is strictly increasing up to ordering.<sup>4</sup> Consider first the process:

$$\theta' = \sum_{n=0}^{\infty} \left( \theta_{\pi_n} 1_{\llbracket \pi_n \rrbracket} + \theta_{\pi_n^+} 1_{\llbracket \pi_n, \pi_{n+1} \rrbracket} \right) \quad (\text{A.18})$$

which is predictable because each  $\pi_n$  is predictable (cf. Jacod and Shiryaev (2003, 1.2.12)). Then, set:

$$\theta'' = \theta' + \sum_{n=0}^{\infty} (\theta_{\tau_n^+} - \theta'_{\tau_n^+}) (1_{\llbracket \tau_n, \sigma_{n+1} \rrbracket} \cap \llbracket \tau_{n+1} \rrbracket + 1_{\llbracket \tau_n, \sigma_{n+1} \rrbracket} \setminus \llbracket \tau_{n+1} \rrbracket}) \quad (\text{A.19})$$

which is again predictable, since  $\llbracket \sigma_n \rrbracket \setminus \llbracket \tau_n \rrbracket \subset \cup_{k > 0} \llbracket \pi_k \rrbracket$ . Thus,  $\theta'$  constructs an approximation on the accessible parts of  $\llbracket \sigma_n \rrbracket$ , and  $\theta''$  adjusts it on the totally inaccessible parts of  $\llbracket \sigma_n \rrbracket$ . This two-step procedure is necessary, because the accessible part of  $\llbracket \sigma_n \rrbracket$  is not identified by a single predictable time.

$\theta''$  satisfies, for all  $n$ :

$$\theta''_{\pi_n^+} = \theta_{\pi_n^+}, \theta''_{\tau_n^+} = \theta_{\tau_n^+} \quad (\text{A.20})$$

$$\theta''_{\pi_n} = \theta_{\pi_n} \quad (\text{A.21})$$

$$\theta''_{\pi_n^-} = \theta_{\pi_{n-1}^+}, \theta''_{\tau_n^-} = \theta_{\tau_{n-1}^+} \quad (\text{A.22})$$

$$\theta_{\tau_n} = \theta_{\tau_n^-}, \theta''_{\tau_n} = \theta''_{\tau_n^-} \quad (\text{A.23})$$

(A.20), (A.21) and (A.22) hold by construction, and (A.23) holds because  $\tau_n$  is totally inaccessible and  $\theta, \theta''$  are predictable. Finally, note that (A.21) implies that  $\theta_{\rho_m} = \theta''_{\rho_m}$  on the accessible part of  $\rho_m$ .

**Step 2:** Prove the equality:

$$\int_{\llbracket \sigma_{n-1}, \sigma_n \rrbracket} S d\theta - \int_{\llbracket \sigma_{n-1}, \sigma_n \rrbracket} S d\theta'' = \int_{\llbracket \sigma_{n-1}, \sigma_n \rrbracket} S d\theta - (\theta_{\sigma_n^-} - \theta_{\sigma_{n-1}^+}) (S_{\sigma_n^-} + \Delta S_{\sigma_n} 1_{\llbracket \tau_n \rrbracket}) \quad (\text{A.24})$$

To see this, observe that:

$$\begin{aligned} \int_{\llbracket \sigma_{n-1}, \sigma_n \rrbracket} S d\theta &= S_{\sigma_n^-} (\theta_{\sigma_n} - \theta_{\sigma_n^-}) + S_{\sigma_n} (\theta_{\sigma_n^+} - \theta_{\sigma_n}) + \int_{\llbracket \sigma_{n-1}, \sigma_n \rrbracket} S d\theta \\ \int_{\llbracket \sigma_{n-1}, \sigma_n \rrbracket} S d\theta'' &= S_{\sigma_n^-} (\theta''_{\sigma_n} - \theta''_{\sigma_n^-}) + S_{\sigma_n} (\theta_{\sigma_n^+} - \theta''_{\sigma_n}) \end{aligned}$$

where the second equality exploits (A.20). Hence, the following equality holds:

$$\int_{\llbracket \sigma_{n-1}, \sigma_n \rrbracket} S d\theta - \int_{\llbracket \sigma_{n-1}, \sigma_n \rrbracket} S d\theta'' = S_{\sigma_n^-} (\theta_{\sigma_n} - \theta''_{\sigma_n} - \theta_{\sigma_n^-} + \theta''_{\sigma_n^-}) - S_{\sigma_n} (\theta_{\sigma_n} - \theta''_{\sigma_n}) + \int_{\llbracket \sigma_{n-1}, \sigma_n \rrbracket} S d\theta$$

<sup>4</sup> Set  $C_0 = \emptyset$ ,  $\pi'_n = \min\{t : (\omega, t) \in B \setminus C_n\}$ , and  $C_n = \cup_{0 \leq k < n} \llbracket \pi'_k \rrbracket$ , which is predictable because  $B \setminus C_n$  is predictable and  $\llbracket \pi'_n \rrbracket \subset B \setminus C_n$ , by 1.2.38 of Jacod and Shiryaev (2003).

First, recall that  $\theta''_{\sigma_n^-} = \theta_{\sigma_{n-1}^+}$ . On  $[\![\tau_n]\!] = [\![\sigma_n]\!] \cap A_n$  the first term in the right-hand side vanishes by (A.23), and the claim follows. On  $[\![\sigma_n]\!] \setminus [\![\tau_n]\!]$ , the second term in the right hand side vanishes by (A.21), while the first one reduces to  $S_{\sigma_n^-}(\theta''_{\sigma_n^-} - \theta_{\sigma_n^-})$ .

**Step 3:** Prove the estimate:

$$\left| \int_{[\![\rho_m, \rho_{m+1}]\!]} Sd\theta - \int_{[\![\rho_m, \rho_{m+1}]\!]} Sd\theta'' \right| \leq \delta(\|\theta\|_{\rho_{m+1}^-} - \|\theta\|_{\rho_m^+}) + \varepsilon 2^{-(m+1)} \quad (\text{A.25})$$

First, rewrite (A.24) as:

$$\int_{[\![\sigma_{n-1}, \sigma_n]\!]} Sd\theta - \int_{[\![\sigma_{n-1}, \sigma_n]\!]} Sd\theta'' = \int_{[\![\sigma_{n-1}, \sigma_n]\!]} (S - S'_{\sigma_n})d\theta$$

where  $S'_{\sigma_n} = S_{\sigma_n} 1_{[\![\tau_n]\!]} + S_{\sigma_{n-1}} 1_{[\![\sigma_n]\!] \setminus [\![\tau_n]\!]}$ . Denote by  $N_m = \max\{n : \sigma_n < \rho_{m+1}\}$ , which implies that  $\sigma_{N_m+1} = \rho_{m+1}$ . By (A.24) it follows that:

$$\begin{aligned} \int_{[\![\rho_m, \rho_{m+1}]\!]} Sd\theta - \int_{[\![\rho_m, \rho_{m+1}]\!]} Sd\theta'' &= \sum_{n=1}^{N_m+1} \left( \int_{[\![\sigma_{n-1}, \sigma_n]\!]} Sd\theta - \int_{[\![\sigma_{n-1}, \sigma_n]\!]} Sd\theta'' \right) \\ &= \left( \sum_{n=1}^{N_m} \int_{[\![\sigma_{n-1}, \sigma_n]\!]} (S - S'_{\sigma_n})d\theta \right) + \int_{[\![\sigma_{N_m}, \rho_{m+1}]\!]} (S - S_{\rho_m})d\theta - \Delta S_{\rho_m} 1_{[\![\tau_{N_m+1}]\!]} (\theta_{\rho_{m+1}^-} - \theta_{\sigma_{N_m}^+}) \end{aligned}$$

Now, observe that  $|S - S_{\rho_{m+1}^-}| \leq \delta$  and  $|S - S'_{\sigma_n}| \leq \delta$  for  $n \leq N_m$ , because  $|S - S_{\rho_m}| \leq \delta$  on  $[\![\rho_m, \rho_{m+1}]\!]$ . Also,  $\|\Delta S_{\rho_m}\| \leq 2M$  because  $|S| \leq M$ , and  $|\theta_{\rho_{m+1}^-} - \theta_{\sigma_{N_m}^+}| \leq \varepsilon_m$ , and the claim follows.

**Step 4:** Conclusion. Setting  $\delta = \varepsilon/M$ , and recalling that  $\|\theta\|$  is bounded by  $M$ , it follows that:

$$\begin{aligned} \left| \int_{[0, T]} Sd\theta - \int_{[0, T]} Sd\theta'' \right| &\leq \sum_{m \geq 1} \left| \int_{[\![\rho_{m-1}, \rho_m]\!]} Sd\theta - \int_{[\![\rho_{m-1}, \rho_m]\!]} Sd\theta'' \right| \\ &\leq \frac{\varepsilon}{M} \sum_{m \geq 1} (\|\theta\|_{\rho_m^-} - \|\theta\|_{\rho_{m-1}^+}) + \varepsilon \sum_{m \geq 1} 2^{-(m+1)} \leq 2\varepsilon \end{aligned}$$

□

**Corollary A.12** *Let  $\theta \in \mathcal{P}_V$  and assume that  $(S, \kappa) \in \mathcal{D}$  are locally bounded. Then for all  $\varepsilon > 0$  there exists  $\theta''$  as in Proposition A.10, which satisfies  $|\theta'' - \theta| \leq \varepsilon$ ,  $|\int Sd\theta'' - \int Sd\theta| + |\int \kappa d\|\theta''\| - \int \kappa d\|\theta\|| \leq \varepsilon$ , and  $\|\theta''\| \leq \|\theta\|$  pointwise on  $[0, T]$  (outside a  $P$ -zero set).*

Repeating the above proof in a deterministic setting yields the following deterministic statement.<sup>5</sup>

**Corollary A.13** *Let  $\theta$  be a finite variation function, and  $S$  a càdlàg function. Then, for all  $\varepsilon > 0$  there exists a sequence of time instants  $(\tau_n)_{n \geq 0}$  such that the integrand  $\theta''$  defined in (A.13) satisfies  $|\theta'' - \theta| \leq \varepsilon$ ,  $|\int Sd\theta'' - \int Sd\theta| \leq \varepsilon$  and  $\|\theta''\| \leq \|\theta\|$  pointwise.*

<sup>5</sup> But a pathwise application of Corollary A.13 does not prove Proposition A.10, which requires the predictability of  $\theta''$ .

A simple consequence is that the integral  $I$  in (A.2) is the unique extension of the simple integrals (A.7) which satisfies a Lebesgue-type theorem with respect to pointwise convergence of the integrator.

**Proposition A.14** *The integral (A.2) is the unique extension of (A.7) to the class of all finite variation functions  $\theta$  which satisfies iii) in Theorem A.9.*

*Proof* Take  $\theta_n$  given by Corollary A.13 with the choice  $\varepsilon = 1/n$ . Let  $\tilde{I}$  be an arbitrary extension of the restriction of  $I$  to simple  $\theta$  which satisfies iii) of Theorem A.9. Then necessarily  $\tilde{I}(S, \theta) = \lim_{n \rightarrow \infty} \tilde{I}(S, \theta_n)$  and this latter limit is  $I(S, \theta)$ , by Corollary A.13.  $\square$

When  $S$  is a *semimartingale*, integration by parts links the predictable Stieltjes integral to the usual stochastic integral:

**Proposition A.15** *Let  $\theta \in \mathcal{P}_V$  and  $S$  a càdlàg semimartingale. Then:*

$$\int_0^T S_u d\theta_u = \theta_T S_T - \theta_0 S_0 - \int_0^T \theta_u dS_u, \quad (\text{A.26})$$

where the first integral is in the predictable Stieltjes sense and the second one is a usual stochastic integral.

*Proof* Without loss of generality, assume  $\theta_0 = \theta_T = 0$ . The stochastic integral exists as  $\theta$  is locally bounded by Proposition A.11. Consider first  $\theta$  simple. By linearity of the integrals it suffices to treat the left- and right-continuous cases separately and localization allows us to assume  $\|\theta\|, \theta$  bounded.

Let us first take  $\theta$  left-continuous with finitely many jumps  $(\tau_j)_{0 \leq j \leq n}$ . We have

$$I_T(S, \theta) = \sum_{j=0}^n S_{\tau_j} (\theta_{\tau_{j+1}} - \theta_{\tau_j}) = - \sum_{j=0}^n \theta_{\tau_j} (S_{\tau_j} - S_{\tau_{j-1}}) = - \int_0^T \theta_u dS_u, \quad (\text{A.27})$$

by the definition of (elementary) stochastic integrals, with the convention  $\tau_{-1} = 0$ . This shows the statement for  $\theta$  and extends easily to arbitrary simple left-continuous  $\theta$ .

Take now  $\theta$  right-continuous. We may suppose (see the proof of Theorem A.10) that the jump times  $\tau_j$  are predictable with announcing sequences  $\tau_{j-1} \leq \tau_j^k < \tau_j$ ,  $k \geq 1$ . Set

$$\gamma(k) := \sum_{j=0}^{\infty} [\theta_{\tau_j} 1_{\llbracket \tau_j, \tau_{j+1}^k \rrbracket} + E(\theta_{\tau_{j+1}} | \mathcal{F}_{\tau_{j+1}^k}^-) 1_{\llbracket \tau_{j+1}^k, \tau_{j+1} \rrbracket}].$$

Note that  $\theta_{\tau_{j+1}}$  is  $\mathcal{F}_{\tau_{j+1}^-}$ -measurable (see 1.2.4 of Jacod and Shiryaev (2003)), hence the conditional expectation converges a.s. to  $\theta_{\tau_{j+1}}$  as  $k \rightarrow \infty$ , by the martingale convergence theorem. Hence  $\gamma(k) \rightarrow \theta$  pointwise (possibly outside a  $P$ -zero set).

It follows from the previous step by the dominated convergence theorem for stochastic integrals and by Theorem A.9 iii) that

$$- \int_0^T \theta_u dS_u = \lim_{k \rightarrow \infty} - \int_0^T \gamma_u(k) dS_u = \lim_{k \rightarrow \infty} I(S, \gamma(k)) = I(S, \theta).$$

For general  $\theta \in \mathcal{P}_V$ , take an approximation  $\theta^n$  constructed in Theorem A.10 with  $\varepsilon := 1/n$ . The above arguments imply that  $I_T(S, \theta^n) = -\int_0^T \theta_u^n dS_u$  for all  $n$ .  $I_T(S, \theta^n) \rightarrow I_T(S, \theta)$  by Theorem A.10 and the stochastic integrals of  $\theta^n$  converge to that of  $\theta$  by dominated convergence (note that  $|\theta_n| \leq |\theta| + 1/n$ ).  $\square$

## B Appendix

For completeness, this appendix recalls the statements of some now classical tools in Mathematical Finance. First, a compactness lemma for bounded sets in  $L_+^0$ :

**Lemma B.1 (Delbaen and Schachermayer (1994, Lemma A1.1))** *Let  $(f_n)_{n \geq 1}$  be a sequence of  $[0, \infty)$ -valued measurable functions on a probability space  $(\Omega, \mathcal{F}, P)$ . There exists a sequence  $g_n \in \text{conv}((f_k)_{k \geq n})$  such that  $(g_n)_{n \geq 1}$  converges almost surely to a  $[0, \infty)$ -valued function  $g$ . If  $\text{conv}((f_n)_{n \geq 1})$  is bounded in  $L^0$ , then  $g$  is finite almost surely.*

Second, an important consequence of the Krein-Smulian Theorem (cf. Delbaen and Schachermayer (1994), Kabanov and Last (2002, Lemma 4.4)):

**Lemma B.2** *Let  $C \subset L^\infty$  be a convex set. Then  $C$  is  $\sigma(L^\infty, L^1)$ -closed if and only if  $C \cap \{Z : \|Z\|_\infty \leq x\}$  is closed in probability for all  $x > 0$ .*

Third, a version of the Kreps (1981)-Yan (1980) separation theorem (cf. Schachermayer (1994) or Kabanov and Stricker (2001b)):

**Theorem B.3** *Let  $-L_+^\infty \subset C \subset L^\infty$  be a convex cone, closed in the  $\sigma(L^\infty, L^1)$ -topology, such that  $C \cap L_+^\infty = \{0\}$ . Then there exists a probability  $Q$  equivalent to  $P$  such that  $E_Q[C] \leq 0$ .*

Finally, a compactness lemma for finite variation processes (cf. Guasoni (2002, Lemma 3.4), Campi and Schachermayer (2006, Proposition 14)). Note that Lemma A.2 bypasses the uses of section theorems.

**Lemma B.4** *Consider  $(\theta^n)_{n \geq 1} \subset \mathcal{P}_V$  such that  $(\|\theta^n\|_T)_{n \geq 1}$  is bounded in  $L^0$ . Then there exists  $\theta \in \mathcal{P}_V$  and a sequence  $(\eta^n)_{n \geq 1} \subset \mathcal{P}_V$  of convex combinations  $\eta^n \in \text{conv}(\theta^n, \theta^{n+1}, \dots)$  converging to  $\theta$  pointwise such that also  $\|\eta^n\| \rightarrow \|\eta\|$  pointwise.*

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