Go over \((T \circ S) \text{ is 1-1 if } T, S \text{ are 1-1}\).

\[ f(x) = mx + b \text{ is linear } \Rightarrow b = 0. \]

**Inverse of matrices**

If \(A\) is \(m \times n\), then \(A^T\) is the unique matrix such that \(AA^T = A^TA = I\). 

\(A^T\) may or may not exist.

We saw that a 2x2 matrix \(A = [\begin{array}{cc} a & b \\ c & d \end{array}]\) is invertible if and only if \(ad - bc \neq 0\) and \(A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\).

Now, what about 3x3 matrices.

\[
\text{Ex. } \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
\[ A^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \]

How did we find \( A^{-1} \)? Well what did each of these matrices do when we multiply them by a second matrix \( B \)?

The first swaps rows, the second scales and adds the first row to the second, and the third scales the first row.

All of these are elementary row operations! And they can all be undone by \( A^{-1} \).

Since each of these matrices represents an elementary row operation, we call them elementary matrices.
Now, if $A$ is invertible, $A\vec{x} = \vec{b}$ always has a unique solution, regardless of what $\vec{b}$ is. Why? We get $\vec{x} = A^{-1}\vec{b}$ for any $\vec{b}$.

But this means $A$ has a pivot in every row. Since $A$ is square, it means $A$ looks like

\[
\begin{bmatrix}
\star & \star & \star \\
0 & 0 & 0 \\
\vdots & \ddots & \vdots \\
0 & 0 & 0
\end{bmatrix}
\]

reduced to echelon form. Reduce it more then and we get $I$!

So there is some sequence of row operations $E_1, E_2, \ldots, E_p$ that turns $A$ into $I$. I.e.

\[(E_pE_{p-1}E_2E_1)A = I\]
But this must mean \( A^{-1} = E_p - E_1 \), since it gives us \( I \) and \( A^{-1} \) is unique. So what we've found is a way to find \( A^{-1} \). Do the row operations to \( AA^{-1} = I \) and get

\[
(E_p - E_1)A A^{-1} = (E_p - E_1)I
\]

So,

\[
A^{-1} = E_p - E_1
\]

\text{Ex}

\[
\begin{bmatrix}
1 & -2 & 1 \\
4 & -7 & 3 \\
-2 & 6 & -4
\end{bmatrix}
\]
So short of just finding $A^{-1}$, how do we know $A$ is invertible?

We’ve already seen 1) $A^{-1}A = I$

but this also means 2) $A$ has a pivot in every column

and 3) $A$ has a pivot in every row

now, 2) $\Rightarrow$ 4) Columns of $A$ are linearly independent

5) $A\overline{x} = \overline{0}$ has only 1 solution

6) $T(\overline{x}) = A\overline{x}$ is 1-1

7) $A\overline{x} = b$ has at most 1 solution for all $b$

and 3) $\Rightarrow$ 8) Columns of $A$ span $\mathbb{R}^n$

9) $A\overline{x} = b$ is always consistent for any $b$

10) $T(\overline{x}) = A\overline{x}$ is onto
So these are all equivalent ways of saying $A$ is invertible.

If $T(x) = Ax$ and $A$ is invertible, then $T^{-1}$ exists and is given by $T^{-1}(x) = A^{-1}x$. And of course it's unique since $A^{-1}$ is unique.

We see $T^{-1}(T(x)) = x = T(T^{-1}(x))$. 