9

Properties of differentiable functions

9.1 When the derivative function exists

When a function is not only continuous on an interval but also differentiable on it, there is much more to say about the behavior the function on this interval.

Theorem: Let \( f(x) \) be differentiable at \( x_0 \). If \( f'(x_0) > 0 \), then the function is locally increasing at \( x_0 \). Namely, a neighborhood of \( x_0 \) exists in which \( f(x_0 + \delta x) > f(x_0) \) if \( \delta x > 0 \), and \( f(x_0 + \delta x) < f(x_0) \) if \( \delta x < 0 \). If \( f'(x_0) < 0 \), then the function is locally decreasing at \( x_0 \). Namely, there is a neighborhood of \( x_0 \) in which \( f(x_0 + \delta x) < f(x_0) \) if \( \delta x > 0 \), and \( f(x_0 + \delta x) > f(x_0) \) if \( \delta x < 0 \).

What this theorem appears to be saying is that near the point of tangency the graph follows the trend of the tangent line. See the figure below.

Think about this theorem this way: a climbing airplane can not reverse direction to come down without continuing briefly to go forward and up. Because the airplane moves in a differentiable trajectory it can not change its flight direction instantaneously. On the other hand, the ability to execute nearly sudden changes of direction is of great advantage in airplanes (as well as nimble birds and insects) that need to carry out sharp and abrupt evasion maneuvers to flee heavier and less agile pursuers. See the figure above to the right.

Proof: We denote, as before, \( \delta y = f(x_0 + \delta x) - f(x_0) \) for \( \delta x = x - x_0 \), and write this \( y \) increment as

\[
\delta y = f'(x_0)\delta x + \left[ \frac{f(x_0 + \delta x) - f(x_0)}{\delta x} - f'(x_0) \right] \delta x
\]

in which

\[
\frac{f(x_0 + \delta x) - f(x_0)}{\delta x} - f'(x_0) \to 0 \quad \text{as} \quad \delta x \to 0.
\]

Hence for a sufficiently small \( \delta x \) there is a wilfull small \( \epsilon > 0 \) such that

\[
\left| \frac{f(x_0 + \delta x) - f(x_0)}{\delta x} - f'(x_0) \right| < \epsilon.
\]

Equivalently

\[
f'(x_0) - \epsilon < \frac{\delta y}{\delta x} < f'(x_0) + \epsilon.
\]
Suppose \( f(x_0) > 0 \). If \( x \) is restricted to the neighborhood of \( x_0 \) where \( f'(x_0) - \epsilon > 0 \), then on this interval \( \delta y/\delta x > 0 \). Suppose \( f(x_0) < 0 \). If \( x \) is restricted to the neighborhood of \( x_0 \) where \( f'(x_0) + \epsilon < 0 \), then on this interval \( \delta y/\delta x < 0 \). End of proof.

This theorem holds true even if

\[
\lim_{\delta x \to 0} \frac{f(x_0 + \delta x) - f(x_0)}{\delta x} = \infty \quad \text{or} \quad \lim_{\delta x \to 0} \frac{f(x_0 + \delta x) - f(x_0)}{\delta x} = -\infty
\]

namely, even in the event of a vertical tangent line at \( x_0 \).

If \( f'(x_0) = 0 \) then it may happen that there is no interval to the right of \( x_0 \) or to its left inside which \( f(x) \) is of one sign. Such is the, notorious, rapidly oscillating function

\[
f(x) = \begin{cases} 
  x^2 \sin \frac{1}{x}, & x \neq 0 \\
  0, & x = 0 
\end{cases}
\]

for which we have

\[
f'(x_0 = 0) = \lim_{h \to 0} \frac{h^2 \sin(1/h) - 0}{h} = 0
\]

meaning that, formally, the function can boast having a tangent line, the horizontal \( x \)-axis, at \( x_0 = 0 \). This function is such that

\[
f(x) = x^2 \text{ if } x = \frac{2}{\pi(4n - 3)}, \quad \text{and} \quad f(x) = -x^2 \text{ if } x = \frac{2}{\pi(4n - 1)}
\]

for any natural \( n > 0 \), and the function vanishes an endless number of times within any interval containing \( x_0 = 0 \).

**Theorem :** Let function \( f(x) \) be differentiable on \([a, b]\), and such that \( f(a) = f(b) = 0 \) and \( f'(a)f'(b) > 0 \). Then point \( a < \xi < b \), exists at which \( f(\xi) = 0 \).

**Proof:** Suppose that \( f'(a) > 0 \) and \( f'(b) > 0 \). Then, according to the previous theorem, \( \epsilon_1 > 0 \) and \( \epsilon_2 > 0 \) exist so that \( f(a + \epsilon_1) > 0 \) and \( f(b - \epsilon_2) < 0 \). Function \( f \) is continuous and by the intermediate value theorem it changes sign at \( \xi \) at which \( f(\xi) = 0 \). End of proof. See the figure below.