PHASE TRANSITION DYNAMICS WITH MEMORY *

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Abstract.
We consider the following partial integro-differential equations which generalize the classical phase field equations with a non-conserved order parameter and describe the process of phase transitions where memory effects are present,

\[
\begin{align*}
    u_t + \epsilon^2 \phi_t &= \int_0^t a_1(t - \tau) \Delta u(\tau) d\tau \\
    \epsilon^2 \phi_t &= \int_0^t a_2(t - \tau) [ \epsilon^2 \Delta \phi + f(\phi) + \epsilon \ u ](\tau) d\tau,
\end{align*}
\]

where \( \epsilon \) is a small parameter. The functions \( u \) and \( \phi \) represent the temperature field and order parameter respectively. The kernels \( a_1 \) and \( a_2 \) are assumed to be piecewise continuous, differentiable at the origin, scalar-valued functions on \((0, \infty)\) with \( a_1(\infty) = a_2(\infty) = 0 \), independent of \( \epsilon \) and such that they satisfy the following conditions \( \int_0^\infty a_i(t) \, dt < \infty \), \( \int_0^\infty \tilde{a}_i(s) \, ds < \infty \) and \( \int_0^\infty \tilde{a}_i(s) \, ds \, ds < \infty \). By means of a formal asymptotic analysis we show that to the leading order and under suitable assumption on the kernels, the integro-differential system behave like a system of partial differential equations obtained by considering appropriate exponentially decreasing kernels.

1. Introduction. In this paper we consider the following system of partial integro-differential equations

\[
\begin{align*}
    u_t + \epsilon^2 \phi_t &= \int_0^t a_1(t - \tau) \Delta u(\tau) d\tau \\
    \epsilon^2 \phi_t &= \int_0^t a_2(t - \tau) [ \epsilon^2 \Delta \phi + f(\phi) + \epsilon \ u ](\tau) d\tau,
\end{align*}
\]

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in a bounded region \( \Omega \subset \mathbb{R}^2 \) with Dirichlet and Neumann boundary conditions for the temperature field, \( u \), and the order parameter, \( \phi \), respectively and for \( u \) vanishing initially. In (1) \( \epsilon \) is a small parameter, \( \epsilon \ll 1 \), \( f(\phi) \) is a real odd function with a positive maximum equal to \( \phi^* \), a negative minimum equal to \( -\phi^* \) and precisely three zeros in the closed interval \([-a, a]\) located at 0 and \( \pm a \), where \( a \) is a positive constant. The kernels \( a_1 \) and \( a_2 \) are assumed to be piecewise continuous, differentiable at the origin, non-negative, scalar-valued functions on \((0, \infty)\) vanishing at \( \infty \), moreover, they are assumed to be independent of \( \epsilon \) and such that \( \int_0^\infty a_i(t) \, dt < \infty \), \( \int_0^\infty \tilde{a}_i(s) \, ds < \infty \) and \( \int_0^\infty \tilde{a}_i(s) \, s \, ds < \infty \) where \( \tilde{a}_i(s) \) is the inverse Laplace transform of \( a_i(t) \) for \( i = 1, 2 \). System (1) generalizes the classical phase field equations with a non-conserved order parameter

\[
\begin{align*}
 u_t + \epsilon^2 \phi_t &= \Delta u, \\
 \epsilon^2 \phi_t &= \epsilon^2 \Delta \phi + f(\phi) + \epsilon \, u,
\end{align*}
\]

to which it reduces if \( a_1(t) = a_2(t) = \delta(t) \), and describes the process of phase transitions where memory effects are present. For a description of the classical phase field equations see [1]. The first equation in (1) is based on the balance of heat equation for a non-Fourrier process in which the expression for the heat flux is given by a convolution in time between the temperature gradient and the kernel \( a_1 \) [2], [3]. The second equation is a phenomenological equation based on energetic penalization driving the evolution of the system toward equilibrium states. More specifically, the functional derivative of a free energy, \( \delta F_\epsilon(\phi)/\delta \phi \), is considered as a generalized force indicative of the tendency of the free energy to decay towards a minimum. The second equation in (1) is obtained by assuming that the response of \( \phi \) to the tendency of the free energy to decay towards a minimum is given by [4], [5].

\[ \tau \phi_t = -\int_0^t a_2(t - t') \frac{\delta F_\epsilon}{\delta \phi} (t') \, dt', \]

For exponentially decreasing kernels \( a_i(t) = \alpha_i \, e^{-\gamma_i \, t}, i = 1, 2 \), where \( \alpha_i \) and \( \gamma_i \) are non-negative constants, system (1) reduces to

\[
\begin{align*}
 u_{tt} + \epsilon^2 \phi_{tt} + \gamma_1 \, u_t + \epsilon^2 \gamma_1 \, \phi_t &= \alpha \, \Delta u, \\
 \epsilon^2 \phi_{tt} + \epsilon^2 \gamma_2 \, \phi_t &= \epsilon^2 \Delta \phi + f(\phi) + \epsilon \, u,
\end{align*}
\]

(3)
by differentiating both equations, rearranging terms and rescaling by means of the transformation $t \rightarrow \alpha_2^{-\frac{1}{n}} t$, $\gamma_2 \rightarrow \alpha_2^\frac{1}{n} \gamma_2$, $\gamma_1 \rightarrow \alpha_2^\frac{1}{n} \gamma_1$ and $\alpha := \frac{\alpha_1}{\alpha_2}$.

We assume that there exist a solution $\{u(x, t), \phi(x, t)\}$ of (1) defined for all small $\epsilon$, every $x \in \Omega$ and every $t \in [0, T]$ which contains an internal layer. We also assume, for such solutions, that for all small $\epsilon \geq 0$ and all $t \in [0, T]$, the domain $\Omega$ can be divided into two open regions $\Omega_+(t; \epsilon)$ and $\Omega_-(t, \epsilon)$ with a curve $\Gamma(t; \epsilon)$, separating between them. This interface defined by

$$\Gamma(t; \epsilon) := \{x \in \Omega : \phi(x, t; \epsilon) = 0\},$$

is assumed to be smooth, which implies that its curvature and its velocity are bounded independently of $\epsilon$. The function $\phi$ is assumed to vary continuously across the interface, far from the interface tending to 1 when $x \in \Omega_+(t; \epsilon)$, -1 when $x \in \Omega_-(t, \epsilon)$, with rapid spatial variation close to the interface.

In order to describe the evolution of solutions of the type refered to in the last paragraph for systems like either (2) or (3), the evolution of the interface with time has been largely studied by means of formal asymptotic analysis [6], [7], [8]. For (2), fronts evolve according to the flow by mean curvature equation [9], [10],

$$\nabla v = \kappa,$$

where $v$ is the normal velocity of the interface and $\kappa$ its curvature. If $y = S(x, t)$ is the Cartesian description of the interface, equation (5) reads

$$S_t = \frac{S_{xx}}{1 + S_x^2}.$$

This same equation has been obtained before for the Allen-Cahn equation [11],

$$\epsilon^2 \phi_t = \epsilon^2 \Delta \phi + f(\phi).$$

For the Klein-Gordon equation

$$\epsilon^2 \phi_{tt} + \epsilon^2 \gamma_2 \phi_t = \epsilon^2 \Delta \phi + f(\phi)$$
with $\gamma_2 = 0$, Neu [12] showed that the evolution of the interface is governed by the Born-Infeld equation

$$
(7) \quad (1 - S_t^2) S_{xx} + 2 S_x S_t S_{xt} - (1 + S_x^2) S_{tt} = 0.
$$

For (6) with $\gamma_2 > 0$ and for (3) Rotstein et. al. [4], [13] showed that fronts move according to an extended version of the Born-Infeld equation given by

$$
(1 - \alpha_2 S_t^2) S_{xx} + 2 \alpha_2 S_x S_t S_{xt} - \alpha_2 (1 + S_x^2) S_{tt} = 0
$$

(8)

$$
\gamma_2 S_t (1 + S_x^2 - \alpha_2 S_t^2) = 0.
$$

In local (geometric) coordinates equation (8) reads

$$
(9) \quad \frac{v_t}{1 - \alpha_2 v^2} + \gamma_2 v = \kappa.
$$

where $v$, $v_t$, and $\kappa$ are the normal velocity, normal acceleration and curvature of the interface respectively. Equation (8) has been studied in [14].

In this manuscript we show that in the asymptotic limit studied and for the class of kernels considered system (1) is equivalent, to the leading order, to (3); i.e., for (1) the evolution of interfaces is described by either (8) and (9) where, in this case,

$$
\alpha_2 = \left( \int_0^\infty \tilde{\alpha}_2(\tau) \ d\tau \right)^{-1},
$$

where $\tilde{\alpha}_2(\tau)$ is the inverse Laplace transform of $\alpha_2(t)$, and

$$
\gamma_2 = \alpha_2^2 \left( \int_0^\infty \tilde{\alpha}_2(\tau) \tau \ d\tau \right),
$$

2. Asymptotic analysis.

2.1. Assumptions and definitions. For the asymptotic analysis, using local coordinates, we set $x = (x_1, x_2)$ and call $d(x)$ the distance from $x$ to $\Gamma$; i.e., $d(x) = \text{dist}(x, \Gamma)$. We next define a local orthogonal coordinate system $(r, s)$ in a neighborhood of $\Gamma$ in the following way
\text{(10)} \quad r(x, t; \epsilon) = \begin{cases} 
  d(x) & \text{if } \phi(x) > 0 \\
  -d(x) & \text{if } \phi(x) < 0, 
\end{cases}

and \( s(x, t; \epsilon) \), a smooth function of \( t \), such that on \( \Gamma(t; \epsilon) \) it measures arclength from some point which moves normally to \( \Gamma \) as \( t \) varies. The assumed initial smoothness of \( \Gamma(t; \epsilon) \) implies that \( r \) is a smooth function, at least, in a sufficiently small neighborhood of \( \Gamma \).

The outer expansions of \( \phi \) and \( u \) are assumed to have the form

\text{(11)} \quad \phi = \phi(x, t; \epsilon) = \phi^0(x, t) + \epsilon \phi^1(x, t) + \epsilon^2 \phi^2(x, t) + \mathcal{O}(\epsilon^3)

and

\text{(12)} \quad u = u(x, t; \epsilon) = u^0(x, t) + \epsilon u^1(x, t) + \epsilon^2 u^2(x, t) + \mathcal{O}(\epsilon^3).

In order to determine the inner expansions we first define the inner variable

\text{(13)} \quad z(x, t; \epsilon) := \frac{r(x, t; \epsilon)}{\epsilon}

and then assume the inner expansions to be given by

\text{(14)} \quad \phi = \Phi(z, s, t; \epsilon) = \Phi^0(z, s, t) + \epsilon \Phi^1(z, s, t) + \mathcal{O}(\epsilon^2)

and

\text{(15)} \quad u = U(z, s, t; \epsilon) = U^0(z, s, t) + \epsilon U^1(z, s, t) + \mathcal{O}(\epsilon^2).

The very definition of \( \Gamma \) requires \( \Phi(0, s, t; \epsilon) = 0 \). In what follows we will use the following notation to refer to any variable \( g \) evaluated by approaching \( \Gamma \) from either side (\( r > 0 \) or \( r < 0 \)):

\text{(16)} \quad g \big|_{r=\pm} = \lim_{{r \to 0^\pm}} g(r, s, t; \epsilon),
\begin{equation}
g_{r} \mid r= \lim_{r \to 0} g_{r}(r, s, t; \epsilon).
\end{equation}

The following relations between the inner and outer variables obtained in [7] are assumed to hold as \( \rho \to \pm \infty \).

\begin{equation}
G^{0}(\rho, s, t) = g^{0}(0^{\pm}, s, t),
\end{equation}

\begin{equation}
G^{1}(\rho, s, t) = g^{1}(0^{\pm}, s, t) + \rho g^{0}_{r}(0^{\pm}, s, t).
\end{equation}

2.2. Derivation of the equations of motion. Let us look at the system (1) assuming that \( a = 0 \) initially and along the boundary.

2.2.1. Outer problems. Substituting (11) and (12) into (1) and equating coefficients of the corresponding powers of \( \epsilon \) we obtain the \( \mathcal{O}(1) \) and \( \mathcal{O}(\epsilon) \) outer problems respectively for points where the interface has not yet arrived:

\begin{equation}
\begin{aligned}
&u_{1}^{0} = a_{1} \ast \Delta u^{0}, \\
&a_{2} \ast f(\phi^{0}) = 0,
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
&u_{1}^{1} = a_{1} \ast \Delta u^{1}, \\
&a_{2} \ast [f'(\phi^{0}) \phi^{1} + u^{0}] = 0.
\end{aligned}
\end{equation}

The solution of (20), given the assumed initially and boundary conditions for \( u_{i} \) is \( u^{0} \equiv 0, \phi^{0} = \pm 1 \). The solution of (21) is \( u^{1} \equiv 0, \phi^{1} \equiv 0 \).

2.2.2. Inner problems. Let us look at system (1) and write the kernels \( a_{1}(t) \) and \( a_{2}(t) \) as Laplace transforms of suitable functions \( \tilde{a}_{1}(t) \) and \( \tilde{a}_{2}(t) \):

\begin{equation}
\begin{aligned}
a_{i}(t) &= \int_{0}^{\infty} \tilde{a}_{i}(\tau) e^{-\tau t} d\tau.
\end{aligned}
\end{equation}

Substituting (22) into (1), rearranging terms and calling
\[(23)\quad \chi(t; \tau) = \int_0^t e^{-\tau (t-t')} \left( \epsilon^2 \Delta \phi + f(\phi) + \epsilon u \right)(t') \, dt',\]

and

\[(24)\quad v(t; \tau) = \int_0^t e^{-\tau (t-t')} \Delta u(t') \, dt',\]

System (1) becomes, for all \( \tau \in [0, \infty), \)

\[
\begin{aligned}
\frac{d}{dt} u_t + \epsilon^2 \frac{d^2}{dt^2} \int_0^\infty \tilde{\alpha}_1(\tau) v(t; \tau) d\tau,
\frac{d}{dt} v_t + \tau v = \Delta u, \\
\epsilon^2 \frac{d^2}{dt^2} \int_0^\infty \tilde{\alpha}(\tau) \chi(t; \tau) d\tau,
\chi_t(t; \tau) + \tau \chi(t; \tau) = \epsilon^2 \Delta \phi + f(\phi) + \epsilon u,
\end{aligned}
\]

From the solution of the outer problems (20) and (21) and for points where the moving front has not yet arrived, we have

\[(26)\quad \chi^0(t; \tau) = 0, \quad \text{and} \quad v^0(t; \tau) = 0,\]

and

\[(27)\quad \chi^1(t; \tau) = 0, \quad \text{and} \quad v^1(t; \tau) = 0,\]

System (25) expressed in the \((z, s, t)\) coordinates, after differentiating the first and third equations with respect to \(t\) and calling \(v = V(z, s, t; \tau, \epsilon)\) and \(\chi = \Upsilon(z, s, t; \tau, \epsilon)\), becomes

\[
\begin{aligned}
\epsilon r_t^2 U_{zz} + \epsilon \left[ 2 r_t U_{zt} + r_{tt} U_z \right] + \epsilon^2 \left[ U_{tt} + 2 s_t U_{st} + s_t^2 U_{ss} + s_{tt} U_s + 2 r_t s_t U_{zs} + r_t^2 \Phi_{zz} \right] = \\
\quad = \int_0^\infty \tilde{\alpha}_1(\tau) \left[ \epsilon^2 V_t + \epsilon r_t V_z + \epsilon^2 s_t V_s \right] d\tau + \mathcal{O}(\epsilon^3),
\end{aligned}
\]

\[
\begin{aligned}
\epsilon r_t^2 V_z + \epsilon^2 \left[ V_t + s_t V_s + \tau V \right] = U_{zz} + \epsilon \Delta r U_z + \epsilon^2 \left[ U_{ss} [\nabla s] + U_s \Delta s \right],
\end{aligned}
\]

\[
\begin{aligned}
\epsilon r_t^2 \Phi_{zz} + \epsilon^2 \left[ 2 r_t \Phi_{zt} + r_{tt} \Phi_{z} + 2 r_t s_t \Phi_{zs} + r_t^2 \Phi_{zz} \right] = \int_0^\infty \tilde{\alpha}_2(\tau) \left[ \epsilon \Upsilon_t + r_t \Upsilon_z + \epsilon s_t \Upsilon_s \right] d\tau + \mathcal{O}(\epsilon^3),
\end{aligned}
\]

\[
\begin{aligned}
\epsilon \Upsilon_t + \epsilon \left[ \Upsilon_t + s_t \Upsilon_s + \tau \Upsilon \right] = \epsilon \left[ \Phi_{zz} + f(\Phi) \right] + \epsilon^2 \left[ \Delta r \Phi_z + U \right] + \mathcal{O}(\epsilon^3),
\end{aligned}
\]

\[(28)\]
for $\tau \in [0, \infty)$. From (26) and (27) we have
\[\lim_{z \to \infty} \Upsilon^0(x, z, t; \tau) = 0, \quad \text{and} \quad \lim_{z \to \infty} V^0(x, z, t; \tau) = 0,\]
and
\[\lim_{z \to \infty} \Upsilon^1(x, z, t; \tau) = 0, \quad \text{and} \quad \lim_{z \to \infty} V^1(x, z, t; \tau) = 0.\]

Substituting (14) and (15) into (28) and equating coefficients of the corresponding powers of $\epsilon$ we obtain the $\mathcal{O}(1)$, $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ problems respectively.

**$\mathcal{O}(1)$:**
\[
\begin{aligned}
&\quad r_t^2 U_{zz}^0 = 0, \\
&\quad U_{zz}^0 = 0, \\
&\quad \int_0^\infty \tilde{\alpha}_2(\tau) \, r_t \, \Upsilon_z^0 \, d\tau = 0, \\
&\quad r_t \, \Upsilon_z^0 = 0, \\
\end{aligned}
\]

**$\mathcal{O}(\epsilon)$:**
\[
\begin{aligned}
&\quad 2 \, r_t \, U_{zt}^0 + r_t^2 \, U_{zz}^1 \, r_t \, U_z^0 = \int_0^\infty \tilde{\alpha}_1(\tau) \, r_t \, V_z^0 \, d\tau = 0, \\
&\quad r_t \, V_z^0 = U_{zz}^1 + \Delta r \, U_{zz}^0, \\
&\quad r_t^2 \, \Phi_{zz}^0 = \int_0^\infty \tilde{\alpha}_2(\tau) \left[ \Upsilon_z^0 + \int \Upsilon_s^1 + s_1 \, \Upsilon_s^0 \right], \\
&\quad \Upsilon_z^0 + r_t \, \Upsilon_z^1 + s_1 \, \Upsilon_s^0 + \tau \, \Upsilon^0 = \Phi_{zz}^0 + f(\Phi^0),
\end{aligned}
\]

**$\mathcal{O}(\epsilon^2)$:**
\[
\begin{aligned}
&
\begin{cases}
 r_t \ u_{zz}^2 + 2 \ r_t \ u_{zt}^1 + 2 \ r_t \ u_{zt}^1 + U_{tt}^1 + U_{tt}^0 + 2 \ s_t \ u_{st}^0 + s_t^2 \ u_{ss}^0 + s_t \ u_{ss}^0 + 2 \ r_t \ s_t \ u_{zz}^0 \\
+ r_t^2 \ \Phi_{zz}^0 = \int_0^\infty \ \tilde{\alpha}_1(\tau)[ \ V_t^0 + r_t \ V_t^1 + s_t \ V_s^0] \ d\tau,
\end{cases}
\end{aligned}
\]
\begin{aligned}
&
\begin{cases}
 r_t \ V_t^1 + V_s^0 + s_t \ V_s^0 + \tau \ V^0 = U_{zz}^2 + \Delta \ r \ U_{zz}^1 + U_{ss}^0[\nabla s] + U_s^0 \ \Delta s,
\end{cases}
\end{aligned}
\]
\begin{aligned}
&
\begin{cases}
 r_t^2 \ \Phi_{zz}^1 + 2 \ r_t \ \Phi_{zt}^0 + r_{tt} \ \Phi_{zt}^0 + 2 \ r_t \ s_t \ \Phi_{zz}^0 = \int_0^\infty \ \tilde{\alpha}_2(\tau)[ \ \Upsilon_t^1 + r_t \ \Upsilon_z^2 + s_t \ \Upsilon_s^1] \ d\tau,
\end{cases}
\end{aligned}
\]
\begin{aligned}
&
\begin{cases}
 \Upsilon_t^1 + r_t \ \Upsilon_z^2 + s_t \ \Upsilon_s^1 + \tau \ \Upsilon^1 = \Phi_{zz}^1 + f(\Phi^0) \ \Phi^1 + \Delta \ r \ \Phi_{zz}^0 + U_s^0,
\end{cases}
\end{aligned}
\]
\end{equation}
\end{align*}

for $\tau \in [0, \infty)$. We call

\begin{equation}
\alpha_i = \left( \int_0^\infty \ \tilde{\alpha}_i(\tau) \ d\tau \right)^{-1},
\end{equation}

for $i = 1, 2$. A bounded solution of the first two equations in (31) satisfying the matching conditions (18) is $U^1 \equiv 0$. From the third and fourth equations in (31) we have $\Upsilon^0(x, z, t; \tau) = 0$ for $\tau \in [0, \infty)$. Integrating with respect to $z$ (assuming that $r_t \neq 0$) and applying condition (29) yields

\begin{equation}
\Upsilon^0 \equiv 0,
\end{equation}

for $\tau \in [0, \infty)$. Replacing $U^1 \equiv 0$ in the first two equations in (32), multiplying the second and fourth equations in (32) by $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ respectively, and integrating with respect to $\tau$ yields

\begin{equation}
\begin{cases}
 (\alpha_1 - r_t^2) \ U_{zz}^1 = 0,
(1 - \alpha_2 \ r_t^2) \ \Phi_{zz}^0 + f(\Phi^0) = 0,
\end{cases}
\end{equation}

Assuming that $r_t^2 \neq \alpha_1$, the bounded solution of the first equation in (36) satisfying the matching conditions (18) is $U^1 \equiv 0$. To solve the second equation in (36), we assume that $r_t^2 \neq \alpha_2$ and we define the new variable

\begin{equation}
\xi := \frac{z}{(1 - \alpha_2 \ r_t^2)^{\bar{r}}},
\end{equation}

In terms of \((\xi, s, t)\), equation the second equation in (36) reads

\[
(38) \quad \Phi^0_{\xi\xi} + f(\Phi^0) = 0,
\]
whose solution is \(\Phi^0 = \Psi(\xi)\), the unique solution of \(\Psi'' + f(\Psi) = 0\), \(\Psi(\pm \infty) = \pm 1\), \(\psi(0) = 0\). Thus

\[
(39) \quad \Phi^0 = \Phi^0 \left( \frac{z}{(1 - \alpha_2 r_t^2)^{1/2}} \right),
\]
which satisfies (18). From the fourth equation in (32) we have \(r_t \, \gamma^1_z = \alpha_2 r_t^2 \, \Phi^0_{zz}\). Integrating with respect to \(z\) and applying condition (30) we get

\[
(40) \quad \gamma^1 = -\alpha_2 r_t^2 \, \Phi^0_z.
\]
Substituting (39) into the third and fourth equations in (33), multiplying the fourth equation by \(\tilde{\alpha}_2\) and integrating with respect to \(\tau\) yields

\[
(41) \quad (1 - \alpha_2 r_t^2) \, \Phi^1_{zz} + f'(\Phi^0) \, \Phi^1 = (\alpha_2 r_{tt} + \gamma_2 \, \alpha_2 \, r_t - \Delta r) \, \Phi^0_z + 2 \, r_t \, \Phi^0_{zt},
\]
where

\[
(42) \quad \gamma_2 = \alpha_2^2 \left( \int_0^{\infty} \tilde{\alpha}_2(\tau) \, \tau \, d\tau \right).
\]
Equation (41) expressed in the \((\xi, s, t)\) coordinate system reads (see appendix)

\[
\Phi^1_{\xi\xi} + f'(\Phi^0) \, \Phi^1 =
\]

\[
(43) \quad \frac{2 \, r_t^2 \, r_{tt}}{(1 - \alpha_2 r_t^2)^{1/2}} \left( \xi \, \Phi^0_{\xi\xi} + \Phi^0_{\xi} \right) + \frac{r_{tt} + \gamma_2 \, r_t - \Delta r}{(1 - \alpha_2 r_t^2)^{1/2}} \, \Phi^0_{\xi}.
\]
It is straightforward to check that \(\Psi(\xi)\) satisfies the homogeneous equation
\[ \Phi_{\xi \xi} + f'(\Phi^0) \Phi^1 = 0. \] That means that the operator \( \Lambda := \frac{\partial^2}{\partial \xi^2} + f'(\Psi^\prime(\xi)) \) has a simple eigenvalue at the origin with \( \Psi^\prime \) as the corresponding eigenfunction. The solvability condition for the equation (43) now gives

\[
\frac{2 \ r_t^2 \ r_{tt}}{(1 - \alpha_2 \ r_t^2)^2} \int_{-\infty}^{\infty} (\xi \ \Psi'' + \Psi'') \ \Psi' \ d\xi +
\]

(44)

\[
\frac{r_{tt} + \gamma_2 \ r_t - \Delta r}{(1 - \alpha_2 \ r_t^2)^{1/2}} \int_{-\infty}^{\infty} (\Psi')^2 \ d\xi = 0.
\]

A simple calculation shows that \( \int_{-\infty}^{\infty} \xi \Psi' \Psi'' \ d\xi = -\frac{1}{2} \int_{-\infty}^{\infty} (\Psi')^2 \ d\xi \). Hence multiplying equation (44) by \( (1 - \alpha_2 \ r_t^2)^{3/2} \) and rearranging terms one obtains

(45)

\[
\frac{r_{tt}}{1 - \alpha_2 \ r_t^2} + \gamma_2 \ r_t = \kappa.
\]

Taking into consideration that on the interface \( \Delta r = \kappa \), the curvature of the interface, and that \( r_t = -v \), its normal velocity \[8\], equation (45) becomes (9).

### 2.3. Conclusions

In this paper we showed that to the leading order and under suitable assumptions on the kernels \( a_1 \) and \( a_2 \), the law governing the evolution of interfaces for the integro-differential equation (1) is the same as for the differential equation (3). It is easy to see that \( \gamma_2 \) is given by

\[
\gamma_2 = -\frac{a_2'(0)}{(a_2(0))^2}.
\]

To solve the \( \mathcal{O}(\epsilon) \) problem we have assumed that \( |r_t| \neq \sqrt{\alpha_1} \) (\( \alpha_1 \) being similar to \( \alpha_2 \)). On the other hand, from equation (9) we can easily see that \( |r_t| < \frac{1}{\sqrt{\alpha_2}} \). Therefore, if \( \alpha_1 \alpha_2 \geq 1 \) the former assumption does not add further restrictions on the interfacial motion.

The theory described above can be extended to the case where \( \gamma_1 = \alpha = \frac{1}{\xi} \) and \( \alpha_2 = 1 \) in (3), then we obtain the following system of partial differential equations where the first equation can be considered as a hyperbolically perturbed energy balance equation,
\[
\begin{align*}
&\epsilon u_{tt} + \epsilon^3 \phi_{tt} + \epsilon u_t + \epsilon^2 \phi_t = \Delta u, \\
&\epsilon^2 \phi_{tt} + \epsilon^2 \gamma_2 \phi_t = \epsilon^2 \Delta \phi + f(\phi) + \epsilon u,
\end{align*}
\]

Calculations have been made which show that for equation (46) fronts move according to (9) but here there is no restriction on \( r \). This is valid also for (1) with a short memory kernel of type \( a_t(t/\epsilon) \) with \( a_t(0) = \frac{1}{\epsilon} \) [15].

REFERENCES