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From Counting to Measuring

1.1 Natural Numbers

Number[1] is the **name** given to a quantity. Not every aggregate or collection of individuals can have a peculiar name all to itself since then there would be no end to the vocabulary. To confront the enormity of numbers we must resort to some system of compounding for the names of large collections.

English has proper names for all the **natural** numbers from 1 to 10. For 20 it reserves the special name score[2], for 100 the special name hundred[3], and for 1000 the special name thousand[4]. English does not have a special name for ten thousand but it has *million* and *billion*. English does have a special name, half[5], for one[6] part of two[7] equal parts. In quantitative chemistry *mole* is the hefty number $60221367 \cdot 10^{16}$, which is the number of Hydrogen atoms in one gram of this matter.

Numbers above ten are, otherwise, compounds[8]. Thus, $25 = 2 \cdot 10 + 5$ is pronounced twenty five shortly for twenty plus five. A very large number such as 10^{36}, not in common use, is pronounced ten-to-the-thirty-six. A very small number such as 10^{-72}, not in common use, is pronounced ten-to-the-minus-seventy-two. Astronomical measures are awkwardly given in terms of terrestrial distances and we resort for them to units such as light-year, which is is approximately equal to 5.87×10^{12} miles. A *parsec* is equal to 3.26 light-years, approximately.

English has also a variety of appellations for a collection of individuals of unsure magnitude: set, crew, group, throng, bunch, crowd, multitude, host, few, flock, pride, mob, many, much[9], myriad, company, swarm, drove, horde, herd, gang, and also zillion.

Out of spoken names for counting[10] huge quantities, man, in less mathematical times, resorted to obvious similes of immensity: "I will multiply thy seed as the stars of the heaven, and as the sand which is upon the sea shore."— Genesis 22.17. "And Joseph laid up corn as

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1 The letter *b* in *number* is excrescent and we, indeed, have *numeric* not *numberic*.

2 Originally meaning a scratched mark on a tally to separate groups of twenty.

3 Possibly meaning ten pairs of hands.

4 The first part of *thousand* means swollen as it does in *toe, tow, two, to, do, duo, thaw, doze,* and *toss*.

5 Originally meaning *side* as in *help*—to be by the side, *behalf* (be-half), and *calf*.

6 we pronounce *one* as 'wun' but we have *only* and *atone*—to unite.

7 The *w* in *two* is muted but it is sounded in *twenty*.

8 The number *twelve* is the compound two+left or two+elevated, namely above ten. Also, *dozen* is apparently the compound two+ten. *First* is the compound (be)for-is-it, *second* is related to *sectioned*, and *both* is the compound bi-is, bi-it, or by-is.

9 It is not by coincidence that *many, more, mere, most, much, among, mature, measure* and *multi-*, all start with the letter *m*.

10 previously *telling* from which we have the bank *teller*, or dealer, telling banknotes apart.

the sand of the sea, very much, until they left off numbering; for it was without number."—Genesis 41.49.

The ancient but perfectly preserved Hebrew has retained the descriptive meaning of all the number-names from one to ten. Biblical Hebrew calls the number one 'echad'—united; it calls the number two 'shnaim'—different; it calls the number three 'shalosh'—lank, extended; it calls the number four 'arba' —varied; it calls the number seven 'sheva'—plentiful or abundant; it calls the number eight 'shmone'—full, big or fat; it calls the number nine 'tesha'—forceful; and it calls the number ten 'eser'—rich. Possibly because the ancient numbers, at least the large ones were at first imprecise. The only Hebrew number-name that can be tentatively traced to the human hand is five which it calls 'chamesh' possibly meaning the cluster of fingers in the clenched fist, lending credence to the often expressed opinion that the preeminence of the decimal system has to do more with our innate inability to distinguish and correctly name on sight quantities larger than ten, rather than the number of fingers on both our hands. And, in fact, the Hebrew 'chamesh' may be a group in general rather than specifically that of the five fingers on one hand.

Hebrew calls one hundred 'mea' – collected; one thousand 'elef' – superior, elevated, elephant-like; and ten thousand 'revava' – a riven multitude. These names were in spoken and literary use thousands of years before the modern introduction of the arithmetical symbols 1,2,3,4,5,6,7,8,9, which together with the zero, written as 0, make up the present **positional** method for writing any[1] number.

Algebra assigns a single letter of the alphabet to designate an arbitrary number and uses them to expresses innate truthful facts about the combination of quantities, or numbers. Different letters refer to different numbers, with = (two equal and parallel strokes) used to denote equality[2]. We recognize two arithmetical operations: **addition** denoted by +, and **multiplication**, denoted by · or ×, and realize that the arithmetic of numbers obeys the following intuitively correct rules of reckoning:

1. Addition is **commutative**, namely $a + b = b + a$ for any pair of numbers a, and b.
2. Addition is **associative**, namely $(a + b) + c = a + (b + c)$ for any numbers a, b, c.
3. Multiplication is **commutative**, namely $ab = ba$ for any pair of numbers a, and b.
4. Multiplication is **associative**, namely $(ab)c = a(bc)$ for any numbers a, b, c.
5. Multiplication is **distributive** with respect to addition, namely $a(b + c) = ab + ac$.
6. For any number a it is true that $1a = a$.

We may write ab instead of the full $a{\cdot}b$ since algebra, contrary to English, refrains from using more than one letter to denote one number[3] .

1 It is just a variant of *one*.

2 equ-al-ity is a compounded variation of *ego*—, self.

3 Digits, and other keyboard (ASCII) symbols (excepting a hyphen), are commonly not permitted in a registered personal name, which may not include less than two, and more than 25, letters. So, A1 is unacceptable as a legal name, but Aone is; A++ is not, but Aplusplus is; and there is little chance that Aaaa will be permitted, as well as O.K.,and any other silly, frivolous or embarrassing name imposed on an unsaspecting child. A number

To this day the word-number and the symbol-number coexist in separate peaceful lives. The assertion that God is one, 'echad' or unified in Hebrew, is never God is 1; possibly the first but never the 1st. Similarly William Shakespeare's Twelfth Night is never 12th Night and Alexander Dumas' The Three Musketeers is never The 3 Musketeers.

The English *four* is a **cardinal** number that names a quantity of such plurality or of so may items, but *fourth* (probably the compound four-is, four-it, four-at or four-the) is an **ordinal** number referring to a single item assigned to a specific position in an ordered list.

Exercises.

1. Prove that the sum of two odd numbers is an even number.
2. Prove that if m is odd, then m^2 is odd.
3. Prove that if m and n are odd, then $m^2 + n^2$ is even.
3. Let n be any natural number. Is $n(n + 1)$ odd or even?

1.2 The Zero

Zero is formally induced into the family of numbers by defining it to be such that for any number a it is true that $a + 0 = a$, taken under the assumption that it is unique and that it obeys all the rules of arithmetic. It results that zero has this singular property that $a0 = 0$ for any a. Indeed, since $0 + 1 = 1$, and since $a(0 + 1) = a0 + a1$, then $a1 = a0 + a1$, or $a = a0 + a$, implying that $a0 = 0$.

It results that if $ab = 0$, then at least one of the numbers a, b is zero.

1.3 Fractions or Rational Numbers

A lump of meat can be cut into arbitrary portions, a stick can be chopped into arbitrary sections, a string can be marked at irregular intervals, a stone can be fractured into random bricks, and a heap of flour can be scooped by willful fills. All occasioning measures of an optional division of one whole, having lead man, from earliest times, to look upon certain collections or sets of objects as being a unit – again the Hebrew 'echad' – and to speak of portions thereof as a half, a third, a quarter and so on; and these words existed in the spoken and written language long before they were rendered the modern arithmetical form of one-over-two, 1/2, one-over-three, 1/3, etc. Hebrew calls one-half 'chetzi' — a cleave.

For the sake of arithmetical universality mathematics looks now upon fractions as numbers, rational numbers, and it formerly induces them into the family of numbers by granting that they can be arithmetically operated upon in accordance with all rules and actions of arithmetic. Rational number $1/a,\ a \neq 0$, the **multiplicative** inverse of a, is defined by the condition that $a(1/a) = 1$, and it is further agreed that $b/a = b(1/a)$, so that $a/a = 1$ and $a/1 = a(1/1) = a$. Algebraically speaking 2/3=2(1/3) is the contrived solution of $3x = 2$. By the assumption that rationals add and multiply commutatively and associatively, that their multiplication is distributive with respect to addition, and that $1{\cdot}(a/b) = a/b$, it readily results that

$$\frac{1}{a}\cdot\frac{1}{b} = \frac{1}{b}\cdot\frac{1}{a} = \frac{ab}{ab}\cdot\frac{1}{a}\cdot\frac{1}{b} = \frac{1}{ab},$$

may be ornamentally used as a mere tag or logo—999 may be just a brand-name.

that

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{b}+\frac{1}{a}=\frac{b}{b}\cdot\frac{1}{a}+\frac{a}{a}\cdot\frac{1}{b}=\frac{a}{ab}+\frac{b}{ab}=\frac{1}{ab}(a+b)=\frac{a+b}{ab}$$

or more generally

$$\frac{a}{b}+\frac{c}{d}=\frac{c}{d}+\frac{a}{b}=\frac{ad}{bd}+\frac{cb}{db}=\frac{ad+cb}{db}$$

with which the product and the sum of two rational numbers are written as a single rational number.

The fact that $a/a=1$ implies that 40320/362880 is but 1/9, but the existence of an unlimited number of (relative) primes raises the possibility of fractions-rational numbers-with vast numerators and denominators having no common factor. Such fractions may be hard to record, difficult to handle, and be of an exaggerated fractional practical interest. Consider

$$(\frac{18}{17})^{17}=\frac{2185911559738696531968}{827240261886336764177}=$$

$$2.64241437518310962025748571165 9\ldots=2+\frac{6}{10}+\frac{4}{100}+\frac{2}{1000}+\frac{4}{10,000}+\frac{1}{100,000}+\cdots$$

which upon rounding-off is reduced to

$$\frac{2185910000000000000000}{827240000000000000000}=\frac{218591}{82724}=2.642413\ldots$$

deviating from the original value by barely 10^{-6}.

In practice we need often come to terms with the distinction between crude reality and the unlimited theoretical refinement of mathematics. Every analog measuring device has a limit to the fineness of its dial gradation and every digital device can display so many digits and no more.

The decimal representation of a rational number may include an endless string of digits, possibly cyclically repeating.

Theorem: *If the decimal representation of a number is repeating, then the number represented is rational*

Proof. By example. Consider

$$0.63636363\ldots=\frac{63}{100}(1+\frac{1}{10^2}+\frac{1}{10^4}+\frac{1}{10^6}+\ldots).$$

The sum of the endless geometric series in the parentheses is 100/99=1.0101010101... and hence

$$0.63636363\ldots=\frac{63}{100}\frac{100}{99}=\frac{9\cdot 7}{9\cdot 11}=\frac{7}{11}.$$

End of proof. But look at this

$$\frac{1}{19}=0.0\ 526315789473684210\ 526315789473684210\ 526315789473684210\ldots$$

Exercises.

1. Write $a/b \div p/q$ as a rational number.
2. Write $(1/a + 1/b) \div (1/p + 1/q)$ as a rational number.
3. Find the fractional form of $0.\overline{12345}$.
4. Find the fractional form of $0.\overline{100000}$.
5. Write out $(0.\overline{100000})^2$.
6. What is the ultimate value of $0.99999999\ldots$?

1.4 Negative Numbers

To avoid having to say no to the solution of $x + 1 = 0$, mathematics has created another compound number of added meaning, -1, negative[1] one, so that $1 + (-1) = 0$. Actually, it proposes that with every number a we associate a negating **additive** inverse $-a$ so that $a+(-a) = 0$. Granting these new negative numbers the ability to add and multiply according to the rules of the **positive**[2] numbers one readily obtains, for instance, that $1 + (-2) = 1 + 1 + (-1) + (-2) = 2 + (-2) + (-1) = -1$.

From $1{\cdot}(-1) = -1$, $(-1){\cdot}0 = 0$, $-1{\cdot}(1 + (-1)) = 0$ it results that $(-1)(-1) = 1$, then that $-a = (-1){\cdot}a$. Hence, $(-a) \cdot (-b) = (-1)a(-1)b = (-1)(-1)ab = 1 \cdot ab = ab$.

Exercises.

1. Prove that $-0 = 0$, and that $-(-a) = a$.
2. Explain why $-2^2 = -4$ but $(-2)^2 = 4$.

1.5 Geometrical Interpretation of the Negatives

Negative, verbally, and often vaguely, connotes *opposite* (as the opposite of happy is sad; the opposite of love is hate; the opposite of sweet is bitter; the opposite of beautiful is ugly.) Consider a measurable thing, and, say that (positive) number a refers to a straight walk of a steps. The product $1 \cdot a$ may be accordingly given the meaning of "keep facing your direction and walk a steps." In this geometrical context we may consider the number -1 as an *operator* issuing the directive: "reverse your direction" or "turn around 180 degrees." Thus, $-5 = (-1) \cdot 5$ is the instruction "turn around and walk five steps in the opposite direction." In this sense, $(-1) \cdot (-1) = 1$ is a succession of two reversions that returns the walker into facing back his original direction. Thus $-(-5) = (-1)(-1)5$ is the instruction to execute two half turns so as to revert to the original direction and then to walk five steps. Generally speaking: the opposite of the opposite is the original thing.

1.6 Generalization of the negative sense and the invention of the imaginary number

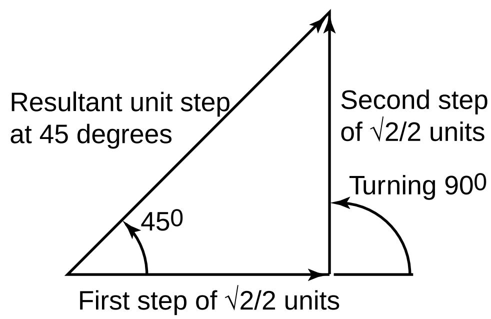
The interpretation of -1 as the operator of turning 180 degrees raises immediately in our mind the possibility of utter generalization, or the invention of a host of other numbers to be interpreted as rotation operators of any given degree. But it is not so. Say we decide

1 To negate essentially means to knock off, as in the give and take of negotiation.

2 That is, possessing the actuality of a thing posed.

to introduce the number i (it may seem unfortunate that a letter is used to denote a number but this is actually the case in mathematics and we have to accept it as such) to mean a counter clockwise rotation of 90 degrees. The arithmetical operation $i5 = 5i$ means, then, turning counter clockwise 90 degrees and walking 5 steps. Obviously, $i \cdot i = i^2 = -1$ being the expression the mere fact that a succession of two 90^0 rotations results in a half turn. Using the numbers 1 (meaning rotation through 0 degrees,) i (meaning rotation through 90 degrees,) and -1 (meaning rotation through 180 degrees,) we can express any arbitrary rotation and no new numbers are needed. The *complex* number $j = \sqrt{2}/2 + \sqrt{2}/2 \cdot i = \sqrt{2}/2(1+i)$, for example, is the operator of turning counter clockwise 45 degrees, or $j \cdot j = j^2 = i$, a fact that we readily verify. Indeed, $[\sqrt{2}/2(1+i)]^2 = 1/2(1+2i-1) = i$. The number i is said to be *imaginary* and is the solution of the quadratic equation $x^2 = -1$.

See the figure below.



Turning 45 degrees and taking one step

Exercises.

1. Write $1/i = a + ib$ and fix a and b.
2. Is it true that $\sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)}$?
3. What is the value of $\sqrt{-1}\sqrt{\frac{1}{-1}}$?

1.7 Irrational numbers

There are rational numbers of no practical existence. Could there be numbers that do not exist rationally even theoretically in spite of the fact that they may have an implicit algebraic description and represent some measurement of concrete nature? One readily encounters such instances. To find the side x of a square field of two area units we need to come up with x so that $x^2 = 2$, which we formally write as $x = \sqrt{2}$, with the $\surd$ being a stylized r standing for *radical.* This x cannot be measured with a yardstick of finite gradation no matter how fine, since no $x = p/q$ exists being the square root of 2.

Theorem: $\sqrt{2}$ *is irrational.*

proof. First we notice that $x = \sqrt{2}$ is not an integer. Then we assume it to be the fraction $x = p/q$ in lowest terms (this can always be done since any number has only a finite number of factors.) Under this assumption $p^2/q = 2q$, which is absurd since the right hand side of the equation is an integer, but the left is not.

Theorem: *if* $n > 1$, $k > 1$ *are integers, and* x *is such that* $x^n = k$, *then* x *is either an integer or irrational.*

proof. If x is not an integer then let it be the fraction $x = p/q$ in lowest terms. Under this assumption $p^n/q = kq^{n-1}$, which is impossible since p^n/q can not equal an integer.

The length of the hypotenuse of a right triangle of unit sides is also $\sqrt{2}$. It is possible thus for us to go into a lumberyard and ask for a board of length $\sqrt{2}$ feet the same way we walk into a bakery and naturally ask for two bagels, or the same way we walk into a grocery and rationally ask for 3/4 pounds of sugar. To satisfy our request the lumberman constructs a right triangle with unit sides and cuts for us its hypotenuse. A problem may now arise as to the price of the piece that is cut from stock that costs, say, $1 per foot. But this problem is resolved as soon as we are informed that the yard's pricing policy is to charge for the next entire footage. To decide the charge the lumberman places his yardstick over the cut piece, determines that its length is more than one foot but less than two feet and we pay for it $2.

The name of the length, or rather the measure of the length, of our piece of lumber is squerootwo, written arithmetically as $\sqrt{2}$.

Theorem: $\sqrt{2}$ *is unique.*

proof. By contradiction. Assume there are two positive roots to $x^2 = 2$ so that ${x_1}^2 = 2$ and ${x_2}^2 = 2$. Subtraction yields ${x_2}^2 - {x_1}^2 = (x_2 - x_1)(x_2 + x_1) = 0$, and since $x_1 > 0$ and $x_2 > 0$ it results that $x_2 - x_1 = 0$ and $x_1 = x_2$. End of proof.

Exercises.

1. Write $\sqrt{5 + 2\sqrt{6}} = \sqrt{m} + \sqrt{n}$ and fix numbers m, n.
2. Prove that $3\sqrt{2}$ is irrational.
3. Prove that $3 + \sqrt{2}$ is irrational.
4. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.
5. Prove that $\sqrt[3]{3} > \sqrt[4]{4} > \sqrt[5]{5} \cdots$.

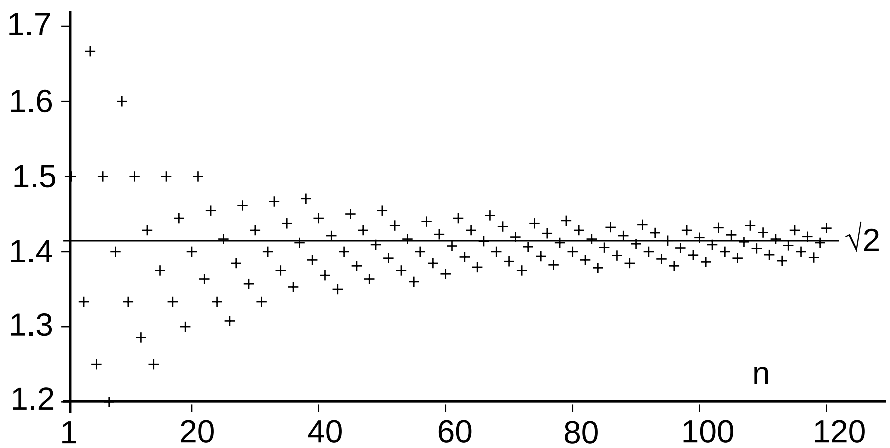
1.8 Rational approximations to $\sqrt{2}$

Even though $\sqrt{2}$ is not rational still rational numbers can be found that are as close to $\sqrt{2}$ as desired or required. A simple routine for constructing such numbers consists of starting with any good rational approximation p/q to $\sqrt{2}$, then adding one to p if $(p/q)^2 < 2$ and adding one to q if $(p/q)^2 > 2$. Starting with 3/2 we obtain the sequence

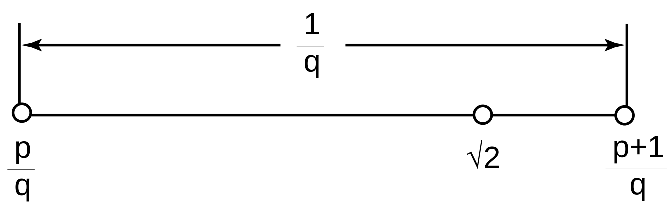
$$\frac{3}{2}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \frac{5}{4}, \frac{6}{4}, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{8}{6}, \frac{9}{6}, \frac{9}{7}, \frac{10}{7} \dots$$

where $(10/7)^2 = 2(1 + 1/49)$.

The figure below shows the thus generated n approximations, each marked as + and elevated proportionally to the value of the approximating rational. As n is increased the computed rational approximations throng ever nearer the horizontal line of height $\sqrt{2}$, albeit sluggishly. Yet we can glean from this long sequence some very good approximations to $\sqrt{2}$ such as 577/408 for which $(577/408)^2 = 2.000006000$. Going up to 4-digit approximations we find $(3363/2378)^2 = 2.000000180$, and $(8119/5741)^2 = 1.999999970$. Among the 5-digit approximations we find $(19601/13860)^2 = 2.000000005$ and $(47321/33461)^2 = 1.999999999$.



To understand the convergence mechanism of this algorithm consider this: Let p/q be the last fraction less then $\sqrt{2}$, namely, such that $p/q < \sqrt{2}$, but $(p+1)/q > \sqrt{2}$. Then, $(p+1)/q - \sqrt{2} < 1/q$, and $\sqrt{2} - p/q < 1/q$; and q is increasing with the step index n. See the figure below.



This procedure for systematically producing ever better rational approximations to $\sqrt{2}$ is utterly simple but slow. We will do better with the **bisection** method that is still simple yet of general interest.

1.7 The method of bisections

In our numerical chase after good rational approximation to the symbolic $\sqrt{2}$ we will be looking for the point where $r(x) = x^2 - 2$ changes sign, actually, as x increases, from minus to plus.

Definition. *Function $f(x)$ is said to change sign at x_0 if positive number ϵ exists such that $f(x) > 0$ (or $f(x) < 0$) whenever $x_0 - \epsilon < x < x_0$, and $f(x) < 0$ (or $f(x) > 0$) whenever $x_0 < x < x_0 + \epsilon$.*

We start with the two rational approximations a_1 and b_1 to $\sqrt{2}$ so that $a_1^2 < 2$ and $b_1^2 > 2$. This guarantees that $a_1 < \sqrt{2} < b_1$. Then we propose to consider the next approximation $x = (a_1 + b_1)/2$, half way between a_1 and b_1. If it turns out that $x^2 > 2$, then now $a_2 < \sqrt{2} < b_2$ with $a_2 = a_1$ and $b_2 = x$. Otherwise, if it happens that $x^2 < 2$ then still $a_2 < \sqrt{2} < b2$ but now with $a_2 = x$ and $b_2 = b_1$ and we proceed to test $x = (a_2 + b_2)/2$ and so on, trapping thus $\sqrt{2}$ in ever narrower straits. After each bisection the interval of certainty for the location of $\sqrt{2}$ is halved. In fact, after n bisections $\sqrt{2}$ is hemmed to the confines of an interval of size $2^{-n+1}(b_1 - a_1)$ which can be reduced, with a sufficiently large n, to any degree of refinement.

For methodical reasons we prefer to look upon the computation of $\sqrt{2}$ as the search for the positive root of $r(x) = x^2 - 2 = 0$. We denote by n the bisection index so that after n bisections $a_n < x < b_n$, a_n being the current left lower bound (or lower estimation) on $\sqrt{2}$, and b_n the current right upper bound (or upper estimation) on $\sqrt{2}$. For $r(x) = x^2 - 2$ it happens that $r(a_n) < 0$ and $r(b_n) > 0$.

Starting with $a_1 = 9/7$ and $b_1 = 10/7$ we establish by successive bisections the following upper and lower estimates for $\sqrt{2}$:

n	$r(a_n) < 0$	a_n $\quad$ b_n	$r(b_n) > 0$	$b_n - a_n$
1	-0.347	$9/7 < x < 10/7$	0.0408	1/7
		next x $= \frac{1}{2}\left(\frac{9}{7} + \frac{10}{7}\right) = \frac{19}{14}$, next r $= -0.158$		
2	-0.158	$19/14 < x < 20/14$	0.0408	1/14
		next $x = \frac{1}{2}\left(\frac{19}{14} + \frac{20}{14}\right) = \frac{39}{28}$, next r $= -0.0599$		
3	-0.0599	$39/28 < x < 40/28$	0.0408	1/28
		next $x = \frac{1}{2}\left(\frac{39}{28} + \frac{40}{28}\right) = \frac{79}{56}$, next r $= -0.00989$		
4	-0.00989	$79/56 < x < 80/56$	0.0408	1/56
		next $x = \frac{1}{2}\left(\frac{79}{56} + \frac{80}{56}\right) = \frac{159}{112}$, next r $= 0.0154$		
5	-0.00989	$158/112 < x < 159/112$	0.0154	1/112
		next $x = \frac{1}{2}\left(\frac{158}{112} + \frac{159}{112}\right) = \frac{317}{224}$, next r $= 0.00273$		
6	-0.00989	$316/224 < x < 317/224$	0.00273	1/224

We see that all these approximations are of the form

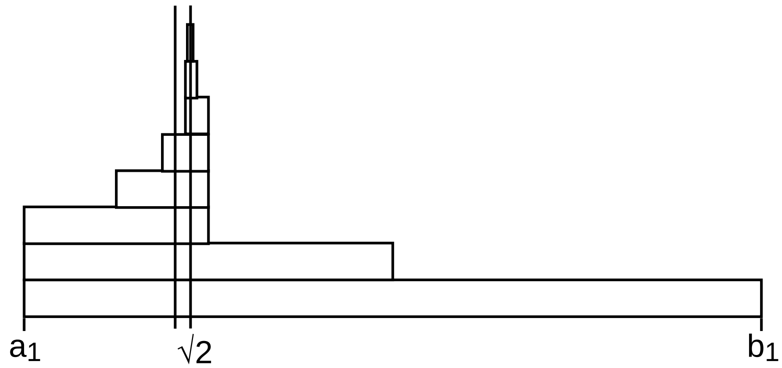
$$-\frac{1}{b} < \sqrt{2} - \frac{a}{b} < \frac{1}{b}.$$

Exercises.

1. Carry out six bisections to improve the lower and upper estimates of the root of $r(x) = x^5 + x - 1 = 0$. Start with $0 < x < 1$.

The figure below is a graphic rendering of the bisection algorithm with the stacked boxes representing the successive halved intervals. The nth interval is contained in all previous

$n-1$ intervals—the intervals are nested, and it is apparent that there is only one number, here $\sqrt{2}$, that is included in all intervals.



Indeed, any other number that differs from $\sqrt{2}$, as the one marked in the above drawing by the other vertical line, is eventually excluded with a sufficiently small spread $b_n - a_n$.

The method of bisection generates two endless, or infinite, **sequences**. One, $a_1, a_2, a_3, \ldots, a_n$ increasing (actually non decreasing), $a_{n+1} \geq a_n$, and the other $b_1, b_2, b_3, \ldots, b_n$, decreasing (actually non increasing), $b_{n+1} \leq b_n$. Both sequences are **convergent**, they both converge to $\sqrt{2}$ as n, the bisection index, is ever increased. Sequence $\{a_n\}_{n=1}^{\infty}$ converges to $\sqrt{2}$ from below, and sequence $\{b_n\}_{n=1}^{\infty}$ from above.

In the language of calculus we say that a_n and b_n **tend** both to $\sqrt{2}$ as n is increased. Symbolically put, $a_n \to \sqrt{2}$ and $b_n \to \sqrt{2}$ as $n \to \infty$ such that $b_n - a_n = (b_1 - a_1)2^{-n+1} \to 0$ as $n \to \infty$. This is also stated by saying that $\sqrt{2}$ is the **limit** of both $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$. In the symbolism of calculus

$$\lim_{n\to\infty} a_n = \sqrt{2},\ \lim_{n\to\infty} b_n = \sqrt{2},\ \lim_{n\to\infty}(b_n - a_n) = 0.$$

All this is summarized and generalized in the following

Lemma (Cantor): *Let $\{a_n\}_{n=1}^{\infty}$ be an increasing (non decreasing) sequence, $a_{n+1} \geq a_n$, and $\{b_n\}_{n=1}^{\infty}$ a decreasing (non increasing) sequence, $b_{n+1} \leq b_n$, with the two sequences such that $b_n > a_n$ for all n, and such that $\lim_{n\to\infty}(b_n - a_n) = 0$. Then a single real number l exists so that $a_n \leq l \leq b_n$ for any natural n.*

Proof. The intervals $b_n - a_n$ are nested (think about a box within a box within a box, forever), and hence if a number is located in the $(n+1)$th interval it is located in the nth interval, and consequently in all previous intervals. Assume there are two numbers (there are actually endlessly many numbers in any nonzero interval) l_1 and l_2, $l_2 > l_1$ that satisfy $a_n \leq l_1 \leq b_n$ and $a_n \leq l_2 \leq b_n$. This implies that $l_2 - l_1 \leq b_n - a_n$ for all natural n. But $b_n - a_n \to 0$ as $n \to \infty$ and $l_1 = l_2$. End of proof.

Realistically speaking, the interval $a_n \leq x \leq b_n$, $b_n > a_n$ contains endlessly many numbers no matter how small $b_n - a_n$ is, but the difference between any two numbers l_1, l_2 picked in the interval may be practically negligible.

But we need to be careful in our arguments. Actually, all we can say with certainty is that the method of bisections traps the number at which $r(x)$ changes sign. The fact that $r(x)$ changes sign at x_0 does not necessarily mean that $r(x_0) = 0$. In fact, if $r(x)$ is discontinuous at x_0, if it experiences a jump at this point, then $r(x)$ changes sign at x_0 but it is not zero there, as seen in the figure below to the left and to the right. We still need to prove, what is numerically suggested in the table, that $r(a_n) \to 0$ and $r(b_n) \to 0$ as $n \to \infty$. We formally show this to be true by writing $r(a_n) = a_n^2 - 2,\ r(b_n) = b_n^2 - 2$ and have by subtraction

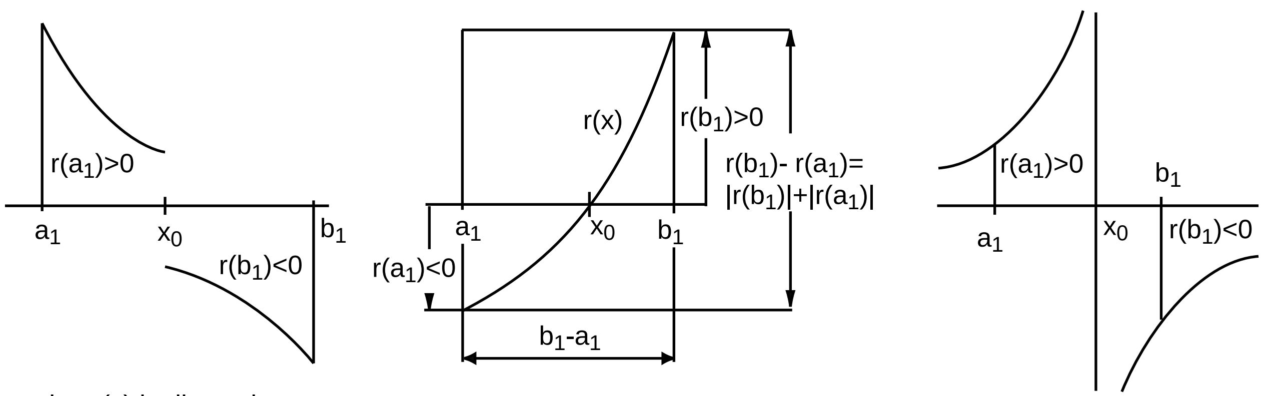
$$r(b_n) - r(a_n) = b_n^2 - a_n^2 = (b_n - a_n)(b_n + a_n).$$

Numbers a_n and b_n are positive and such that $b_n > a_n$. Also $-r(a_n) = |r(a_n)| > 0$, and $r(b_n) = |r(b_n)| > 0$. Hence

$$r(b_n) - r(a_n) = |r(b_n)| + |r(a_n)| < (b_n - a_n)2b_n$$

and since $|r(a_n)| > 0$ and $|r(b_n)| > 0$, both $r(a_n)$ and $r(b_n)$ must tend to zero as $b_n - a_n$ tends to zero.

Now we designate $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ as being $\sqrt{2}$, the irrational number such that $r(\sqrt{2}) = 0$. Thus, from the point of view of bisection we may look upon the existence of $\sqrt{2}$ in the sense of a limit—in the sense of the possibility of constructing rational approximations to the solution of $r(x) = x^2 - 2 = 0$ to any degree of accuracy.



function r(x) is discontinuous at x_0 function r(x) is continuous at x_0 function r(x) is discontinuous at x_0

Theorem: *Every irrational number can be approximated as closely as desired by upper and lower rational bounds. Namely, for any irrational number s and any willfully small real number $0 < \epsilon < \epsilon_0$ there are two rational numbers a and b, $b > a$, that satisfy the conditions $b - a < \epsilon$ and $a < s < b$.*

Proof. Start with the two rational numbers a_1 and b_1, $a_1 < s < b_1$, and continue with the method of bisections to produce after n steps the two numbers a_n and b_n, $a_n < s < b_n$ such that $b_n - a_n = (b_1 - a_1)/2^{n-1}$. End of proof.

To be practical, this algorithm for sharpening the bounds on irrational number s must, of course, include a device for a clear cut decision as to whether a rational number is greater than s or smaller than s.

By the way, it is calculated with certainty that $333/106 < \pi < 355/115$.

A number, rational or irrational, is said to be real. The system of real numbers is said to be a **field**. It is, actually, an **ordered** field by virtue of the fact that between any two real numbers a and b one, and only one, of the conditions $a > b$, namely a is **bigger** than b, $a = b$, namely a is **equal** to b, and $a < b$, namely a is **less** than b, always holds. A number greater than 0 is said to be positive, a number less than 0 is said to be negative. A member of the endless list $1, 2, 3, 4 \ldots$ is called a **natural** number. A number in the endless list $\ldots - 3, -2, -1, 0, 1, 2, 3 \ldots$ is called an **integer**.

For real numbers the basic rules of arithmetic rise from six to nine:

7. There exists a number 0, zero, such that $a + 0 = a$.

8. There exists a number $-a$, the additive inverse of a, such that $a + (-a) = 0$.

9. For any number $a \neq 0$ there exists a number $1/a$, the multiplicative inverse of a, such that $(1/a) \cdot a = 1$. The multiplicative inverse of a is also written as a^{-1}.

The inequality Theorem:

1. *If $a > b$ then for any c, $a \pm c > b \pm c$*
2. *If $a > b$, then for any $c > 0$, $ac > bc$ and $a/c > b/c$*
3. *If $a > b$, then $-a < -b$*
4. *If $a > 0$, $b > 0$ and $a > b$, then $a^2 > b^2$ but $1/a < 1/b$*
5. *If $a > 0$, $b > 0$ and $a > b$, then $\sqrt{a} > \sqrt{b}$*
6. *If $a > 0$, $b > 0$ and $a > b$, then $a^r > b^r$ for any rational $r > 0$*
7. *If $a > 0$, $b > 0$ and $a > b$, then $a^r < b^r$ for any rational $r < 0$*
8. *If $a > b$ and $b > c$, then $a > c$*
9. *If $a > 1$, then $a^2 > a$*
10. *If $0 < a < 1$, then $a^2 < a$*
11. *All these statements remain true if $>$ and $<$ are replaced by $\geq$ and $\leq$ throughout*

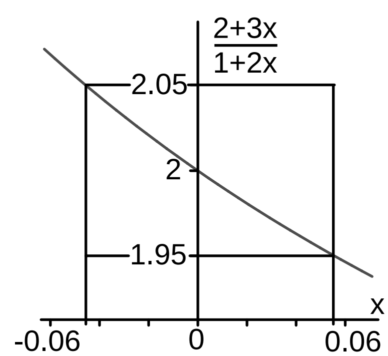
Exercises.

1. Consider the two rational numbers $r_1 = p_1/q_1 > 0$ and $r_2 = p_2/q_2 > 0$, $r_2 > r_1$. Show that $r = (p_1 + p_2)/(q_1 + q_2)$ is such that $r_1 < r < r_2$.

2. What are the bounds on $x, x > -1/2$ if

$$2 - \frac{1}{n} < \frac{2 + 3x}{1 + 2x} < 2 + \frac{1}{n}.$$

Do first the specific case of $n = 20$. See the figure to the right.



The absolute value $|a|$ of of real number a is such that $|a| = a$ if $a \geq 0$, and $|a| = -a$ if $a \leq 0$.

Absolute Value Theorem:

1. *If $a = b$, then $|a| = |b|$*
2. $|a| = |-a|$
3. $|ab| = |a| \cdot |b|$

4. $|a+b| \leq |a| + |b|$

Triangular inequality 4 turns into an equality if a and b are of the same sign.

Exercises.

1. Let a and b be two rational numbers. Prove that $a + \sqrt{2}{\cdot}b$ is irrational. Generalize to $a + r{\cdot}b$ for any irrational r.

2. Show that between any two rational numbers a and b, $b > a$, there is an irrational number. Hint: consider the number $c = a + \sqrt{2}(b-a)/2$.

3. Argue that there is an endless number of irrationals.

4. Prove that between any two real numbers there is a rational number, also an irrational number.

5. Prove that if number a is such that $a > -1$ and also $a \neq 0$, then $(1+a)^n > 1 + na$ for any natural number $n > 1$.

6. Show that if $x > 0$, then $x + 1/x \geq 2$.

7. What is the error in $10^{\sqrt{2}}$ if $\sqrt{2}$ is replaced by 1.4? By 1.414?

8. Suppose that $k > 1$ is a natural number. Prove that, if $a < c < b$ for rational numbers a, b, c, then $k^a < k^c < k^b$.

9. Prove by induction that $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$.