Last time we talked about solutions to homogeneous equations, $A\hat{x} = 0$.

We saw the set of solutions was the span of a set of finitely many vectors given by the free variables.

What happens if we change it from $A\hat{x} = 0$ to $A\hat{x} = b$?

\[
E \quad A = \begin{bmatrix} -3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad b = \begin{bmatrix} 7 \\ -1 \\ -9 \end{bmatrix}
\]

\[
[ A \mid b ] = \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
\hat{x} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{9}{13} \\ 0 \end{bmatrix} x_3
\]

which is not in the span of anything.
Geometrically:

- line through origin
- not through origin

Note: Any two solutions differ by a solution to $A\hat{x} = \hat{0}$. So every solution looks like $\hat{x} = \hat{p} + \hat{x}_H$, where $\hat{p}$ is an arbitrary solution to $A\hat{x} = \hat{b}$ and $\hat{x}_H$ is a solution to $A\hat{x} = \hat{0}$. 
Def: Vectors \( \vec{v}_1, \ldots, \vec{v}_p \) are linearly independent if there do not exist nonzero weights \( c_1, \ldots, c_p \) such that
\[ c_1 \vec{v}_1 + \cdots + c_p \vec{v}_p = \vec{0}. \]
OR \( A\vec{x} = \vec{0} \) has only the trivial solution where
\[ A = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_p \end{bmatrix}. \]

Otherwise, we say \( \vec{v}_1, \ldots, \vec{v}_p \) are linearly dependent.

Ex: \[
\begin{bmatrix} 1 & -2 \end{bmatrix}, \begin{bmatrix} -2 & 4 \end{bmatrix}, \begin{bmatrix} -3 \end{bmatrix}, \begin{bmatrix} 5 \end{bmatrix}
\]

Ex: \[
\begin{bmatrix} \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{4}{3} \end{bmatrix}
\]
Ex: \[
\begin{bmatrix}
1 & 2 & -1 \\
5 & 8 & 0
\end{bmatrix}
\]

Ex: \[
\begin{bmatrix}
0 & 11 \\
0 & 2
\end{bmatrix}
\]

Ex: \[
\begin{bmatrix}
1 & 3 & 5 \\
2 & 4 & 6
\end{bmatrix}
\]