So today we're going to turn from the indefinite integral to the definite integral.

The idea of the definite integral is we have some function \( f(x) \) from \( a \) to \( b \), and we want to measure the area between \( f(x) \) and the \( x \)-axis.

The symbol for this area is \( \int_{a}^{b} f(x) \, dx \).

So how do we find this area?

Well, let's start with a simple example:
Ex. \( \int_0^2 4 \, dx \)

This is just the area of a rectangle, so \( \int_0^2 4 \, dx = \text{base} \cdot \text{height} = 2 \cdot 4 = 8 \).

Ex. \( \int_0^5 f(x) \, dx \)

We can break up the area into 3 parts, so it shouldn't be surprising we get

\[
\int_0^5 f(x) \, dx = \int_0^2 f(x) \, dx + \int_2^4 f(x) \, dx + \int_4^5 f(x) \, dx
\]

Shape:

- \( \Delta \) 2 2
- \( \square \) 2
- \( \triangle \) 1

Area: \( \frac{1}{2} \cdot 2 \cdot 2 + 2 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2 + 1 \cdot 2 \)

\[=\frac{1}{2} \cdot 4 + 4 + \frac{1}{2} + 2 = 9 \]
Ex: \[\int_{0}^{2} h(x) \, dx\]

\[= \int_{0}^{1} h(x) \, dx + \int_{1}^{2} h(x) \, dx\]

Now when we say the area between \(h(x)\)
and the \(x\)-axis we really mean signed-area.
In other words, area above is positive
and below is negative. So,

\[\int_{0}^{1} h(x) \, dx = \frac{1}{2} + \frac{1}{2} = 0\]

So far our examples have been given by
nice graphs made of triangles and rectangles.
But what if it's a random function and
isn't so nice?

Well maybe we can approximate by
rectangles?
Riemann sums:

\[ f_w \]

\[ f(x_1) \Delta x + f(x_2) \Delta x + \ldots + f(x_n) \Delta x \]

\( N = 2 \)

Left Riemann sum

\[ f(x_1) \Delta x + f(x_2) \Delta x \]

\( N = 4 \)

The more rectangles we use, the closer to the area. So

\[ \int f(x) \, dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i) \Delta x \]

\( (n \to \infty) \)

the finite version of \( \int f(x) \, dx \)

Also, we could do the same thing from the right.
It turns out Riemann sums are actually very good ways of approximating \( \int_a^b f(x) \, dx \).

If \( f(x) \) is increasing (or decreasing) on \([a, b]\), then the sum is off by at most \( |f(b) - f(a)| \cdot \frac{b-a}{n} \sum \Delta x \).

If we have time, we'll cover the properties of definite integrals.