When we started the course, we talked about how we want to take a function like position and not only find the average rate of change (average velocity) but also the instantaneous rate of change (velocity) at any point.

Slope of secant = average rate of change on \([a, b]\)

So we see that

slope of tangent at \(a\) = inst. rate of change at \(a\) = \(\lim_{x \to a} \frac{f(x) - f(a)}{x-a}\)

Slope of tangent = instantaneous rate of change at \(a\).
Ex. Find the equation of the tangent line to \( f(x) = -16x^2 + 96x \) at the point \((1, 80)\).

\[
\begin{align*}
  m_{\text{tan}} &= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \\
  &= \lim_{x \to 1} \frac{-16x^2 + 96x - 80}{x - 1} \\
  &= \lim_{x \to 1} \frac{-16(x^2 - 6x + 5)}{x - 1} \\
  &= \lim_{x \to 1} \frac{-16(x - 1)(x - 5)}{x - 1} \\
  &= -16 \lim_{x \to 1} (x - 5) \\
  &= -16 \cdot 4 \\
  &= -64
\end{align*}
\]

So the equation of the tangent line is \( y - 80 = 64(x - 1) \) or \( y = 64x + 16 \).
Now, there's a second way to write out this limit with just the point we're finding the slope at.

\[ \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{a+h \to a} \frac{f(a+h) - f(a)}{a+h - a} \]

\[ = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \]

Ex. Find the equation of the tangent line to \( g(x) = x^2 + 3x \) at 1 using the second limit.
Now, as our point changes, our tangent line changes, and so does the slope. So the slope is actually a function of the point \( x \). We call this function the **derivative** of \( f \) and write it as \( f'(x) \) ( \( f \) "prime").

But formally,

\[
f''(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

as long as \( f(x) \) exists and the limit exists.

If \( f'(x) \) does in fact exist, we say \( f \) is differentiable at \( x \) and the process of finding \( f'(x) \) is called differentiation.

**Ex.** Compute \( f''(1) \) if \( f(x) = -16x^3 + 96x \).
Note as far as notation,

\[ f'(x) = \frac{df}{dx} \quad \text{and} \quad f'(a) = \frac{df}{dx}\bigg|_{x=a} = \frac{df}{dx} \bigg|_{x=a} \]

or if \( y = f(x) \),

\[ f'(x) = y'(x) = \frac{df}{dx} = \frac{dy}{dx} \]

Ex. Find the derivative of \( f(x) = \sqrt{x} \).

Ex. Find the derivatives of \( 1, x, x^2, x^3, x^4 \).

Ex. Find the derivatives of \( \frac{1}{x}, \frac{1}{x^2} \).