year olds." After all, these students are just as bright and just as able as you and I were when we were entering college. The difference is that these new students have never learned how to learn. So we had better step into the breach and show them how to do it. There *is* a difference between high school learning and university learning. Let's show the students what that difference is and then show them how to make the transition.

There are important philosophical and educational issues at play here. In America at the close of the millennium we attempt, as much as possible, to educate everyone. Whereas forty years ago this meant that "everyone" went to high school, now it means that "everyone" goes to college. It is vital in a free society that all citizens, regardless of financial resources, have an opportunity to pursue a college education. But society is set up in such a way now that a large percentage of young people go to college regardless of their interests or goals. For this we pay a price. We can rely less than we would like on the preparedness of our freshmen. We also can rely somewhat less on their attitudes and motivation.

What this means in practice is that, quite often, especially when teaching freshmen—and especially at a public institution—we are not necessarily teaching a very select group. Many public institutions these days have an open admissions policy. Anyone with a high school diploma has the right to attend the state university. From the taxpayers' point of view, such an admission policy makes perfectly good sense. If you are a professor at a state institution then you must make peace with the realities connected with such a policy. You must learn to adjust your expectations. You must learn a little patience, and learn to be flexible.

There are decided differences between the reality of what a high school education is and the dream of what a university education ought to be. The former is often a passive activity which does not challenge students to think creatively or incisively. The latter should be a stimulating and demanding rite that will turn students into productive citizens who will think creatively and profoundly and ask difficult questions of our leaders when appropriate (see Section 3.5 for more on the differences between high school and college). You, the college instructor, can and should play a critical role in helping students effect the transition from the first of these environments to the second.

I am not about to recommend that college math teachers spend their evenings reading position papers on motivational psychology. I *am* recommending that the college mathematics teacher exercise some tolerance. Students *will* rise to a challenge, provided that the teacher starts with small challenges and works up to big ones. If students stumble at the first few challenges then they need encouragement, not derision. Exercising patience requires no more effort than exercising your vocal chords with an insulting remark.

I have found persistence to be one of the most powerful pedagogical tools in my arsenal (see also Section 2.3). If my students aren't getting some idea I say, "OK, let's do it again." If that doesn't work, we do it some more. If I won't give up then the students are also unlikely to give up. They trust me; they know that I am trying to do something good for them.

The professor's attitude toward the class is apparent from his every word, every gesture, every action. If you are arrogant, if you despise your students, if

you feel that you are above the task of teaching this course, then your students will get the message immediately. And what are you accomplishing by evincing these attitudes? Does it make you feel superior? More accomplished? More secure? More important? It should not. Teaching is an essential part of what we do. Society sees us as teachers and, increasingly, the university sees us as teachers. Why not see yourself as a teacher? It will increase your self-esteem. And it will increase the esteem that others have for you. Being a good teacher does not detract from your mathematical attributes. It augments them.

1.3 Prepare

Some people rationalize not teaching well by saying (either to themselves or to others), "My time is too valuable. I am not going to spend it preparing my calculus lecture. I am so smart that I can just walk into the classroom and wing it. And the students will benefit from watching a mathematician think on his feet." (As a student, I actually had professors who announced this nonsense to the class on a regular basis. And, as you can imagine, these were professors who royally botched up their lectures on a regular basis.)

It is true that most of us can walk into the room most of the time and mostly wing it. But most of us will not be very successful if we do so. Thirty minutes can be sufficient time for an experienced instructor to prepare a calculus lecture. A novice instructor, especially one teaching an unfamiliar subject for the first time, may need considerably more preparation time. Make sure that you have the definitions and theorems straight. Read through the examples to make sure that there are no unpleasant surprises. It is a good idea to have a single page of notes containing the key points. To write out every word that you will say, write out a separate page of anticipated questions, have auxiliary pages of extra examples, have inspirational quotes drawn from the works of Thomas Carlyle, make up a new notational system, make up your own exotic examples, and so forth, is primarily an exercise in self-abuse. Over-preparation can actually stultify a lecture or a class. But you've got to know your stuff.

I cannot emphasize too strongly the fact that preparation is of utmost importance if you are going to deliver a stimulating class. However it is also true that the more you prepare the more you lose your spontaneity. You must strike a balance between (i) knowing the material cold and (ii) being able to "talk things through" with your audience.

My own experience is that there is a "right amount" of preparation that is suitable for each type of course. I want to be confident that I'm not going to screw up in the middle of lecture. But I also want to be actually thinking the ideas through as I present them. I want to feel that my lecture or class has an edge. It *is* possible to over-prepare. To continue to prepare after you have already prepared sufficiently is a bit like hitting yourself in the head with a hammer because it feels so good when you stop.

You must be sufficiently confident that you can field questions on the fly, can modify your lecture (again on the fly) to suit circumstances, can tolerate a diversion to address a point that has been raised. The ability to do these things

well is largely a product of experience. But you can *cultivate* this ability too. You cannot learn to play the piano by accident. And you will not learn to teach well by accident. You must be aware—in detail—of what it is that you are trying to master and then consciously hone that skill.

If you do not prepare—I mean *really* do not prepare—and louse up two or three classes in a row, then you will experience one or more of the following consequences: **(i)** Students will take up your time after class and during your office hour (in order to complain and ask questions), **(ii)** Students will stop coming to class, **(iii)** Students will complain to the undergraduate director and to the chairman, **(iv)** Students will (if you are really bad) complain to the dean and write letters to the student newspaper, **(v)** Students will write bad teaching evaluations for your course.

Now student teaching evaluations are not gospel (see Section 2.9). They contain some remarks that are of value and some that are not. Getting bad teaching evaluations does not necessarily mean that you did a bad job. And I know that the dean will only slap me on the wrist if he gets a complaint about my teaching (however, if there are ten complaints, then I had better look out). Finally, I know that the chairman will give me the benefit of the doubt and allow me every opportunity to put any difficult situation in perspective. But if I spend thirty minutes preparing each of my classes then I will avoid all this grief and, in general, find the teaching experience pleasurable rather than painful. What could be simpler?

As well as preparing for a class, you would be wise to debrief yourself after class. Ask yourself how it went. Were you sufficiently well prepared? Did you handle questions well? Did you present that difficult concept as clearly as you had hoped? Was there room for improvement? Be as tough on yourself as you would be after any exercise that you genuinely care about—from playing the piano to engaging in a tennis match. It will result in real improvement in your teaching.

Read your teaching evaluations (Section 2.9). Many are insipid. Others are puerile. Most, however, are thoughtful and well-meant. If ten of your students say that your writing is unclear, or that you talk too quickly, or that you are impatient with questions, then maybe there is a problem that you should address. Teaching is a yoga. Your mantra is "Am I getting through to them?"

It is a good idea to try to anticipate questions that students will ask. But you cannot do this artificially, as a platonic exercise late at night over a cup of coffee. It comes with experience. Assuming that you have adopted the attitude that you actually care whether your students learn something, then after several years of teaching you will know by instinct what points are confusing and why. This instinct enables you to prepare a cogent lecture—to know what to emphasize, where to slow down, where to provide extra examples. It helps you to be receptive to student questions. It helps you to have a good attitude in the classroom.

An easy way to cut down on your preparation time for a class is to present examples straight out of the book. The weak students will appreciate this repetition. Most students will not, and you will probably be criticized for this policy. On the other hand, it is rather tricky to make up good examples of maximum-minimum problems or graphing problems or applications of Stokes's theorem.

It can be time-consuming as well. If you need more examples for your calculus class, then pick up another calculus book and borrow some. Develop a file of examples that you can dip into each time you teach calculus. You will learn quickly that making up your own examples is hard work. Do you ever wonder why most calculus books are so disappointing? All right, *you* try to make up eight good examples to illustrate the divergence theorem.

1.4 Clarity

When you teach a mathematics class, clarity (or lack thereof) manifests itself in many forms. If you are the most brilliant, and even the most well prepared, mathematics teacher in the world, but you stand facing the blackboard and mumbling to yourself, then you are not being clear. If instead you shout at the top of your lungs so that all can hear, but your handwriting is cryptic, then you are not being clear. If your voice is clear, your handwriting clear, but your blackboard technique nonexistent, then you are not being clear. If your voice is beautiful, your handwriting artistic, your blackboard technique flawless, but you are completely disorganized, then you are not being clear. If you speak clearly, write clearly, have good blackboard technique, are well organized, but speak with a foreign accent, then don't worry. You are being clear.

Here is the point: Mathematics is hard. Do not make it harder by putting artificial barriers between yourself and your students. If you are shy and simply cannot face your audience, then perhaps you chose the wrong profession. More seriously, be extremely well prepared. *Make* yourself confident. Calculus is one of the most powerful analytic tools that has ever been created. It is a privilege to be able to pass it along to the next generation. Be proud of what you are doing. It is no less an event for you to teach the fundamental theorem of calculus to a group of freshmen in the 1990s than it was for Archimedes to teach his students how to calculate the area inside a circle.

I have atrocious handwriting. When my departmental librarian got her first written message from me she thought it had been written in traditional Chinese characters. But when I lecture I slow down. I write deliberately and clearly. I *want* my audience to understand me and to respect me and I take steps to see that this actually happens.

Suppose that you are in the middle of a lecture and you are making a very important point. How can you drive it home? How can you get the students' attention? We all know that students drift into a malaise in which they are copying and not thinking (after all, we were once students and did the same).[2] How do you wake them up? It's easy. Pause. State the point clearly and simply. Write it clearly and simply. Say "This is important." Repeat the point. One of Mozart's most decisive tools in his compositions was repetition of a particularly beautiful passage. We can benefit from his example.

Ask whether there are any questions. Repeat the point again. Assure students that this point will be on the exam, and that it will come up over and over

[2]Surely you have heard the old saw about the concepts passing from the professor's lecture to the students' notes without ever entering the students' brains.

Chapter 2

Practical Matters

What a waste it is to lose one's mind. Or not to have a mind as being truly wasteful. How true that is.

Dan Quayle, former Vice President of the U.S.

In God we trust; all others bring data.

A Statistician

All innovations succeed in the hands of the innovator, and none succeed in other hands.

David S. Moore

You may as well put the sidewalk where the students walk, because they'll walk there anyway.

William James Lewis

The pleasure of learning, and knowing, though not the keenest, is yet the least perishable of pleasures; the least subject to external things, and the play of chance, and the wear of time. And as a prudent man puts money by to serve as a provision for the material wants of his old age, so too he needs to lay up against the end of his days provision for the intellect.

A. E. Houseman

People who become legends in their own time usually have very little time left.

John D. MacDonald

The most irresponsible thing that a civilized adult can do is to stand up in a crowded room and shout "Proof!"

James Carlson

Everybody will be famous for fifteen minutes.

Andy Warhol

An expert is just some guy from out of town.

Mark Twain

2.0 Chapter Overview

Like many activities in life, fine teaching is composed of many technical components. When everything works properly, then the whole is considerably greater than the sum of its parts. However, if some of the crucial parts are rusty or, worse, non-functional, then the whole will creak and drag and not do a good job of it.

The novice instructor should probably read every section in this chapter. The more experienced instructor may wish to pick out particular sections for concentrated effort.

must take each idea in the course and rebuild it in his own mind. This is nothing new. Go read Beth and Piaget [BPA]—they discuss this notion in detail. An awareness of this concept will help you to shape your teaching methods. If the students cannot understand what you are talking about, then it is unlikely that they will take the ideas home and think about them. If the students watch you state and prove a lot of abstruse theorems, and in the process become terminally depressed, then it is unlikely that they will take the ideas home and analyze them and internalize them. If the students watch you flounder around, unable to complete an example coherently or explain a concept neatly, then it is unlikely that they will take the ideas home and rebuild them in their own minds.

If instead you kindle the students' curiosity, plant in them a desire to learn, show them something they have never seen before and *make them realize that it is something they have never seen before*—and certainly never understood before—then there is a real likelihood that they will leave class turning the new thoughts over in their minds, talk among themselves about it, ask questions, and come back to you with their own ideas. *That* is teaching.

Even if you know how to use your voice well with a small audience, and to capture their attention and get them excited about learning, there are special problems with the large classes that are used in the teaching of calculus (for instance) at many universities. Refer to Section 2.14 for more on this matter.

2.2 Eye Contact

We all know certain people who invariably emerge as the leader of any group conversation. Such people seem to sparkle with wit, erudition, and presence. They usually pick the topic and they usually aim the discussion. They have a sense of humor, and they are intelligent. What is their secret?

It is partly a matter of attention and awareness. The sort of person I am describing has an inborn curiosity; he is aware of you, and interested in you, and genuinely eager to learn about your opinions and experiences and interests. When you ask yourself what makes another person interesting, an honest answer would have to entail that such a person is outer directed, and cares about others.[1]

This is obviously a talent that is partly inborn and partly cultivated. Some of the trick is to show genuine interest in what other people have to say before bounding ahead with what you have to say. Another part is to talk about subjects, and to tell anecdotes, that you know will interest other people. Being charming and witty helps too, but in this section I want to concentrate on more mechanical features of repartee.

Many of the devices that make for an engaging conversationalist also make for an engaging teacher. A review of the last paragraphs, and of the rest of this book, will bear out this assertion. In this section I will discuss the importance of *eye contact.*

Telling a good joke while staring at the floor with your thumb in your ear will not have the same effect as telling the joke while looking at your listener,

[1] A boring person is one who talks about himself. An interesting person is one who talks about you.

engaging his attention, and reacting to the listener while the listener is reacting to you. A good joke teller has his audience starting to chuckle half way through the joke and just dying for the punch line. Getting a good laugh is then a foregone conclusion.

Giving a good lecture or class is serious business, and is not the same as telling a joke. But many of the moves are the same. If you want to hold your audience's attention then you must look at your audience. You must engage not one person but all. You must learn to use your body as a tool. Step forward and back. Force the eyes of the audience to follow you. A good lecturer speaks to individuals in the audience, to grouplets in the audience, and to the whole audience. Like a movie camera, you must zoom in and zoom out to get the effects that you wish to achieve. A ninety minute movie filmed at the same constant focal length would be dreadfully boring. Ditto for a lecture.

Some people are very shy about establishing eye contact. It is a device that you must consciously cultivate. The end result is worth it. The teacher who can establish eye contact is also the teacher who is confident, who is well prepared, and who conducts a good class.

2.3 Blackboard Technique

Write neatly. Write either in very plain longhand or print. Be sure that your handwriting is large enough. Be sure that it is dark enough. Endeavor to write straight across the blackboard in a horizontal line. Proceed in a linear fashion. Don't have a lot of insertions, arrows, and diagonally written asides.

Don't put too much material on each board. The ideas stand out more vividly if they are not hemmed in by a lot of adjacent material. In particular, it is difficult for students to pay attention when the teacher fills the board with long line after long line of print. An excellent guitarist once said that the silences in his music were at least as important as the notes. When you are laying material out on a blackboard, the same can be said of the blank spaces.

Some people find it useful to divide the blackboard into boxes. This practice makes it easier for the lecturer to organize what he is writing, and also makes it *much, much easier* for the students in the audience to organize the material in their minds and in their notes.

Label your equations so that you can refer to them verbally. Draw sketches neatly. Use horizontal and vertical lines to set off related bodies of material.

You can control your output more accurately by keeping the length of each line short. Think of the blackboard as being divided into several boxes and write your lecture by putting one idea in each box. To repeat: If necessary, *actually divide the blackboard into boxes.*

If the classroom has sliding blackboards, think ahead about how to use them so that the most (and most recent) material is visible at one time. For those combinatorial theorists among you, or those experts on the game of NIM, this should be fun.

If you are right-handed, consider writing first on the right-hand blackboard and then working left. The reason? That way you are never standing in front of

what you've written. Good teaching consists in large part of a lot of little details like this. You shouldn't be pathological about these details, but if you are aware that they are there then you will pick up on them.

Every now and then during your presentation, you should stand aside and pause. Don't say *anything*. This gives you an opportunity to collect your thoughts and catch your breath. You can verify the accuracy of what you have written. It gives the students an opportunity to catch up and to ponder what they've heard. They may decide to formulate questions. If instead you are barging ahead at full speed for the entire hour, then students never have a moment to think about what they are hearing. They cannot interact with you because you are not interacting with them.

Try to think ahead. Material that needs to be kept—and not erased—should be written (probably in a box) on a blackboard to the far left or far right where it is out of the way but can be referred to easily. You may wish to reserve another box on the blackboard for asides or remarks. Some instructors put material that needs to be seen for the entire hour on an overhead slide. This frees up all the blackboards, and keeps those important equations or definitions front and center.

These ideas are another facet of the precept that you know the material cold so that you can concentrate on your delivery. Just as an actor knows his lines cold so that he can make bold entrances and exits, and not trip over his feet, so you must be able to focus a significant portion of your brain on the *conveying of the information.*

If your lesson will involve one or more difficult figures then practice them on a sheet of paper in advance. Remember that you are a mathematical role model for the students. If you make it appear that it is difficult for *you* to draw a hyperboloid of one sheet, then how are the students supposed to be able to do it? Of course you can prepare the figure ahead of time on an overhead slide (or even photocopy it straight out of a book, or straight from `Mathematica` or `Maple`, onto a transparency). This solves the problem of having a nice figure to show the students. It does not solve the problem of *showing the students how to draw the figure.* As a result, it puts a barrier between them and the ultimate goal of learning how to *read* the graph. If necessary, consult a colleague who is artistically adept for tips on how to draw difficult figures.

You will find students quite resistive to learning to graph—especially in three dimensions. I learned a useful teaching device when last I taught multi-variable calculus, and it became clear to me when I was showing my students how to graph. That device is *persistence.* I made it clear to them that anything that gave them a pain in the neck was going to appear repeatedly in subsequent work. For example, almost all of them did rather poorly on the graphing portion of the first midterm. So I gave them several followup quizzes to help them hone their graphing skills. After each quiz they all gave me their "Is that finally the end of graphing?" look. But after examining their work I said, "Nope; not good enough yet." And on we went. Over and over again I graphed functions in three dimensions. I went through every step. And I did it *at the blackboard*, just as I expected them to do it with pencil and paper. Graphing appeared again on the final. And I *told* them that it would so appear. In fact I told them that the best way to study for the final was to find everything on the first two midterms

that they hated, and that this material would certainly be on the final. They believed me, and it worked.

If you cannot organize the steps of a maximum-minimum problem, then can you really expect the students to do so? In the best of all possible worlds, the students' work is but a pale shadow of your own. So your work should be the platonic ideal. Sometimes, in presenting an example or solving a problem at the blackboard, you may inadvertently gloss from one step to another. Or you might make a straightforward presentation look like a bag of tricks. This practice is very confusing for students, especially the ones who lack confidence. By organizing the solution in a step-by-step format you can avoid these slips.

After you have filled a board, it should be neat enough and clear enough that you could snap a Polaroid® snapshot and read the presentation from the Polaroid®. In particular, you should not lecture by writing a few words, erasing those, and then writing some more words on top of the erased old words. Students cannot follow such a presentation. I cannot emphasize this point too strongly: Write from left to right and from top to bottom. *Do not erase.* When the first box is filled, proceed to the second. *Do not erase.* Only when all blackboards are full should you go back and begin erasing. Students must be given time to stare at what they've just seen as well as what is currently being written. Keep as much material as possible visible at all times.

BUT: When it is time to erase, be sure to erase thoroughly. It is well worth spending a few extra moments being sure that the blackboard is sparkling clean before you begin a new block of material. For if you endeavor to write over a sloppily erased blackboard then your writing will be obscure at best. This is really a psychological issue. Of course the students can squint and strain and figure out what you are writing (even if it is a virtually unreadable palimpsest), but it bums them out to have to do so. Try to make their job as easy as possible.

Do not stand in front of what you are writing. Either stretch out your arm and write to the side or step aside frequently. Read aloud to the class as you write. Make the mathematics happen before their eyes and *be sure that they can see everything.* Every once in a while, pause and step aside to catch your breath and to let them catch up.

Here is a common error that is made even by the most seasoned professionals. Imagine that you do an example that begins with the phrase "Find the local maxima and minima of the function . . ." And so forth. Say that you've worked the example. Now suppose that the next example begins with the same phrase. It is a dreadful mistake to erase all but that first phrase and begin the new example on the fly, as it were.

Why is this a mistake?—it *seems* perfectly logical. But the students are taking notes! How can they keep up when you pull a stunt like this? *Slow yourself down.* Write the words again. If a student gets two sentences behind then he may as well be two paragraphs behind. Give frequent respites for catch up.

And now a coda: How much of what you are saying should you write? In my experience, the answer is "As much as possible." When you are transmitting sophisticated technical ideas verbally, students have trouble keeping up. Many

of them are not native English speakers. They need a little help. Write down everything except asides (actually, some asides are worth recording as well). Say the words as you write them. This is also a device for slowing yourself down. Most of us tend to talk far too fast—at least about mathematics. Slowing yourself down and writing deliberately will help you to keep your handwriting clear and will make the lesson as a whole appear to be neat and clean. If the *appearance* is neat and clean then perhaps the *ideas* are neat and clean—at least that's what you want the students to think.

The flip side of the last paragraph is that the tendency to talk too rapidly may cause you to write too rapidly (and therefore sloppily). Thus periodically checking the quality of your handwriting on the blackboard can serve as a means of telling whether either your verbal or written delivery is too speedy.

Let me reiterate one of my most fundamental precepts. There is a real psychological barrier for the instructor to overcome when learning blackboard technique, and voice control. When we understand very deeply what we are talking about, then it all seems quite trivial. We can convince ourselves rather easily—at least at a subconscious level—that it is embarrassing to stand in front of a group and enunciate whatever mundane material is the topic of the day. Thus we are inclined to race through it, both verbally and in the way that we render it on the blackboard. *Be conscious of this trap and do not fall into it.* I have never been criticized for being too clear, whether I was giving a calculus lecture to freshmen or a seminar lecture at the Mathematisches Forschungsinstitut Oberwolfach. *Slow down.* Be deliberate. Enunciate. Explain.

Many of us, at the beginning of the class, rattle on verbally at some length before we finally persuade ourselves that we had better start writing something on the board. Please don't do this. Start writing the material from the very outset. If you want the students to notice it, and write it down, and get it straight, then you had better set the example by writing it.

Writing material neatly and slowly is a subtle way of telling the students that this material is important. If you are taking the trouble to write it down deliberately, then it must be worth writing deliberately. Conversely, if you scribble some incoherent gibberish, or scribble nothing at all, then what signal are you sending to the students?

2.4 Body Language

If you skulk into your classroom, stand slouching in front of the class with a furtive and disreputable expression, wear slovenly clothing, and give off a ripe odor that permeates the room well into the fifth row, then you are sending numerous negative signals to your class. It sounds trite to say it, but dress neatly and attractively when you go to teach. Stand erect and look dignified. Master the basics of personal grooming. Attending to these mundane matters really does make a difference. If you want the class to behave as though *you* are in charge, then look and act the part. It is true that Grigory Rasputin broke all these precepts and still managed to rule a country, but he had many other talents that are foreign to most mathematicians.

Think about the last time you attended a colloquium party or other mathematics social function. In my experience, there is always a group of eight or ten people who don't have the good sense to go home at the propitious time (obviously the only reason I know this is that sometimes I've been one of those people). These stragglers usually pull chairs into a ring and sit and stare at each other for an hour or so before the host finally says a cheery, "Well gang, time to go!" This "circle of death" is a strange and embarrassing artifact of mathematical—and perhaps academic—social life. One cannot help but suppose that people who would display this sort of social ineptitude might be equally gauche in other circumstances, such as when they teach. *Do* endeavor to be aware of the world around you. Endeavor to get to know your students, and to get to know who they are. Try to understand their needs, and their goals. Use your *body* (as well as your wit) to show them that you are there to help them learn the material, to be their guide in this new territory.

A friend of mine, fresh from Romania and with minimal English skills, once sat in a lunch room in Chicago endeavoring to formulate his order. He stared forlornly at "Ruben Sandwich", "Chile Size", "Three Pigs in a Blanket", "Montecristo Sandwich", and other solecisms on the menu trying to determine what in the world they could be. His dictionary was of little help. Finally he found a menu item that he could translate: "bacon, lettuce, and tomato sandwich". He happily signaled the server and painstakingly ordered "a bacon, lettuce, and tomato sandwich and a Coke." The server scribbled down the order and drawled "Ya want mayo on your BLT?"

Has it ever occurred to you that we treat our calculus students in just the way that that server treated my Romanian friend? The students are struggling with the chain rule and the mean value theorem and related rates in just the way that my friend struggled with the concept of Ruben sandwich, and we often answer (because we are so accustomed to the ideas) "Ya want mayo on your BLT?" It is important that we develop some sensitivity to the student's point of view, his values, and his vocabulary. We need not *admire* these attributes, but we must *deal with them.* That is the teacher's job.

Not long ago I asked a young assistant professor how he taught the Mean Value Theorem to freshmen. He began by saying, "First I tell them about the Axiom of Choice ..." I could not resist interrupting and asking "Why?" He seemed to have some vague idea that the Axiom of Choice had something to do with the completeness of the real numbers (this turns out to be a misapprehension). But so what if it does? I wouldn't say a word about the completeness of the reals when explaining the Mean Value Theorem to eighteen year olds, much less about the Axiom of Choice. In fact I'm quite sure that I wouldn't even prove the Mean Value Theorem. I would concentrate on getting them to understand what the theorem *says.* In short, while my friend would be asking the freshmen "Ya want mayo on your BLT?" I would be serving them a nice hot sandwich on a platter.

It is best—really—if your students can think of you as a human being, part of the same species to which they belong. Intellectually you may be on an entirely different plane. Nevertheless, when you are not running bulls in Pamplona or swapping gestalts with the Maharishi, you are in fact just the calculus instructor.

It is an important job, and you should endeavor to carry it out with dignity and professionalism.

2.5 Homework

In most lower-division courses, and many upper-division ones, it is by way of the homework that you have the greatest direct interaction with your students. When students waylay you after class or come to your office hour, it is usually to ask you about a homework problem. This is why the exercise sets in a textbook are often the most important part of the book (textbook authors do not seem to have caught on to this observation yet) *and* why it is critical that homework assignments be sensibly constructed.

Let me stress again that I am not trying to sell you a time-consuming attitude or habit. If you take twenty minutes to compose a homework assignment then you are probably taking too much time. But consider the following precepts:

- Do not make the homework assignment too long.
- Do not make the homework assignment too short.
- Check over the problems you assign to confirm that there are no notational or obvious typographical errors. (Students can waste great amounts of time trying to fathom typos that are trivial to you and me. As a result, they become quite frustrated and angry. Doing this sort of checking shows them that you are on their side.)
- Be sure that the assignment touches on all of the most important topics.
- Be sure that the homework assignment drills the students on the material that you want them to learn and the material that you will be testing them on.
- *Make sure that at least some of the homework problems are graded.*
- Plan ahead. The exams that you give should be based only on material that the students have seen in the classes and in the homework.

If homework does not count and is not graded, then students will not do it. That is a fact. I realize that many of us have neither the time nor the inclination to spend long hours each evening grading homework. Many universities and colleges these days simply do not have the resources to provide enough graders for lower-division courses. But there are compromises that you can make. For example, you can tell the students that, of ten problems on the homework assignment, just three will be graded. But don't tell them which three. This device will force most of the serious students to do *all* the homework problems, but it requires much less grader time to get the grading done.

If the last suggestion will not work for you, then you can give weekly quizzes that you yourself will grade. The amount of your time involved will be little,

and it is a device to force students to keep up with the work. Incidentally, this device also gives you a gentle way to keep your finger on the pulse of the class.

Consider the implementing following policy to help get your students more interested in doing the homework. Students can and do benefit from collaboration, just as we mathematicians do in our research. While you probably do not want to encourage collaboration on exams, you may wish to encourage it on homework. Of course I'm not talking about "I'll copy yours this week and you can copy mine next week." Instead, I'm talking about an intelligent exchange of information among equals.

Some studies have shown that one reason that Oriental students in this country tend to do very well in their mathematics classes (and there are surely many reasons) is that they work in groups. More precisely, they first work hard as individuals. Then they get together and compare results. In short, they collaborate in much the same way that mature mathematicians collaborate. They are willing to say, "I can do this but I cannot do that. What can you contribute?" At the same time, the studies indicate that certain other elements of the student population are either loath to work in groups or are unaware of the benefits of this activity. These strata tend to do poorly in mathematics classes. See [TRE] for details.

Some of the more interesting teaching reform projects, including those from Harvard and Duke, are specifically designed to encourage students to learn mathematics through group activities. Reports on these experiments are encouraging.

If you do decide to encourage group work in your classes, then you will have to make peace between said collaboration and your grading policies. If homework is not collected, then there is no problem and you can separate the good students from the bad through exams and quizzes. If instead homework is collected, then you will have to consider carefully how to tell whose work is whose, or at least how to divide up the credit.

2.6 Office Hours

At most universities the instructor is required to hold three or more office hours per week. Choose three hours that are convenient for you or convenient for the students or both. Monday/Wednesday/Friday at 11:00 A.M. is, on most campuses, one of the most popular times for classes. If you schedule your office hour at that time then many students will not be able to attend. One good strategy is to stagger your office hours, so that they are at different times on different days. Another is to make an office hour from half past the hour to half past the hour, so that a student's class is likely to overlap only half of it rather than all of it.

Of course you cannot select a time for office hours that will please *everyone*, so don't even attempt to do so. Set your office hours, and announce them, and explain to the students that you can make appointments for those who cannot attend the regularly scheduled hours. Such an announcement will not appreciably increase the number of visitations from your students, and it is just good business to set such a policy.

Promise students that you will be there during your office hour. And be there. Students should be made to understand that they need not wait for a natural or personal disaster in order to come to your office hour. It is perfectly all right for a student to come to your office hour and say "I don't get problem 6." or "The chain rule makes no sense to me."

During your office hour, you will usually not be overwhelmed with students (except perhaps just before an exam). In fact it is a general rule of thumb that, the larger the class, the smaller the percentage of students who will come to your office hour. But those who do show up will appreciate your attentions. Of the hours that you have designated, you can spend some of them catching up on your correspondence, making up the next homework assignment, or reading the *Notices* or the *Monthly* or the *Mathematical Intelligencer*.

If you have sufficient space in your office, it is a good idea to have a table and a couple of chairs set up in a special part of the office—*away from your desk and your papers and books and personal artifacts*—where you will consult with students. What is the reason for this affectation? First of all, you don't want students inadvertently walking away with your papers or your correspondence. Second, you don't want them spilling coffee on your latest manuscript or your new book that you purchased at great expense from Marcel Dekker. Third, students are by nature careless. They may put their feet on your desk or use your telephone or grab your fountain pen. Rather than appear to be an old fuddy-duddy and constantly be scolding, it is so much easier to have a special venue in which to "hold court".

When a student comes to my office expressing befuddlement over a particular type of problem, I have a powerful and decisive weapon that I unleash. I begin by asking, "Do you have a half hour or so?" If the answer is "Yes", then I sit the student down and say, "Try a problem of the kind you are having trouble with. When you get stuck, tell me." Of course the student invariably gets stuck, and I give him a little help. I might need to intervene three or four times during the first problem that the student does for me. But the second problem may require only two interventions, and the third only one. By the time the fourth problem rolls around, the student's newfound confidence is irrepressible, and the transaction is a great success. The student goes away pleased and happy that he has now mastered a heretofore mystifying mathematical idea. Of course I always tell the student, "If you get home, and you find that you are still confused, then come back and we'll do this again."

On days when your office hour is not crowded, and you only have a couple of customers, I highly recommend that you try this teaching technique. It's good business, and it always produces satisfied customers. Word gets around in the class that the professor is not such a bad guy after all. Perhaps, as a result, a few other students drop by for help.

You want to convey to students that the office hour is a particular time that you have set aside for them. If you consult with students while sitting at your desk and glancing at your mail, or scribbling notes for an upcoming seminar, or answering phone calls, then you in fact will *not* convince them of your dedication. Instead, if you hold court in the special part of your office that you have set aside for consultations, then your students will understand that this time is theirs. If

you really want to do it right, then let your voice mail pick up on your phone calls during your office hour. For those sixty minutes, give yourself to your students.

The office hour is your opportunity to get to know at least some of your students personally. Of course I do not mean by this that you should get involved in their *personal lives*. Problems about their love lives or their parents or their social diseases should be referred to the professional counselors that are on the staff of every college. What I mean is that you should take the opportunity to get to know some of your students as people, and to let them get to know you as well.

This activity has several beneficial side effects, both for you and for them. When you are lecturing, you can have certain individuals in the room in mind as you formulate your remarks. You can make reference (*without* mentioning any names) to questions that came up during office hour. It is reassuring to the average student (the type that *does not* go to office hour) to know that conscientious students (the type that *do* go to office hour) have some of the same questions that they have.

This point is in fact worth developing. Some components of teaching may be compared with certain components of psychotherapy. One big aspect of therapy—certainly an aspect that is exploited by popular psychology and self-help books—is to assure the patient that he is not alone. There are thousands of people with exactly the same problems, suffering in just the same ways. And they have been treated successfully.

Just so, when you teach you must give both tacit and explicit reassurances to students that their questions and confusions are not theirs alone. An eighteen year old is scared to death that he is the only person in the room who doesn't understand why the numerator in the quotient rule has the form that it has—or why it does not seem to be symmetric in its arguments. Such a student would not dare ask about it in front of a room full of his peers. The student may not even be sure how to articulate the question, so surely will not want to flounder about in front of the entire class. At the same time the student may be afraid to come to your office hour and, alone but in *your* august presence, ask for a clarification.

Thus you must signal to students that questions are a good thing. When a student asks a question in class that might be of general interest, I not only repeat it but I often state that I am glad this question was raised. I carefully record the question on the blackboard. Several people have visited me privately, I add, and asked variants of the same question. If there is a question that should be asked but has not been, then I ask it myself. I say that if this point is unclear to them (the students) then they should come see me in my office hour and get it straightened out. You don't need to give away door prizes to drum up business at your office hour. However, it is psychologically important for students to know that you are available, whether they actually come to see you or not.

I have said repeatedly in this book that persistence is an important attribute for the successful teacher. Another such attribute is patience. If a student finds the nerve to ask a question in class or during office hour—*even if it is a question that I have answered before*, in fact *even if I have answered it several times before*—then I treat the questioner with respect and I pay due homage to the

question and I answer it. I *never* say, "I've already answered that question. Go home and read your notes." Such a rejoinder would be counterproductive, and would discourage further question-asking in class.

I often announce to my classes that students may drop by my office even when it is not my office hour. If I am not busy, I'll be happy to talk to them. In practice, this charity does not appreciably increase the flow of business. There are always students who strictly respect your designated office hour and there are always those who drop by whenever they please. But making an announcement of this nature is one of those little details that contribute to a favorable student attitude. For it sends a signal to the class that you care, and that you truly want to help them learn mathematics. If you do make such an announcement, be courteous to those who take you up on it. If you are busy and must send the student away, do so with respect and suggest another time for the student to return.

I once had a colleague who, whenever a student would show up at his door, would crawl under his desk until the student went away. This is certainly a memorable way to deal with students, but not one that I would recommend. When a student comes to your office, make him feel welcome. Act as though you are happy that he dropped by. Endeavor to adopt the same cheery tone that you would assume if a good friend paid a surprise visit. Such an action on your part will put the student at ease, and will make the transaction go smoothly and productively.

The office hour is a way to step out of your role as instructor and let the students know that you are a person. It is a way to become acquainted with some of your students. Any good public speaker "works the audience" before his speech. Holding your office hour is one way to work the audience. You will also get a feeling during the office hour for how the class is doing, what problems and concerns have arisen, how the pace is working. It is wrong, and self-defeating, to view your office hour as a dreary duty. It is a teaching tool that you should use wisely.

2.7 Designing a Course

Many of us never have the privilege of actually designing a course. Instead, we are assigned to teach prepackaged courses that the department has already assembled. This will especially be true if your teaching load is primarily "service courses": precalculus, calculus, linear algebra, ordinary differential equations. In non-service courses—upper-division courses or courses that are taught for majors—you may in fact have considerable discretion as to what you will include in the course, and how you will organize it. In what follows, I shall draw a sketch of what input you may have into the structure of a course, and also what input you may not have.

If you are teaching one of the prepackaged service courses, then certainly the content will be pre-specified. And, like it or not, you had better stick to the syllabus or outline that the department provides for you. Your students are taking this course *only because* it is required for their major. If you are an

2.15 Small Classes vs. Large Classes

We all know, deep in our guts, that small classes are a much more stimulating venue for learning than large classes. After all, in a small class students can participate, they can feel much more a part of the process, they can get to know the instructor personally, they are more comfortable asking questions, and they experience a lot of one-on-one interaction.

All true, but (as noted in the last section) there are no studies that show significant improvement in learning, performance, or retention when students are taught in small classes rather than large classes. In fact the main difference that can be objectively verified is that students in small classes feel better about the class, feel more empowered, and have higher self-esteem than those in large classes.

This is certainly a situation in which we instructors must train ourselves to separate objective fact from self-evaluation and intuition. When we argue (with the dean, for example) for smaller calculus classes, we are motivated in part by a concern for student welfare, in part by a distaste for teaching large lectures, and in part by a desire to find objective reasons for hiring more faculty.

If the conclusions of the objective studies are valid, then perhaps we can learn something from them that will inform the way that we teach large lectures. I'm sure that there are few preachers who want smaller flocks, who would prefer to preach to an audience of 30 rather than 300. I wonder why that is so? Can we learn ways to make our students in a calculus class of a few hundred feel more involved, have a better opinion of themselves, feel good about the learning process?

The final word on these questions is certainly not in. But simply being aware of the information in this section and the last should give you food for thought the next time you face down a large class. What is it that makes teaching a large class seem to be difficult? Is it the size of the room, the population of students, the use of a microphone, the feeling that there is less room for error, or what? What can we as math teachers—not preachers or performers—do to make a large class still seem like a family? What can we do to make a large class still feel like a place where learning is taking place?

2.16 Problem Sessions, Review Sessions, and Help Sessions

At many big universities, the large thrice weekly lectures in a lower-division math course are supplemented by once- or twice- weekly "problem sessions" or "help sessions". Usually the lectures are delivered by a professor or instructor while the help sessions are staffed by graduate student teaching assistants (TAs).

Imagine that you are the graduate student in charge of a problem session. It is easy to fall into the trap of not taking the work very seriously. After all, student attendance at these sessions is poor in general and spotty at best. Students seem to be inattentive and their questions are often puerile. But the quality of

any class or help session is largely influenced by the attitudes and efforts of the person in front of the room. If your attitude is to treat the help session casually or carelessly then you will get correspondingly disappointing results from the students. Consider giving weekly quizzes, sending students to the board, and other devices for livening up your problem session (see also Section 3.12). I wish to concentrate here on more mundane matters.

It is arguably more difficult to conduct a good problem session than to give a good lecture or class lesson. For the problem session presents all the difficulties of a class period, and more. At least in a class you are in complete control of the order of topics and can, if you wish, present them from prepared notes. In a problem session, if you really let the students ask what they wish, then you must be ready for anything. And you must be able to think quickly, on your feet, of the best way to present any given topic, give a hint on any problem, or handle any point of confusion. In a class or lecture you can always pull rank and say, "There is no time for questions now. See me in my office hour." (I don't recommend that you say this very often, but it is an option that is available). But help sessions are for questions.

If you are a novice, then it is probably safest to view the help session in the most naive way. Your role is to help students do their homework assignment for *that week.* Thus your preparation for a help session might consist of working all the homework problems for the week, or at least staring at them long enough to be sure that you know how to do them.

Be certain that the techniques that you present are consistent with those used in class and in the book. Some professors require their TAs to attend their classes, just to insure this consistency. Such a professor might even do a spot check of the grader's work, or drop in on help sessions to see how things are going.

I know of at least one professor who works closely with his grader and his TAs by attending, once per week, each problem session for his class *accompanied by the grader!* This requires some extra effort on everybody's part, but it shows real consideration for the student who has questions about the way that his homework was graded (or how the class, as a whole, is being conducted). It goes without saying that in order to use this device to good effect the professor will have to be well-coordinated with the grader on how he wants each homework assignment graded.

When you are helping with a homework problem that is to be handed in, don't give away the store. One reasonable answer to the dreary question, "How do you do number 14?" might be, "I'll do number 16 for you, which is similar." Another reasonable answer might be, "I'll get you started. You do the rest." A third is, "Here is an outline of the basic steps." The truly skillful instructor will turn this question-answer session into a team effort. Gently goading the students with his own prompts and questions, this instructor will resist simply doing the requested problem for the students. The trouble with just solving the problem—and nothing more—is that only the requestor and perhaps a few others will be paying attention. If instead the instructor can generate some repartee, and can get the students to want to pitch in, then there will be considerable student interest and a number of class members will learn from the experience.

There are subtle psychological forces at play in the scenario just described. If each student is worried about protecting his turf, and simply does not want to share what he knows, then you will have a hard time generating useful dialogue in your problem session. If instead the atmosphere is one of learning being a sharing activity, and of giving knowledge in expectation of receiving knowledge, then the problem session can be a worthwhile and nurturing experience for everyone. (We all know of mathematicians who collaborate easily and well, and of others who seem to be thoroughly incapable of collaboration. Perhaps these differences reflect attitudes similar to those being described here.) Of course you as the TA or instructor must set the example. If the signal you send is that *you* are not willing to help, that *you* are not willing to share, that instead you are like the oar master on an ancient galley, then you will get little in the way of cooperation and sharing from your students. If instead the example you set is one of patience and giving and caring, then you are likely to be the beneficiary of an enthusiastic response.

The advice to the TA (five paragraphs ago) to work all the homework problems the night before a problem session is one that I tender hesitantly. I never do this, but I've been teaching math for twenty-five years. I am rarely surprised by any question in a calculus class or help session and, even if I am, I can usually slug my way through whatever new features are present. If I am at a review session for an exam and a student presents a really difficult question then I always have the option of saying, "That's an interesting question, but one that could never be put on the test. Let's discuss it privately."

In your first few years of teaching you will have to strike a balance between being thoroughly prepared (by working all problems in advance) and spending too much time on preparation (see also Section 1.3). Just remember that a large part of your job is **(i)** to show the students how to do the problems and **(ii)** to persuade the students that the problems are doable (by ordinary mortals). If you fumble around and act baffled by the problems, then you are presenting a poor role model and, more to the point, doing your job badly. Students find appealing the fact that I can do all the problems and that, moreover, I invariably know where the difficult spots are and can help them to chart their way through them. This ability can only come with experience. It is the model that you should strive to attain.

2.17 Transparencies

I have already touched upon the topic of overhead transparencies (Section 2.3). With the use of transparencies, you can cover more material than you could by just using a blackboard. By using several overhead projectors, you can create an ambience similar to that achieved by several blackboards. By using color, overlays, photos from books, data printouts, computer-generated graphics, histograms, and the like, you can put on a dazzling display of information.

Many of the principles governing good blackboard technique also apply to overhead slides. But, in this somewhat different environment, they take on a new form. Of course you must be organized and write neatly. Be sure to write

Chapter 3

Spiritual Matters

Even paranoids have enemies.

Delmore Schwartz

Conservatism goes for comfort, reform for truth.

Ralph Waldo Emerson

If anything ail a man, so that he does not perform his functions, if he have a pain in his bowels even,—for that is the seat of sympathy,—he forthwith sets about reforming the world.

Henry David Thoreau

Reformers can be as bigoted and sectarian and as ready to malign each other, as the Church in its darkest periods has been to persecute its dissenters.

Elizabeth Cady Stanton

You can't make a Hamlet without breaking a few egos.

Anon.

If you think education is expensive, try ignorance.

Derek Bok

Bad planning on your part does not automatically constitute an emergency on my part.

Anon.

We, the unwilling, led by the unqualified, have been doing the unbelievable for so long with so little, we now attempt the impossible with nothing.

Anon.

Teaching has ruined more American novelists than drink.

Gore Vidal

3.0 Chapter Overview

This chapter addresses philosophical issues connected with teaching. How do students learn, how do they formulate questions, how should we answer those questions, what is the function of the mathematics teacher? An adequate instructor records the material accurately on the blackboard and then goes home. A truly dynamic instructor interacts with the students, excites their intellectual curiosity, and helps them to discover ideas for themselves. The material in this chapter should help you to pass from the first state to the second.

It doesn't require much effort to show that you care about the class. If they do well on a test, celebrate with them. If they do poorly, commiserate with them. If they cannot do the homework, then help them with it. If they can't understand the concept of orientability of a surface, explain it to them. Show them a Möbius strip. They have probably all seen Möbius strips before, but never realized that they had a serious use. The point is that you should share their pain. You are there to help them learn. Act like it.

I once witnessed a conversation between two mathematicians—one American and one German. The first man was discussing John James *Audubon*, the American naturalist. The second was discussing the *Autobahn*, the famous high-speed German highway. Each scholar prattled away for ten minutes or more, discussing his own topic, blissfully unaware that his interlocutor was discussing something entirely different. How was this possible, you may ask? Well, how many mathematicians do you know?

Has it ever occurred to you that, sometimes, when you are talking to your calculus students, you are no different than the savants in the last paragraph? Here is the poor student trying to tell you that he can't even understand the *logic* in problem #6 on the homework, and you are busy trying to explain to him the circumstances under which the mean value theorem fails. Such a situation doesn't make you either a bad person or a bad teacher. But it certainly makes you less than optimal in helping your student to learn. Part of caring is to step out of your own shoes and, as much as possible, step into the student's. It may be a new experience for you, but it is one that you should seek.

3.4 Breaking the Ice

The first day of class is simultaneously a day of happy anticipation and a day of stress. It is the first of these (assuming that you like to teach) because you are, after a restful summer, jumping into something that you enjoy and that you do well. It is stressful because you don't know what this new group of students is going to be like, or whether they will play ball with you, or whether you can get through to them.

I am a teacher of long experience. On days of exceptional hubris, I convince myself that I am rather a good teacher. Yet most semesters, especially in the fall, I meet a new class with new students and I have to demonstrate to these people that I'm a good guy. We begin as total strangers, and my goal is to turn us into a working group. Usually this takes a while—often several weeks.

Since I so enjoy a class once we have all become friends, I find the period of tooling up to that happy steady state generally too long and too painful. What usually happens is that there is a period of two to five weeks during which the students look at me as though I am from Mars. They don't laugh at my jokes, they don't answer my questions, they don't seem to take me very seriously. If the class is to be a success, then some magical thing must happen to change everyone's attitude.

You should consider ways to make yourself seem like a human being to your students. Being playful, or impish, or making fun of yourself, is certainly one

technique for accomplishing that goal. If that doesn't work for you, or makes you feel uncomfortable, then try something else. Read them some history. Tell them of Bishop Berkeley and his doubts about calculus. Tell them about fractals, or dynamical systems, or wavelets, or why[1] mathematicians don't get the Nobel Prize.[2]

Find some way to open up to your students so that they will open up to you. Some instructors hide their unease behind regimen. They take roll, or put together a seating chart, or ask each student to introduce himself. This routine is fine if you are comfortable with it. My view is that you should show students from day one that you are a person, and that you are going to spend the term doing your best to communicate with them. I don't think that taking roll is a good way to send that signal. It is better if you tell them what the course is about, or describe your grading policies, or give them some clues as to what *you* are like.

To repeat an important theme of this book: If your students are not talking to you it is probably because you are not talking to them. Set the tone on the first day. And never forget it.

3.5 Why Do We Need Mathematics Teachers?

When I wrote the first edition of this book, I had trouble formulating a cogent answer to this question. We all believe that teachers are necessary. Society must believe it too, for it deigns to pay (not very generously) a great many teachers. But *why* are teachers necessary? Two colleagues (Gary Jensen and Meyer Jerison)—both wiser than myself—have supplied a striking and memorable answer:

The teacher

(1) Sets a pace for the students;

(2) Teaches students to read (mathematics);

(3) Helps the students to become engaged in the learning process.

There is considerable wisdom in these simple observations. Let us consider them one by one.

(1) Watch a young person—or even an old person—attempt to learn to play the piano without benefit of a teacher. Such a person usually has neither the

[1]There are actually several versions of this story. The so-called French/American version is that a mathematician (Mittag-Leffler) ran off with Nobel's wife. The Swedish version is that Alfred Nobel was a practical man of the world who wasn't aware of mathematics as a discipline.

[2]If you are tired of the standard Nobel Prize story, then tell the lesser known story of the Mittag-Leffler Prize. Mittag-Leffler set it up, of course, to spite Nobel. He mandated that the medal would be twice as large, and the award twice as grand in several notable aspects. It was only awarded twice because Mittag-Leffler invested the funds in the Italian railroad system and German World War I bonds.

experience is someone worth listening to. Find a mentor, ask him questions, and listen to the answers. And then use this input to craft your own style of teaching.

3.12 How to Ask, How to Answer

If a pollster asked the average American voter, "What do you think of the upcoming election?" then the resulting answer would probably not be very enlightening. If you turn to your calculus class one day and say, "OK, now we've covered Chapters 3 and 4—any questions?" then you will get a bunch of blank looks. By the same token, if a textbook salesman hands a new calculus book to a math professor and says, "What do you think?", the professor will probably say, "I dunno; they all look the same to me." By the same token, students come to professors with questions such as, "Like, you know; I don't think I understand any of this stuff we're doing."

It is a strange facet of the human condition that most of us don't know consciously what we think about most things most of the time. A skilled questioner learns to ask *specific questions* in order to obtain meaningful answers. Rather than asking your class if there are questions about Chapters 3 and 4, ask them instead if they are comfortable with the chain rule, or if they can do related rates problems, or falling body problems. The material in a person's memory is hung on hooks. You must reach for those hooks to get useful answers to your questions.

The same principle applies when you are holding a review session—for a midterm exam, let's say. If you are serious, if you *really* want to help the students, then it is simply not good enough for you to stand before the students and say, "Any questions?" They *do not know* what they want to ask. And, even if they think they know, they are timid about doing so. You must prompt them: "Do you understand integration by parts? Can you do partial fractions? What about the u-substitution? Is Section 7.5 confusing? Was the second homework assignment particularly difficult?" You, the instructor, must understand that your having said these things will (i) jog their memories, and (ii) make it OK for them to ask about these topics. I find that it breaks the ice for me to write a list of topics on the board. This is just one way to get the "review session ball" rolling. Remember: You must poke the students and prod them and, if necessary, embarrass them a little. Never forget the psychological aspects of teaching.

We implement these dicta naturally when writing an exam. You would never set an exam question for freshmen that said, "Tell everything you know about differential calculus." Instead you ask very specific questions. You want to train yourself to do the same when talking or lecturing to students. More, you want to train yourself to do the same in reverse when you are trying to elicit questions from students.

There is a gentle art of getting your students to pose questions. And I don't mean questions like, "Will this be on the test?" I mean the kind of meaty, well-thought-out questions that we all live for. Perhaps the most common com-

plaint that I hear from disillusioned mathematics instructors is that they cannot develop any participation from their lower-division classes (see Section 4.5 on Frustration). The matter of garnering good questions is a non-trivial issue, and one to which an entire separate book could be devoted. You are going to have to find methods that suit your personality, and your teaching style, and that work for you. (See the Appendix to this section for some specific suggestions on how to increase student participation and inject some life into your class.)

The devices that you use can be quite simple. For example, giving a good quiz once or twice per week is a device for focusing student attention on some *particular issues*. The quiz is a little bit like a traffic officer pulling you over and threatening you with a citation. We are all aware—in a general sort of way—of the traffic safety laws. But if a cop gets in your face and starts telling you things that you are doing wrong then suddenly the penny drops.

The devices that you use can also be complex. You could have each student develop an ongoing, long-term project. Such a project might have the property that it must include material based on the ideas from each week of the course. And each student must be prepared to report to you at any time on the status of his project.

You may very well think that quizzes are too trite and semester-long projects are too massive for you to consider. Fine. I use quizzes frequently in my own course, and I'm frankly too lazy to do semester-long projects. Finding a way to get students to participate is something that you must do for yourself. Consider wheedling, threatening, cajoling, joking, challenging, priming. You can get through to your students by making them like you, or by scaring the hell out of them, or by conning them, or by being gruff with them. I am not necessarily recommending any of these. But if you want to be an effective teacher then you must find something that works for you.

As you experiment with ways to liven up your class, bear in mind the nature of the enemy. One enemy is that young adults, for the most part, are quite unsure of themselves. Unlike an experienced mathematician, who in effect makes a career out of asking (often stupid) questions, the student is deathly afraid of looking silly in front of his peers. He is not intellectually mature, and not experienced. He is not expert in the art of discourse (see also Section 3.14).

This last point is worth developing. If you have survived in the academic game, then you have learned to ask questions. You would never go up to a member of the National Academy of Sciences and say "Duh. I was trying to prove an interior regularity theorem for the Laplacian, and I just cannot seem to do it. I tried integrating by parts, but I couldn't decide what to do with the boundary term." Your friend the National Academy member would—justifiably—probably conclude that you were an idiot. A safer way to pose the question would be: "I've been thinking about interior regularity for the Laplacian. I know the classical ideas, but what is the modern approach? What would be a general context in which to fit this type of question?"

If you know something about elliptic partial differential equations, then you are probably not sent into paroxysms of ecstasy by the second question either. But it certainly sounds more intelligent than the first. And it gives the questioner some room to maneuver. Students simply don't have this skill at discourse, so

they resort to the obvious subterfuge—they clam up. Part of your job as teacher is to help your students learn to engage in scholarly discourse. Help them to ask questions. If a student asks a weak question, help him to turn it into a better one. Try to create an atmosphere in which you and the students are co-explorers. Convey that you will sometimes make false starts, and so can they. It's a knack, but you can learn it.

Another enemy, besides the observed fact that students are uncertain and don't want to talk, is that mathematics *can be* (it is not by nature) a dry, forbidding subject. Part of your job as teacher is to make the subject come alive and to motivate the students to want to learn the material. This book supplies a variety of techniques for achieving that goal (Sections 1.7, 1.12, 3.1, 3.3, 3.5, 3.7, 3.12, 3.14).

APPENDIX: SOME SUGGESTIONS FOR ENCOURAGING CLASS PARTICIPATION

This appendix contains several techniques, drawn from the literature or from my own experience, for bringing your class to life. Take them for what they are worth. Some may appeal to you, and some may not. But reading about them may give you ideas of your own. Note that the activities discussed here are designed for classes of manageable size. They do not lend themselves well to a large lecture of 350; see Section 2.14 for a consideration of techniques suitable for that environment.

In lower-division political science courses, it is common for the instructor to begin a class by saying, "Today we are going to be a medieval village. Who wants to be the mayor? Who wants to be the executioner?" And so forth. It is quite natural for a mathematician to react to that type of classroom activity with derision, to observe that it appears to be childish and non-productive. Perhaps, but such devices are a wonderful way to get students involved with the subject matter. What can we do in our math classes that will (i) teach the students something of value and (ii) get them involved with the subject matter? Here are some possible answers.

1) Get students to go to the blackboard. I have noted in Section 3.1 that this is not necessarily the most efficient use of time. But it *is* a way to get the students to participate. If you wish, and if it is feasible in your learning environment, you could record problems on the board before students come into the classroom. Those who wish can go to the board—even before class begins—and work problems. To avoid having the same old students monopolize this activity, you could institute a rule that no student may work a problem at the board twice in one week. Of course the *entire class* should discuss the various solutions that are so recorded.

2) Have students prepare oral reports or mini-lectures. This activity is usually best reserved for the last part of the semester, when everyone is tired and

students are receptive to a change of pace. Since most of the students will be inexperienced in activities of this nature, I recommend that you assign students to each give a fifteen minute lecture on a very specific topic. Time considerations show that this activity is only feasible in a rather small class.

3) Have students take turns writing and grading quizzes. It might be appropriate to assign a team of three students to each quiz. Not only will this activity cause the students to think critically about the material that they are studying, but it will also imbue them with an appreciation for the sorts of things that you, the instructor, must do.

4) If a student *cannot* do a problem, and brings this fact up in class, then have him go to the blackboard and explain what he tried and where he got stuck. It is certainly true that some students will be too shy to pull this off, but most students will be secretly thrilled to be treated like fellow scholars. You can orchestrate a similar activity for a student who *does* know how to do a problem.

5) Use "Minute Notes". These work in the following manner. Once every week or so, ask students to jot down on a slip of paper anything that is bothering them—problems that they cannot do or concepts that they cannot understand or anything else that pertains to the class. You give them just one minute for this task (hence the name). Do it at the beginning of the class hour, and collect the notes right away. Read them on the spot. You will suddenly have a much clearer picture of what is going on in the class, what concerns the students have, where you should go from there.

Perhaps more importantly, you will have given the students a feeling of empowerment. You will have helped them to understand that their input is a constructive part of the class. After a few weeks of Minute Notes, you will generally find that students are much more willing to raise their hands in class and make meaningful contributions to the learning experience.

6) If you are truly daring, then you can design your course so that it is more like a literature course. That is, you give the students regular reading assignments and homework assignments, but you do not lecture directly on a linearly ordered sequence of topics. Instead you come to class each time with an air of, "Well, what shall we talk about today? Who would like to begin?" The idea is that your classroom is a marketplace of ideas. You need to really know your stuff, and have an engaging manner, to pull this off. But it is bound to be great fun.

7) Have guest instructors. To use this tool well, you must work closely with the guests to be sure that they will talk about material that is salient to the class, and will present it at an appropriate level. If you think of the fourteen weeks (give or take) of your course in the same way that I have discussed single lectures or classes (see Section 3.7), then having guest instructors is a way to prevent your course from being an "uninflected monotone". You can also consider roles that graduate students, teaching assistants, and "teacher's aids" (i.e.,

teachers in training doing their practicum in your class) might play in livening up the atmosphere.

8) If you have the resources, and the breadth of acquaintance, or if your department has the contacts, you could bring in guest speakers from industry or government or business. Imagine a calculus class in which you bring in someone working on the NASA space station project to talk about how calculus is used to design the work platform for the engineers in space (I'm not making this up; there really is such a project). Students would really wake up and smell the coffee when confronted with such a class experience.

9) This technique was devised by Jean Pedersen. She asserts that it works extraordinarily well for her. It is called the method of "mathematical POST-IT® notes".

We all know that POST-IT® notes are those little squares of colored paper that easily can be affixed or un-affixed to a document for the purpose of making remarks or memos. The idea for the application of these devices in a math class is that the professor comes to class with a tablet or two of these notes, each having the professor's name (or some other identifiable epithet) stamped on it. Whenever a student asks a good question (not "Will this be on the test?" or "What is this stuff good for?" or some pseudo-question that the student just cooked up), then he is rewarded with a POST-IT® note. "So what?" you ask.

When the next exam comes around, the students are instructed to bring their POST-IT® notes along. They are to affix them to the front of the exam that they hand in. The student then receives two extra points (or some number to be pre-determined) for each POST-IT® note.

Reports are that, when this policy is announced in class, it is as though a jolt of electricity has run through the room. Suddenly hands are waving in the air, and previously uninterested students become the life of the party.

Now let me be the first to admit that this teaching device, like any other, is not perfect. Some students who are already alienated will become more alienated if they are unable to garner any POST-IT® notes. Other students may object that they are being treated like children. Think carefully before you try this, or any, new technique.

10) I have saved the most frivolous suggestion for last. Although you probably will not choose to use it yourself, it may suggest analogous techniques that more naturally suit you and your classroom. And, although the technique is a bit silly, it is currently in use by at least one successful math teacher.

On the first day of class the instructor announces that he is very embarrassed to report that he simply cannot spell. Students should feel free to correct his dreadful spelling. Then he begins to lecture, spelling "line" as "lien" and "book" as "buk". Students are so delighted confidently to be able to correct the professor's spelling that participating constructively in the mathematics portion of the course becomes very natural.

I find this last technique of deliberate misspelling to be a bit dishonest, but it's hard to argue with success. In my own classes, I endeavor to create the feeling that we are all creating the lesson together. I do this with a constant line of patter, much like that used by a magician or an illusionist. With this technique, I have the students talking all the time as well. If a mistake is made in class, then it is *our mistake*, and we fix it together. If a problem is solved correctly, then that is our shared triumph.

The key to bringing your class to life is to become involved with the students and to make learning a shared activity. Perhaps this is one of the great lessons of the reform movement. It is not an ideal learning environment to have the teacher as stick-man preaching before an audience of sponges. Learning should be done symbiotically, and it is up to the instructor to structure his class accordingly.

3.13 Teaching with the Internet

The Internet is a marvelous tool for making information available to a large body of people quickly. For example, if a mathematics department subscribes to an electronic journal, then as many people in the department who wish to do so can read the journal at the same time. Those who want to study a paper carefully can download it, compile it, and print it out. When you prove a new theorem, you can post your abstract (or your entire paper) on an electronic bulletin board. Your results are then instantly known around the world.

The Internet is also a useful teaching device. Create a Web page for your class. Put the class syllabus on the Web page. You could have a page about prerequisites for the course, or ancillary reading, or ways to prepare for exams. Post homework assignments and due dates on your class Web page. Put information about upcoming exams there. If you need to write up a correction to something from class, or disseminate a list of errata to the text, or post homework solutions or exam solutions, then the Internet is just the ticket.

I once read a proposal for an "Internet Mathematics Curriculum". The premise was that, at certain universities with a great many part-time and commuter students, absenteeism is a problem. Students have families and jobs and cannot always make it to class. In the electronic age, modes of communication are changing—so why not take advantage? The proposal was that the professor would still give his lecture, and those who could attend would do so. But there would be assigned note-takers who would post official notes on the Internet. The Internet could also be used to cut through the problem that students will not—or are too shy to—participate in class. The math class would have its own electronic bulletin board(s), and students could post their queries there—anonymously or not. Other students, or the professor or the TA, could answer the queries as they saw fit. Since many students have the same questions, this use of a bulletin board would allow the professor to use his time more efficiently.

The proposal that I just described was not funded. In fact it didn't even make the first cut. I think there is real merit to some of the ideas just described. But

I also think that the concept of an Internet University abrogates much of what the learning process is all about. Classes are held for a good *reason*, and it is this: Many things that we do in life have a ceremonial aspect. We hold funerals to come to grips with someone's passing, and to create a sense of closure; we have graduation ceremonies to pause to think about an important moment in a young person's life; we select people for prizes (the Nobel Prize, the Cannes Film Festival Award, etc.) in part to recognize talented individuals and in part to ponder the human condition and what we are trying to achieve. Just so, we hold classes so that the students will take an hour, go to a special place, sit in a controlled environment, and think in a focused manner about a particular subject under the guidance of an expert. If this were not as important as you and I know it to be, we would not do it.

My point is that Internet classes, while they may have their place, eliminate what is powerful about attending a class. Glancing at prepared lecture notes for your calculus class on your computer screen is (for the student) a bit too casual, and too much like turning on the radio. The student attempting to learn in this manner could be interrupted by the telephone, the doorbell, a pot boiling over, a baby crying, or any number of other exigencies (again, this is why traditional classes are a good thing). A mature and disciplined person with suitable scholarly training might be able to learn successfully from an Internet class. I'm not so sure about inexperienced eighteen-year-old students.

You can use the Internet as the nerve center of your class, to keep everyone informed of up-to-the-minute information and last-minute changes, to post new homework assignments, to post grades, to change your office hours, to give last minute room or seating assignments for the upcoming exam, and so forth. The concept of fielding questions over the Internet, or with *e*-mail, is a fascinating one. The one obvious impediment is that most students don't know how to enter mathematics using the keyboard.[1] This certainly is more efficient than trying to remember to photocopy the information and bring it to class, it avoids the class time wasted when you distribute handouts, and it is more permanent (that is, the material can always be found right there on the Web for the duration of the term).

I believe that the full picture of the value of the Internet as a teaching device is yet to be determined. But I caution you against thinking that it can be a substitute for classroom learning.

3.14 The Art of Discourse

Ask yourself this question: If a student has a successful and fulfilling college education, then what does he take away with him? Twenty years after graduation,

[1]The software product *NetTutor* by Link-Systems is designed to cut through this problem. It presents the student with a white board on which to write his query by hand. Or else the student can click on icons to pull down mathematical symbols. The student can submit a question anonymously or not. The professor can answer questions in real time or at his convenience—and he can do so publicly or privately. The professor also can, with little effort, create a database of frequently asked questions that he can allow the students to access.

what does that student still retain? What intellectual framework does he have to build on?

Comic Don Novello, in his role as Father Guido Sarducci (on the television show *Saturday Night Live*), gave the following answer. "If you majored in Economics, all you remember is 'Supply and Demand'. If you majored in French, all you remember is 'Parlez vous Français?' If you majored in Physics, all you remember is 'Every action has an equal and opposite reaction.' " (He might have added, "If you majored in math, all you remember is 'Take the exponent and put it in front.' ") So Father Sarducci proposed that people not spend four years and $100,000 on a traditional university education. If this is all you are going to retain, argued the good cleric, Father Sarducci will teach it to you in five minutes—and charge you much less. He called his solution "Father Guido Sarducci's Five Minute University."

We who devote our lives to university teaching hope fervently that there is considerably more to higher education than Father Sarducci's droll diatribe would suggest. In this section I am going to endeavor to say what that "more" is.

The naive answer to the question "What does a student get from his education?" is that the student receives career training. Certainly career training has significant value, and should not be dismissed lightly. But if we take the long view then we can see a larger picture. We can see depth and texture. What a student ought to take away from college is (i) critical thinking skills and (ii) knowledge of and experience with discourse (see also the discussion in Section 3.10). These two aspects of education are essential, and they are not disjoint.

In college, a student declares a major. And that is the area in which the student obtains advanced training. But most of the student's courses are *not* in the major. In those other courses, the student is learning philosophical discourse, humanistic discourse, the discourse of social thought, and scientific discourse. The student is learning *different modes* of critical thought.

For example, Renaissance philosophers considered the questions, "What is the world we see and what is the world we experience and what is the world that is *actually out there*? Are they one and the same world? If not, then how do they differ? And how can we tell?" Renaissance mathematicians studied algebraic equations. Renaissance musicians studied the lute. All of these are valuable avenues of inquiry, and they are all quite different. An important part of gaining an education is learning about these different modes of thought.

When we teach undergraduates—especially lower-division students—we are primarily teaching non-majors. It can certainly inform our teaching, and remind us of what we are about, to be cognizant of our goals when we teach. When you teach calculus to a pre-medical student, or finite math to a business student, you are endeavoring to acquaint him with modes of mathematical thinking, with our special method of reasoning and analysis.

In fact, when I teach my undergraduates, I have in mind a much larger and more ambitious goal. I want to teach my students that the world need not be a place in which they are passive observers. They need not spend their lives "letting things happen." Put in other words, we do not—at least should not—live in a world in which some nebulous *other people* generate ideas, and hold