Modeling Second Degree Block

Goal: How do changes in physical properties of the heart "mess up" heart beat timing?

We'll look at two types of 2nd degree block:

1. 2:1 AV Block ⇒ heart skips every other beat
2. Wenckebach Phenomenon ⇒ heart beats normally, skips one beat, and repeats this cycle.

Basic Biology for our model:

Sinoatrial (SA) node is the pacemaker, regularly sending out an electrical signal to the

Atrioventricular (AV) node. It then fires a signal to the rest of the heart to beat if conditions are appropriate.

Key AV node keeps track of state using an electrical potential.
Model

Let \( V_n \) = potential of AV node just after receiving the \( n^{th} \) signal from the SA node.

We assume the signal decays exponentially with rate \( \alpha \)

\[
e^{-\alpha t} V_n = \text{potential} \text{ after the } n^{th} \text{ signal}
\]

We assume the SA node signals every \( \tau \) time units.

With \( V_n \) the potential right after the \( n^{th} \) signal, i.e. at time \( (n\tau)^+ \), the potential right before the \( n^{th} \) signal is then

\[
\hat{V}_n = e^{-\alpha \tau} V_n = c V_n, \quad c = e^{-\alpha \tau}
\]

Notice \( 0 < c < 1 \)!

When the AV node receives the \( n^{th} \) signal, one of two things can occur:

1. If \( c V_n > V_{th} \), where \( V_{th} \) = a threshold voltage
then \( V_{n+1} = cV_n \) (no firing)

2. If \( cV_n < V_{th} \) then \( V_{n+1} = cV_n + u \) (firing)

Here the AV node signals by jumping its potential by \( u \)!

\[
V_{n+1} = \begin{cases} 
  cV_n & cV_n > V_{th} \\
  cV_n + u & cV_n < V_{th} 
\end{cases} = f(V_n)
\]

By sketching \( f \) and using cobwebbing we can get a feel for what happens to \( V_n \) as \( n \to \infty \)

To study equilibria, there are two cases based on the values of \( c, u \)

(recall, the slope \( < 1 \))
So looks like there is one stable SS. Should represent a state where the signal decay balances the response, u!
In a working heart, \( cV_n < V_m \) each step

\[
\Rightarrow V_{n+1} = cV_n + u \Rightarrow \text{guess SS eq that }
\]

\[
\bar{V} = c\bar{V} + u \Rightarrow \bar{V} = \frac{u}{1-c} \quad (\geq 0 \text{ as } 0 < c < 1)
\]

This is only a valid SS if right before a signal the potential is sufficiently low, i.e., \( c\bar{V} < V_{th} \) (\( \Rightarrow \frac{cU}{1-c} < V_{th} \))

\[
\Rightarrow \frac{u}{e^{\alpha t} - 1} \leq V_{th} \quad \text{as } c = e^{-\alpha t}
\]

Notice if \( u \to \infty \) or \( \alpha \to \infty \) or \( \gamma \to \infty \) this is violated.

\( \alpha \) small means signal decays too slowly

\( \gamma \) " " " SA node fires too infrequently

Healthy dynamics \((c\bar{V} < V_{th})\)
Second Degree Block

1. AV node block = skips every other beat

\[ V \]

\[ \text{beats every } a \cdot l \text{!} \]

Period 2 cycle!

How to find when it can occur?

We want a \( \bar{v} \) s.t. \( \bar{v} = f(f(\bar{v})) \) but \( \bar{v} \neq a \cdot \bar{s} \)

\[ \bar{v} \rightarrow c \bar{v} \rightarrow c \bar{v} \rightarrow c^2 \bar{v} \rightarrow c^2 \bar{v} + u \]

Time:

- \((n \cdot \bar{v})^-\)
- \((n \cdot \bar{v})^+\)
- \((n+1) \cdot \bar{v}^-\)
- \((n+1) \cdot \bar{v}^+\)

To possibly get this sequence we need

\[ C \bar{v} > V_{th} \quad (1^{st} \text{ signal time, no response}) \]

\[ C \bar{v} < V_{th} \quad (2^{nd} " " " , \text{ we want a response}) \]

When these hold, \( \bar{v} \) satisfies \( \bar{v} = \frac{C \bar{v} + u}{1 - C^2} \)
Wenckebach Phenomena

For normal beating, we need

\[ c \bar{V} = \frac{cu}{1-c} \leq V_{Th} \quad (\Rightarrow) \quad c \leq \frac{V_{Th}}{V_{Th} + u} \]

What happens if we make \( c \) just a bit bigger than \( \frac{V_{Th}}{V_{Th} + u} \)?

We can get Wenckebach Phenomena!

**Example:** Let \( V_{Th} = 1 \), \( u = 1 \) \( \Rightarrow \) \( c \geq \frac{1}{2} \). So take \( c \) a bit bigger than \( \frac{1}{2} \). \( \Rightarrow \) \( c = e^{-\alpha x} \) \( \Rightarrow \) \( \alpha x = \ln(\frac{1}{2}) \)

If \( \alpha = 1 \) \( \Rightarrow \) \( \alpha = \ln(\frac{1}{2}) \leq \ln(2) \)

\( \frac{1}{c} = 1.99 \) works, as we'll see in the notebook file.

\[ \text{get a cycle!} \]

many steps between repeats!
Note: In real phenomenon heart beats a bit more slowly before the missed beat. (AV node response is no longer instantaneous.)

Summary

Cases at Normal

\[ \tilde{v} = \frac{u}{1-c} \text{ if } \tilde{\nu} = \frac{cu}{1-c} < \nu_{th} \]

(6) 2:1 Block \[ \tilde{v} = \frac{u}{1-c^2} \text{ if } \tilde{\nu} > \nu_{th}, \quad c^2 \tilde{v} < \nu_{th} \]

(6) Outside these we saw Wenckebach Beats, where each beat the value of \( \nu_n \) drifts higher until it is too high to go below \( \nu_{th} \) before the next signal. We therefore end up with a skipped beat!

Conditions on \( c \) that give different cases:

Normal: \[ c < \frac{\nu_{th}}{u+\nu_{th}} \]

2:1 block: \[ c^2 u + \nu_{th} < \nu_{th} \]

\[ \Leftrightarrow \quad c < \frac{\nu_{th}}{(\nu_{th} + u)} \]

and
also requires \[ \frac{Cu}{1-c^2} > V_{Th} \iff C(u+u_{Th}c) > V_{Th} \]

(\iff) \[
\frac{C(u+u_{Th}c)}{u+u_{Th}} > \frac{V_{Th}}{u+u_{Th}}
\]

Notice \[ \frac{u+u_{Th}c}{u+u_{Th}} < 1 \text{ as } c < 1 \]

so it is possible for \[ c > \frac{V_{Th}}{u+u_{Th}} \text{ (un-normal)} \]

but \[ \frac{C(u+u_{Th}c)}{u+u_{Th}} < \frac{V_{Th}}{u+u_{Th}} \]

This should be where Wenkebach Phenomenon arise!