MA565 Homework 5 - Due 03/19/20 in class.

Reading: SEK: Ch 2.1-2.24, 2.3-2.4.2, EK: Ch 5.1-5.6, 5.8 (review) and Ch 7.1-7.2.

1) EK:
Ch 5: 5a, 5e, 6a, 6e.
Ch 7: 4, 5, 6 (note, notation for Michaelis Menten variables is different than the class notes, see sections 7.1-7.2)

2) Sontag, ODE6, pg 168, #4 (again note the notation is different from class: \( e_0 = E_T, s_0 = S_0, s = S \) and \( c = C \)).

3) Read through the Michaelis-Menten approximation Julia notebook I posted, which studies the non-dimensionalized Michaelis-Menten equations from class:

\[
\begin{align*}
\frac{dS}{dt} &= -S(1 - C) + \sigma C \\
\frac{dC}{dt} &= \frac{1}{\varepsilon} \left( S(1 - C) - \kappa C \right).
\end{align*}
\]

a. Assume that \( \varepsilon \) is now a large parameter. To understand the behavior of the system as \( \varepsilon \) grows make the substitution that \( \mu = 1/\varepsilon \) and the change of time \( t = \tau/\mu \) for the new non-dimensional time \( \tau \). What equations do \( dS/d\tau \) and \( dC/d\tau \) satisfy?

b. By developing asymptotic expansions, \( S(\tau) = S^{(0)}(\tau) + \mu S^{(1)}(\tau) + \ldots \) and \( C(\tau) = C^{(0)}(\tau) + \mu C^{(1)}(\tau) + \ldots \), find what equations the leading-order quasi-steady state approximation, i.e. \((S^{(0)}(\tau), C^{(0)}(\tau))\), satisfies as \( \mu \to 0 \).

c. Use the Julia notebook and \((S^{(0)}(\tau), C^{(0)}(\tau))\) to sketch the phase-plane for \((S(\tau), C(\tau))\) when \( \varepsilon \) is very big (i.e. \( \mu \) is very small). Make sure to draw the nullclines, direction arrows showing how solutions move as they cross the nullclines, and example solutions. Which nullcline do solutions rapidly approach and then follow? Using your results from the previous part, explain why.

4) Read through the Fitzhugh-Nagumo (FN) model Julia notebook I posted, where the FN equations are

\[
\begin{align*}
\frac{dX}{dt} &= \frac{1}{\varepsilon} \left( -\frac{X^3}{3} + X + Y \right), \\
\frac{dY}{dt} &= -X.
\end{align*}
\]

a. Find all steady-states of the equations, and their stability.

b. Find the nullclines \( dX/dt = 0 \) and \( dY/dt = 0 \), and determine in what directions the solution is changing as it crosses them. Show your work.

c. Sketch the steady-state(s), nullclines and arrows indicating the direction the solution is moving as it crosses the nullclines.
d. Find the leading order behavior of asymptotic expansions for \((X(t), Y(t))\) as \(\varepsilon \to 0\). That is, expand \(X(t) = X^{(0)}(t) + \varepsilon X^{(1)}(t) + \ldots\) and \(Y(t) = Y^{(0)}(t) + \varepsilon Y^{(1)}(t) + \ldots\), and find what equations \((X^{(0)}(t), Y^{(0)}(t))\) satisfy.

e. Using your phase-plane sketch, the graphs in the Julia notebook, and your solution to the previous part, explain why the solution moves into a steady oscillation that follows branches of the \(dX/dt\) nullcline, but then jumps between different branches when reaching a maxima or minima in the \((X, Y)\) plane.