We've now done everything we need with differentiation for functions of more than one variable. What about integration? How can we extend it to multivariable functions?

Just like we had partial differentiation, we have partial integration.

Example: \[ \int (2xy + 4y^3) \, dx = \text{treat } y \text{ like a constant} \]

\[ \int (2xy + 4y^3) \, dy = \text{treat } x \text{ like a constant} \]

Note: We can only say what the answers are, up to a functions \( F(x) \) and \( G(x) \) resp.
We can even do definite integrals:

\[
\int_0^1 (2xy + y^3) \, dx = \text{bounds are for } x \\
\int_0^1 (2xy + y^3) \, dy = \text{bounds are for } y
\]

Note: \( F(y) \) and \( G(x) \) don't affect the answers, and the solutions are still functions.

Since \( \int_0^1 (2xy + y^3) \, dx \) is a function of \( y \), it's something we can integrate with respect to \( y \), and we'll get a number

\[
\int_0^1 \left( \int_0^1 (2xy + y^3) \, dx \right) \, dy = 0
\]
Similarly, we can calculate
\[ \int_0^1 \int_0^1 (2xy + 4y^3) \, dx \, dy = -\frac{9}{4} \, xy \, dy \]

Notice anything? They're the same!

Kind of like the partial derivatives, the order won't matter.

We call the thing we just found the double integral of \(2xy + 4y^3\). Since order doesn't matter, we can represent this integral as
\[ \iint_R (2xy + 4y^3) \, dA \]
where \( R \) is the rectangle \([0, 1] \times [0, 1] \)

and \( dA = dx \, dy \) or \( dy \, dx \)
We also say we integrated $2xy + 4y^3$ over the region $R$.

We can also write $R$ as

$$R = \{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

Note: It's a good thing we can change the order.

$$\iint_R 2xe^{x^2 + y} \, dA, \quad R = \{(x,y) \mid 0 \leq x \leq 1, -1 \leq y \leq 1\}$$

2) $$\iint_R xy e^{x^2} \, dA, \quad R = \{(x,y) \mid 0 \leq x \leq 1, 1 \leq y \leq 2\}$$

Also, we can use the double integral to find the average value of a function of two variables over a region $R$. 