We ended last time with the following example:

\[ S(r) = k(R^2 - r^2) \]

\[
\text{Blood flow} = \int_0^R 2\pi k (R^2 r - r^3) \, dr \\
= \pi k R^2 R^2 - \frac{1}{2} \pi k R^4 \bigg|_0^R \\
= \frac{1}{2} \pi k R^4
\]

Ex. Suppose there are two types of bacteria in a petri dish. The total amount grows at a rate of \(1000e^{0.2t}\) while the amount of type B grows at a rate of \(500e^{0.1t}\). How much of type A has grown from \(t = 0\) to \(t = 10\)?

\[
A = \int_0^{10} (1000e^{0.2t} - 500e^{0.1t}) \, dt \\
\approx 23,354
\]
We've now done a bunch of definite integrals using basic rules and substitution. But we haven't come close to being able to do every integral.

What about $\int x \ln x \, dx$ or $\int x e^x \, dx$.

Well, we've done the reverse of the basic integrals/derivatives and we've use the chain rule for substitution. We haven't used the product rule.

$$\frac{d}{dx} \left( f(x) g(x) \right) = f(x) g'(x) + g(x) f'(x)$$

$$\Rightarrow \quad f(x) g'(x) = \frac{d}{dx} \left( f(x) g(x) \right) - g(x) f'(x)$$

$$\int f(x) g'(x) \, dx = f(x) g(x) - \int g(x) f'(x) \, dx$$

Let $u = f(x)$, $v = g(x)$. Then $du = f'(x) \, dx$ and $dv = g'(x) \, dx$.
Then we get

$$\int u \, dv = uv - \int v \, du.$$  

This formula is known as integration by parts.

\[ \text{Ex} \quad \int x \ln x \, dx = \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx \]

\[= \frac{\ln x \cdot x^2}{2} - \int \frac{x}{2} \, dx \]

\[= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C \]

\[\text{Ex} \quad \int_1^e x \ln x \, dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} \bigg|_1^e \]

\[= \frac{e^2}{4} + \frac{1}{4} \]

\[\text{Ex} \quad \int x e^x \, dx \quad \text{now we can try} \quad u = e^x \quad v = \int du \quad \text{dv = xdx} \]

\[\text{but then we get} \quad \int x e^x \, dx = \frac{x^2 e^x}{2} - \int \frac{x^2}{2} e^x \, dx \]

\[\text{which is an even more complicated.} \]
So let's try \( u = x \) \( v = e^x \). So,

\[
\int x e^x \, dx = xe^x - \int e^x \, dx
\]

\[
= xe^x - e^x + C
\]

In general, make \( dv \) the easier thing to integrate and \( u \) is everything else.

Example 1)

1) \( \int x^2 \ln x \, dx \)

2) \( \int (\ln x)^2 \, dx \)

3) \( \int x^3 e^x \, dx \)

4) \( \int \ln(x^7) \, dx \)

5) \( \int x^3 e^{x^2} \, dx \)