Front interactions in a three-component system

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Outline

1. Introduction

2. Existence

3. Stability

4. Interaction

5. Future work
Localized structures are solutions to a PDE that are close to a trivial background state, except in one or more localized spatial regions

- Weak interaction regime: well-developed mathematical theory
- Strong interaction regime: no mathematical theory
- Semi-strong interaction regime:
Three component system

\[
\begin{align*}
U_t &= U_{\xi\xi} + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma) \\
\tau V_t &= \frac{1}{\varepsilon^2} V_{\xi\xi} + U - V \\
\theta W_t &= \frac{D^2}{\varepsilon^2} W_{\xi\xi} + U - W
\end{align*}
\]

where \(0 < \varepsilon \ll 1; D > 1; 0 < \tau, \theta\) and \(O(1); \alpha, \beta, \gamma \in \mathbb{R}\) and \(O(1)\) (with respect to \(\varepsilon\)); and \((\xi, t) \in \mathbb{R} \times \mathbb{R}^+\).

- Physical background: gas-discharge experiments by Purwins et al.
- Inspiration: numerical collision experiments by Nishiura et al.
- Motivation: ‘rich behavior’ and ‘transparent structure’ enables rigorous mathematical analysis.
- Goal: understanding the semi-strong interaction regime.
2-front interacting: different initial conditions
Stationary 2-pulse solution
Understood, but not today II

Uniformly traveling 1-pulse solution

\[ \tau, \theta \gg O(1) \]
3-front interacting: different forcing parameter $\gamma$

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**Introduction**

**Existence**

**Stability**

**Interaction**

**Future work**

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Understood, but not today III
Don’t understand

Complex dynamics

\[ \tau, \theta \gg \mathcal{O}(1) \]
Geometric singular perturbation theory

Idea: 5 regions

- Region I: $U = -1 + \mathcal{O}(\varepsilon)$
- Region II: $V = V_0 + \mathcal{O}(\varepsilon)$, $W = W_0 + \mathcal{O}(\varepsilon)$
- Region III: $U = 1 + \mathcal{O}(\varepsilon)$
- Region IV: $V = V_0 + \mathcal{O}(\varepsilon)$, $W = W_0 + \mathcal{O}(\varepsilon)$
- Region V: $U = -1 + \mathcal{O}(\varepsilon)$
Preliminaries

Stationary 1-pulse solution

No movement in time: \( \frac{\partial}{\partial t} \cdot = 0 \)

6-dimensional ODE: nonlinear but autonomous

\[
\begin{align*}
    u_\xi &= p \\
    p_\xi &= -u + u^3 + \varepsilon(\alpha v + \beta w + \gamma) \\
    v_\xi &= \varepsilon q \\
    q_\xi &= \varepsilon(v - u) \\
    w_\xi &= \frac{\varepsilon}{D} r \\
    r_\xi &= \frac{\varepsilon}{D}(w - u)
\end{align*}
\]

Fixed points

\[ P^{\pm}_\varepsilon = (\pm 1, 0, \pm 1, 0, \pm 1, 0) + \mathcal{O}(\varepsilon), \quad P^{0}_\varepsilon = (0, 0, 0, 0, 0, 0) + \mathcal{O}(\varepsilon) \]
Regions II and IV: fast reduced system

\[ \varepsilon \downarrow 0 \]

\[
\begin{align*}
    u_\xi &= p \\
    p_\xi &= -u + u^3
\end{align*}
\]

and \((v, q, w, r) = (v_0, q_0, w_0, r_0)\)

**Hamiltonian**

\[
H(u, p) = \frac{1}{2}(u^2 + p^2) - \frac{1}{4}(u^4 + 1)
\]

**Phase portrait**

\[
\begin{align*}
    u_{\text{het}}(\xi) &= \pm \tanh \left( \frac{1}{2} \sqrt{2} \xi \right), \\
    p_{\text{het}}(\xi) &= \pm \frac{1}{2} \sqrt{2} \text{sech}^2 \left( \frac{1}{2} \sqrt{2} \xi \right)
\end{align*}
\]
Regions I, III, and V: slow reduced system (SRS)

\[ U = \pm 1 \text{ (to leading order)} \]

\[
\begin{align*}
    v_{\xi \xi} &= \varepsilon^2 (v \mp 1) \quad \Rightarrow \quad v(\xi) = A_i \varepsilon^{\xi} + B_i \varepsilon^{\xi} \pm 1 \\
    w_{\xi \xi} &= \frac{\varepsilon^2}{D} (w \mp 1) \quad \Rightarrow \quad w(\xi) = \widetilde{A}_i \varepsilon^{\xi} D + \widetilde{B}_i \varepsilon^{\xi} D \pm 1
\end{align*}
\]

Phase portrait

\[ U = -1 \]

\[ U = 1 \]
Stationary 1-pulse solution

Schematic picture

Analysis

- Fenichel theory
- Melnikov method: $\alpha v_0 + \beta w_0 + \gamma = 0$
- Solve SRS + matching: $\alpha e^{-\varepsilon \xi} + \beta e^{-\frac{\varepsilon}{D} \xi} = \gamma$
- Rigorous
Existence result

Theorem

There exists a stationary 1-pulse solution if is a $\xi^* \in (0, \infty)$ which solves

$$\alpha e^{-\varepsilon \xi^*} + \beta e^{-\varepsilon D \xi^*} = \gamma. \quad (1)$$

Moreover, $\xi^*$ corresponds to the width of the pulse.

Corollary

- Equation (1) has at most 2 solutions.
- If $\text{sgn}(\alpha) = \text{sgn}(\beta)$, then (1) has at most 1 solution.
- If $|\gamma| > |\alpha| + |\beta|$, then (1) has no solutions.
- If $\text{sgn}(\alpha) \neq \text{sgn}(\beta)$ and $|\alpha D| > |\beta|$, then there is a saddle-node bifurcation of stationary 1-pulse solutions.
Stability: Evans function

**Essential spectrum**

Stable and $O(1)$ gap to imaginary axis.

**Discrete spectrum**

\[
\begin{align*}
U(\xi, t) &= u_h(\xi; \varepsilon) + u(\xi)e^{\lambda t}, \\
V(\xi, t) &= v_h(\xi; \varepsilon) + v(\xi)e^{\lambda t}, \\
W(\xi, t) &= w_h(\xi; \varepsilon) + w(\xi)e^{\lambda t},
\end{align*}
\]

**Stability problem: nonautonomous but linear**

\[
\begin{align*}
u_{\xi\xi} + u(1 - 3u_h^2 - \lambda) &= \varepsilon(\alpha v + \beta w) \\
v_{\xi\xi} &= \varepsilon^2((1 + \tau \lambda)v - u) \\
w_{\xi\xi} &= \frac{\varepsilon^2}{D^2}((1 + \theta \lambda)w - u).
\end{align*}
\]
Eigenfunctions: again 5 regions

- Region I: $u = 0 + \mathcal{O}(\varepsilon)$
- Region II: $\tilde{v} = \tilde{v}_0 + \mathcal{O}(\varepsilon)$, $\tilde{w} = \tilde{w}_0 + \mathcal{O}(\varepsilon)$
- Region III: $u = 0 + \mathcal{O}(\varepsilon)$
- Region IV: $\tilde{v} = \tilde{v}_0 + \mathcal{O}(\varepsilon)$, $\tilde{w} = \tilde{w}_0 + \mathcal{O}(\varepsilon)$
- Region V: $u = 0 + \mathcal{O}(\varepsilon)$
Stability theorem

Analysis

- Solving the corresponding FRS and SRS + matching
- Rigorous: Evans function

Theorem

The stationary 1-pulse solution with width $\xi_*$ is stable if and only if

$$\alpha e^{-\varepsilon \xi_*} + \frac{\beta}{D} e^{-\frac{\varepsilon}{D} \xi_*} > 0.$$ 

Corollary

- The 1-pulse solution is stable if $sgn(\alpha) = sgn(\beta) = 1$ and unstable if $sgn(\alpha) = sgn(\beta) = -1$.
- One of the branch of the saddle-node bifurcation is stable, the other one is unstable.
Saddle-node bifurcation

\[ \alpha > 0 > \beta \]

\[ \gamma \]

\[ \gamma_{SN} \]

\[ \alpha + \beta \]

\[ A_{SN} \]

\[ A \]
Front interaction

Question

Given the system-parameters and an initial condition, can we predict how the structure evolves in time?
2-Front Dynamics

Theorem

The distance $\Delta \Gamma$ between two fronts of a 1-pulse solution evolves according to

$$\Delta \Gamma_t = 3\sqrt{2}\varepsilon \left(\alpha e^{-\varepsilon \Delta \Gamma} + \beta e^{-\frac{\varepsilon}{\delta} \Delta \Gamma} - \gamma\right).$$

(2)

- Fixed points of (2) are precisely the solutions to the existence condition (1).
- The fronts $\Delta \Gamma(t)$ asymptote to a stationary 1-pulse solution with width $\Delta \Gamma^1_*$ iff the 1-pulse solution is stable and there is no unstable stationary 1-pulse solution determined by $\Delta \Gamma^2_*$ with $\Delta \Gamma(0) < \Delta \Gamma^2_{*} < \Delta \Gamma^1_{*}$ or $\Delta \Gamma(0) > \Delta \Gamma^2_{*} > \Delta \Gamma^1_{*}$. 

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Existence
Stability
Interaction
Future work

ODE and PDE I
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ODE and PDE II
General $N$-front dynamics

Lemma

- Stationary $N$-front solutions do not exist for $N$ odd.
- Uniformly travelling $N$-front solutions do not exist for $N$ even.
Sketch of proof I

Formally: Easy, introduce 2 co-moving frames and distinguish again 5 different regions $\implies$ ODE analysis.

Rigorous: Hard, real PDE analysis. Proof is based on a Renormalization group method.
Future work

- Interactions for $\tau, \theta$ large
  - Problems with the essential spectrum

- Spatial inhomogeneities
  - Work in progress with K.-I. Ueda & Y. Nishiura

- Two space dimensions
  - Rubicon research at Brown University
Sketch of proof II

Skeleton solution $\Phi_{\Gamma}(\xi)$

$$\Phi_{\Gamma}(\xi) := \begin{pmatrix} U_0(\xi; \Gamma) \\ G_V \ast U_0(\xi; \Gamma) \\ G_W \ast U_0(\xi; \Gamma) \end{pmatrix}$$

- $U_0(\xi, \Gamma) = -1 + \tanh \left( \frac{1}{2} \sqrt{2}(\xi - \Gamma_1) \right) - \tanh \left( \frac{1}{2} \sqrt{2}(\xi - \Gamma_2) \right)$, where $\Gamma_i$ is the location of the $i$-th front ($\Delta \Gamma = \Gamma_2 - \Gamma_1$).
- $G_V = -\frac{1}{2} \varepsilon e^{-\varepsilon|\xi|}$; the Green's function associated to

$$0 = \frac{1}{\varepsilon^2} V_{\xi \xi} + U_0 - V.$$
Sketch of proof III

- Decomposition: $\Phi_2 = \Phi_\Gamma + Z$ (Assumption in initial condition!)
- Going to show:
  - $\Phi_2$ evolves according to (2)
  - $||Z||_\chi = O(\varepsilon)$

$$Z_t + \frac{\partial \Phi_\Gamma}{\partial \Gamma} \Gamma_t = R + L_\Gamma Z + N(Z),$$

where $R =$ residual, $L =$ linear part, $N =$ nonlinear part.
- ‘Freeze’ time $t = t_0$:

$$Z_t + \frac{\partial \Phi_\Gamma}{\partial \Gamma} \Gamma_t = R + L_{\Gamma_0} Z + (L_\Gamma - L_{\Gamma_0}) Z + N(Z)$$

- Define $\pi_{\Gamma_0}$ as the projection on the space spanned by the eigenfunctions of $L_{\Gamma_0}$ associated to the small eigenvalues; $\tilde{\pi}_{\Gamma_0} = I - \pi_{\Gamma_0}$. 
Sketch of proof IV

- Projection by $\pi_{\Gamma^0}$ yields/confirms the ODE for $\Gamma_t$ (since $Z$ can assumed to be ‘perpendicular’ to $\pi_{\Gamma^0}$: $\pi_{\Gamma^0}Z = 0$).
- Projection by $\tilde{\pi}_{\Gamma^0}$ shows that $||Z||_\chi$ stays $O(\varepsilon)$ small for a finite time $\Delta t$ (this is due to the ‘secular growth’).
- $\Delta t = O(\log \varepsilon)$ (Hard work)
- Renormalize: ‘freeze’ a new time: $t_1 = t_0 + \Delta t$ such that $||Z(t_1)||_\chi = O(\varepsilon)$.
- Repeat above procedure
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\[ \Phi(t_0) \]

\[ \Phi(t_0 + \Delta t) \]

\[ \Phi(t_1) \]

\[ \Phi(t_1 + \Delta t) \]

\[ \Gamma^0 \]

\[ \Gamma(t_0 + \Delta t) \]

\[ \Gamma^1 \]

\[ \Gamma(t_1 + \Delta t) \]

\[ \Gamma^2 \]

Future work