1/31/19  Improved Model for Evolutionary Dynamics

Our previous model left out lots of possible influences on the population dynamics, like:

1. A’s more (or less likely) to divide than B’s. This is called a fitness advantage/disadvantage.

2. A’s (or B’s) can mutate to B^k (or A^j)

3. different chances of death for A’s vs B’s

... called

Let’s look at 1. Random drift with constant selection.

We’ll assume A has fitness r≠1 to reproduce, vs fitness 1 for B. If r>1 A’s will be more likely to divide, if r<1 A^j B’s will be less likely to divide.

If \( A=j \) we encode this by taking the probability an A divides to be \( \frac{r^j}{r^j+N-j} \). The prob a B divides is then:

\[
1 - \frac{r^j}{r^j+N-j} = \frac{N-j}{r^j+N-j}.
\]
As before, the prob an A dies is $\frac{j}{N}$, with $\frac{N-j}{N}$ the prob a B dies.

Then, similar to our previous calculation, we find:

$$T_{j-1,j} = \frac{j}{r_j + N-j} N = \beta_j \quad T_{jj} = 1 - T_{j+1,j} - T_{j-1,j}$$

$$T_{j+1,j} = \frac{r_j}{r_j + N-j} \frac{N-j}{N} = \alpha_j$$

Recall $T_{jk} = \text{prob when } A = k$

of transitioning in one step to $A = j$

The probability $B$ goes extinct given $A(0) = i$, $\Phi_j$, then satisfies:

$$\begin{align*}
\Phi_0 &= 0 \\
\Phi_j &= \beta_j \Phi_{j-1} + (1 - \alpha_j - \beta_j) \Phi_j + \alpha_j \Phi_{j+1} \quad i = j, \ldots, N-1 \\
\Phi_N &= 1 
\end{align*}$$

Notice, $\beta_i = \frac{1}{\alpha_i} \implies$ middle difference eqs can be simplified to

$$0 = \Phi_{i+1} - (1 + \frac{1}{r_i}) \Phi_i + \frac{1}{r_i} \Phi_{i-1},$$

which has the characteristic polynomial
\[ \lambda^2 - \left(1+\frac{1}{\tau}\right)\lambda + \frac{1}{\tau} = 0 \implies \lambda = \frac{\left(1+\frac{1}{\tau}\right) \pm \sqrt{\left(1+\frac{1}{\tau}\right)^2 - 4\frac{1}{\tau}}}{2} \]

\[ \lambda = \frac{\left(1+\frac{1}{\tau}\right) \pm \sqrt{\left(1-\frac{1}{\tau}\right)^2}}{2} = \frac{1}{2} \left[ 1+\frac{1}{\tau} \pm \sqrt{1-\frac{1}{\tau^2}} \right] \]

\[ \lambda_1, \lambda_2 = \left\{1, \frac{1}{\tau}\right\} \implies \Phi_i = A(1^i) + B\left(\frac{1}{\tau}\right)^i = A + B \left(\frac{1}{\tau}\right)^i \]

\[ \Phi_0 = 0 = A + B \implies B = -A \quad \Phi_N = 1 = A \left[1 - \left(\frac{1}{\tau}\right)^N\right] \]

\[ \Phi_i = A \left[1 - \left(\frac{1}{\tau}\right)^i\right] \quad \implies A = \frac{1}{1 - \left(\frac{1}{\tau}\right)^N} \]

\[ \Phi_i = \frac{1 - \frac{1}{\tau^i}}{1 - \frac{1}{\tau^N}} \]

We'll see what this looks like shortly!

Suppose \( A \) has a fitness benefit so \( r > 1 \). We consider two special cases.

1. \( N \gg 1 \implies \frac{1}{\tau} N < 1 \implies \Phi_i \approx 1 - \frac{1}{\tau^i} \)

2. \( i = 1 \), which we can think of as the special case that all \( N \) cells are normal (type \( A \)) and suddenly
a mutant of one appears (can A). Then

$$\bar{\Phi}_1 = \frac{1 - \frac{1}{r}}{1 - \frac{1}{r} N} = \text{the fixation probability of the mutant, i.e. the probability it takes over the population with B going extinct.}$$

as \( N \to \infty \) \( \bar{\Phi}_1 \to 1 - \frac{1}{r} \) Notice, there is always a non-zero probability the mutant does not take over the population! Later we'll see this is different than corresponding ODE models!

See IJulia Notebook for examples and plots of the fixation probabilities.