We present here some results for polynomial approximation theory and stability for polynomial approximation methods on a bounded domain. All results are from [6] unless otherwise noted. Such theory is also possible on unbounded domains via the use of Laguerre and Hermite polynomials (see [4] Section 2.6). In the proofs we use $C$ to represent an arbitrary constant, which may change from line to line.

1 Polynomial Approximation Theory for Smooth Functions

We've now seen how to approximate smooth functions via orthogonal polynomials which are eigenfunctions for a singular Sturm–Liouville problem given by:

$$
\mathcal{L}\phi(x) = -\frac{d}{dx} \left(p(x)\frac{d\phi(x)}{dx}\right) + q(x)\phi(x) = \lambda w(x)\phi(x)
$$

\begin{align*}
\alpha_-\phi(-1) + \beta_-\phi'(-1) &= 0, \alpha_-^2 + \beta_-^2 \neq 0 \\
\alpha_+\phi(1) + \beta_+\phi'(1) &= 0, \alpha_+^2 + \beta_+^2 \neq 0 \\
\alpha_-\beta_- &\leq 0, \alpha_+\beta_+ \geq 0
\end{align*}

We seek now to prove results on the convergence rate of a finite dimensional approximation of order $N$. The rate depends on the regularity of the function being approximated, and in particular, we see if we are approximating $u \in C^\infty$ that the convergence rate is spectral, i.e. faster than any algebraic order of $N$. We focus on ultraspherical polynomials, which are Jacobi polynomials (see [6] Section 4.2) with $\alpha = \beta$ under some normalization. So we have an $n$’th degree ultraspherical polynomial $P_n^{(\alpha)}$ solves Equation 1 with

$$
\lambda_n = n(n + 2\alpha + 1) \\
p(x) = (1 - x^2)^{\alpha+1} \\
q(x) = 0 \\
w(x) = (1 - x^2)^\alpha.
$$

In particular, the most widely used polynomial approximations, the Legendre polynomials ($\alpha = \beta = 0$) and the Chebyshev polynomials ($\alpha = \beta = -1/2$) fall into this category.

The continuous expansion

We seek to estimate the distance between $u$ and its spectral expansion

$$
\mathcal{P}_N u(x) = \sum_{n=0}^{N} \hat{u}_n P_n^{(\alpha)}(x), \quad \hat{u}_n = \frac{1}{\gamma_n} (u, P_n^{(\alpha)})_w,
$$

as measured in the weighted Sobolev norm $\|u\|_{H^m_w[-1,1]}$. Recall $w(x)$ is the weight under which $\{P_n^{(\alpha)}(x)\}_{n \in \mathbb{N}}$ is orthogonal, $(u,v)_w := \int_{-1}^{1} u(x)v(x)w(x)dx$, and $\gamma_n := (P_n^{(\alpha)}, P_n^{(\alpha)})_w$. In particular

$$
\gamma_n = \begin{cases} 
\frac{\pi}{2n+1} & \text{for Legendre polynomials} \\
\frac{\pi}{2\tau_n} & \text{for Chebyshev polynomials.}
\end{cases}
$$

We also have that $\|u\|_{H^m_w[-1,1]} = \left(\sum_{i=1}^{m} \left\|u^{(i)}\right\|_{L^2_w}^2\right)^{1/2} = \left(\sum_{i=1}^{m} (u^{(i)}, u^{(i)})_w\right)^{1/2}$. 

1
Theorem 1.1. For any \( u \in H^p_w[-1,1] \), \( p \geq 0 \), there exists \( C \) a constant independent of \( N \) such that
\[
\|u - P_N u\|_{L^2_w[-1,1]} \leq CN^{-p} \|u\|_{H^p_w[-1,1]}.
\]

Proof. By Parseval’s identity,
\[
\|u - P_N u\|_{L^2_w[-1,1]}^2 \leq \sum_{n=N+1}^{\infty} \gamma_n |\hat{u}_n|^2. \tag{2}
\]

Noting that \( P_n^{(\alpha)} \) satisfies the Sturm-Liouville equation
\[
[Q + \lambda_n] p_n^{(\alpha)} = 0
\]
where
\[
Q u = (1 - x^2) \frac{d^2 u}{dx^2} - 2x(1 + \alpha) \frac{du}{dx},
\]
we get
\[
\hat{u}_n = \frac{1}{\gamma_n} \int_{-1}^{1} u(x) P_n^{(\alpha)}(x) w(x) dx \\
= \frac{1}{\gamma_n \lambda_n} \int_{-1}^{1} u(x) \lambda_n P_n^{(\alpha)}(x) w(x) dx \\
= -\frac{1}{\gamma_n \lambda_n} \int_{-1}^{1} u(x) Q P_n^{(\alpha)}(x) w(x) dx \\
= -\frac{1}{\gamma_n \lambda_n} \int_{-1}^{1} Qu(x) P_n^{(\alpha)}(x) w(x) dx
\]
by multiple uses of integration by parts and that \((1 - x^2)\) and \( w(x) \) vanish on the boundary. Doing this repeatedly yields
\[
\hat{u}_n = (-1)^m \frac{1}{\gamma_n \lambda_n^m} \int_{-1}^{1} [Q^m u(x)] P_n^{(\alpha)}(x) w(x) dx.
\]
Then by Holder’s inequality,
\[
|\hat{u}_n|^2 = \frac{1}{\gamma_n^2 \lambda_n^{2m}} \left( \int_{-1}^{1} |Q^m u(x)| P_n^{(\alpha)}(x) w(x) dx \right)^2 \\
\leq \frac{1}{\gamma_n^2 \lambda_n^{2m}} \left( \int_{-1}^{1} |P_n^{(\alpha)}(x)|^2 w(x) dx \right) \left( \int_{-1}^{1} |Q^m u(x)|^2 w(x) dx \right) \\
= \frac{1}{\gamma_n^2 \lambda_n^{2m}} \|Q^m u\|_{L^2_w[-1,1]}^2. \tag{3}
\]

Since \(|x| \leq 1\), we have
\[
\|Qu\|_{L^2_w[-1,1]} \leq C \|u\|_{H^p_w[-1,1]}.
\]
To see this,
\[
\|Qu\|_{L^2_{[-1,1]}}^2 \leq \sup_{x \in [-1,1]} |1 - x^2| \int_{-1}^{1} \left( \frac{d^2 u(x)}{dx^2} \right)^2 w(x) dx + (1 + \alpha) \sup_{x \in [-1,1]} |2x| \int_{-1}^{1} \left( \frac{du(x)}{dx} \right)^2 w(x) dx
\]
\[
\leq \int_{-1}^{1} \left( \frac{d^2 u(x)}{dx^2} \right)^2 w(x) dx + 2(1 + \alpha) \int_{-1}^{1} \left( \frac{du(x)}{dx} \right)^2 w(x) dx
\]
\[
\leq C \left[ \int_{-1}^{1} \left( \frac{d^2 u(x)}{dx^2} \right)^2 w(x) dx + \int_{-1}^{1} \left( \frac{du(x)}{dx} \right)^2 w(x) dx + \int_{-1}^{1} u(x)^2 w(x) dx \right]
\]
\[
= C\|u\|_{H^2_{[-1,1]}}^2
\]

Then inductively,
\[
\|Qu\|_{L^2_{[-1,1]}} \leq C\|u\|_{H^m_{[-1,1]}}^2. \tag{4}
\]

Combining the bounds from Equations 2, 3, and 4, we get
\[
\|u - P_N u\|_{L^2_{[-1,1]}}^2 \leq C\|u\|_{H^m_{[-1,1]}}^2 \sum_{n=N+1}^{\infty} \lambda_n^{-2m} \leq CN^{-4m}\|u\|_{H^m_{[-1,1]}}^2.
\]

Where for the last inequality we used the form of \(\lambda_n\). Taking \(p = 2m\), we are done. It is okay to prove only for even \(p\) due to the tower property of Sobolev spaces, meaning for \(k \leq j\), \(H^j \subset H^k\). This follows via Holder inequality, or alternatively, by the Sobolev embedding theorem.

Now we seek to show that the error decays spectrally not only in the \(L^2\) norm, but in the corresponding weighted Sobolev norm. First we prove the following theorem.

**Theorem 1.2.** Let \(u \in H^p_w[-1,1]\). Then there exists \(C\) independent of \(N\) such that
\[
\left\| \mathcal{P}_N \frac{du}{dx} - \frac{d}{dx} \mathcal{P}_N u \right\|_{H^q_{[-1,1]}} \leq CN^{2q-p+3/2}\|u\|^p_w[-1,1],
\]
where \(1 \leq q \leq p\).

**Proof.** WLOG, we consider the case where \(N\) is even. We first note that
\[
\frac{d}{dx} \mathcal{P}_N u(x) = \sum_{n=0}^{N} \tilde{u}_n(P_n^{(\alpha)}(x))'
\]
\[
\mathcal{P}_N \frac{d}{dx} u(x) = \mathcal{P}_N \sum_{n=0}^{\infty} \tilde{u}_n(P_n^{(\alpha)}(x))' = \sum_{n=0}^{N+1} \tilde{u}_n(P_n^{(\alpha)}(x))'
\]

since taking the derivative moves the degree of the polynomial down 1. Then we have
\[
\mathcal{P}_N \frac{du}{dx} - \frac{d}{dx} \mathcal{P}_N u = \hat{u}_{N+1}(P_{N+1}^{(\alpha)})'.
\]

By Theorem 1.1, we have
\[
|\hat{u}_{N+1}| \leq \|u - \mathcal{P}_N u\|_{L^2_{[-1,1]}} \leq CN^{-p}\|u\|_{H^p_{[-1,1]}}. \tag{5}
\]
Now we observe the form of $\gamma_k$:

$$\gamma_k = \frac{2^{2\alpha+1}}{(2k+2\alpha+1)k!} \frac{[\Gamma(k+\alpha+1)]^2}{\Gamma(k+2\alpha+1)} \quad \text{(Equation 4.10 in [6]).}$$

We can apply Stirling's formula,

$$\Gamma(t) = \sqrt{2\pi e^{-t} t^{t-1/2}} (1 + O(1/t)), \ t \to +\infty$$

to see

$$\frac{C}{k+1/2} \leq \gamma_k \leq \frac{C'}{k+1/2} \quad \text{(Equation 19.9 in [1]).}$$

So $\gamma_k$ is bounded in $k$, and using the orthogonality of $P_k^{(\alpha)}$ and the same computation as in Theorem 5.1 in [6], we arrive at

$$\left\| \left( P_N^{(\alpha)} \right)' \right\|_{L_2^2[-1,1]}^2 = \sum_{k=0}^{N-1} \frac{(2k+2\alpha+1)P_k^{(\alpha)} }{L_2^2[-1,1]} \leq \sum_{k=0}^{N-1} (2k+2\alpha+1) \left\| P_k^{(\alpha)} \right\|_{L_2^2[-1,1]}^2$$

$$= \sum_{k=0}^{N-1} (2k+2\alpha+1) \gamma_k \leq CN^3$$

Combining this with 5 yields

$$\left\| \mathcal{P}_N \frac{du}{dx} - \mathcal{P}_N u \right\|_{L_2^2[-1,1]} \leq CN^{\frac{3}{2} - p} \left\| u \right\|_{H^p([-1,1])}.$$

By the Strum-Liouville equations and Poincare inequality, we get

$$\left| \frac{d^m}{dx^m} P_n^{(\alpha)}(x) \right| \leq CN^2 m |P_n^{(\alpha)}(x)|,$$

for that further differentiation leads to a loss of $N^2$ on the error bound. Thus we obtain

$$\left\| \mathcal{P}_N \frac{du}{dx} - \mathcal{P}_N u \right\|_{H^q([-1,1])} \leq CN^{2q + \frac{3}{2} - p} \left\| u \right\|_{H^p([-1,1])}.$$

We can further obtain the following (see [4]):

**Theorem 1.3.** For any $u \in H^p([-1,1])$, there exists $C$ independent of $N$ such that

$$\left\| u - \mathcal{P}_N u \right\|_{H^q([-1,1])} \leq CN^{\sigma(q,p)} \left\| u \right\|_{H^p([-1,1])}$$

where $0 \leq q \leq 1$

$$\sigma(q,p) = \begin{cases} \frac{3}{2} q - p, & 0 \leq q \leq 1 \\ 2q - p - \frac{1}{2}, & q \geq 1 \end{cases}$$

We note that with the Fourier method we only lost a factor of $N$ in the error estimate when taking derivatives, while here we found with polynomial expansions we lose a factor of $N^2$. 


The discrete expansion

Now we obtain estimates for the discrete expansion, in which we use a quadrature formulas to evaluate the integrals used in evaluating the expansion coefficients. Hence, there is an aliasing error for the approximation of the integrals introduced in the proofs for the following results, making them a bit more complex than in the case of the continuous expansion. For that reason, the comparable results to the continuous case are presented, but the proofs are omitted.

Our approximation here is given by

$$
\mathcal{I}_N u(x) = \sum_{n=0}^{N} \tilde{u}_n P_n^{(\alpha)}(x) = \sum_{j=0}^{N} u(x_j) l_j(x),
$$

where

$$
\tilde{u}_n(x) = \frac{1}{\tilde{\gamma}_n} \sum_{j=0}^{N} u(x_j) P_n^{(\alpha)}(x_j) w_j.
$$

$\tilde{\gamma}_n$ is given for Legendre polynomials by

$$
\tilde{\gamma}_n = \begin{cases}
\frac{2}{2n+1}, & n < N \\
\frac{2}{2n+1}, & n = N \text{ for Gauss and Gauss-Radau quadrature} \\
\frac{2}{2n+1}, & n = N \text{ for Gauss-Lobatto quadrature}
\end{cases}
$$

and for Chebyshev polynomials by

$$
\tilde{\gamma}_n = \begin{cases}
\pi, & n = 0 \\
\frac{\pi}{2}, & 0 < n < N \\
\frac{\pi}{2}, & n = N \text{ for Gauss and Gauss-Radau quadrature} \\
\pi, & n = N \text{ for Gauss-Lobatto quadrature}
\end{cases}
$$

The quadrature points $x_j, j = 1, ..., N$ are given by

$$
\frac{d}{dx} P_N^{(\alpha)}(x_j) = 0 \text{ for } j = 1, ..., N-1
$$

$$
x_0 = -1
$$

$$
x_N = 1,
$$

and the interpolation polynomials $l_j$ and weights $w_j$ are given for each of the 3 quadrature rules in Section 5.4 of [6].

We get that under the assumption of sufficient smoothness (it suffices for $u \in H^1_{w}[-1,1]$), the aliasing error is

$$
\hat{u}_n = \tilde{u}_n + \frac{1}{\tilde{\gamma}_n} \sum_{k>N}^{\infty} [P_n^{(\alpha)}, P_k^{(\alpha)}]_w \hat{u}_k
$$

where $\hat{u}_n$ is the coefficient from the continuous expansion and the discrete inner product $[u, v]_w := \sum_{j=0}^{N} u(x_j) v(x_j) w_j$.

This follows from inserting $u(x) = \sum_{n=0}^{\infty} \tilde{u}_n P_n^{(\alpha)}(x)$ into the expression for $\hat{u}$ and using that the quadrature
Theorem 1.6. There exists $C$ such that for the discrete Legendre expansion, we have

$$
\|u - \mathcal{I}_N u\|_{L^2[-1,1]}^2 = \|u - \mathcal{P}_N u\|_{L^2[-1,1]}^2 + \|A_N u\|_{L^2[-1,1]}^2
$$

where the aliasing error is

$$A_N u(x) = \sum_{n=0}^N \frac{1}{\gamma_n} \left( \sum_{k>N} [P_n^{(\alpha)}, P_k^{(\alpha)}] w \tilde{u}_k \right) P_n^{(\alpha)}(x).$$

Interchanging the summations we get

$$A_N u(x) = \sum_{k>N} (\mathcal{I}_N P_k^{(\alpha)}) \tilde{u}_k,$$

so the aliasing error can be thought of the error introduced by using the interpolation of the basis, $\mathcal{I}_N P_k^{(\alpha)}$ to represent the higher modes.

The following theorem holds for Gauss and Gauss-Lobatto quadrature rules. It is the discrete analogue of Theorem 1.1.

**Theorem 1.1.** We specialize to the Legendre and Chebyshev expansions.

**Theorem 1.4.** For $u \in H^p_w[-1,1]$, where $p > \max\{1,1+\alpha\}$, there exists $C$ depending on $\alpha$ and $p$ but not $N$ such that

$$
\|u - \mathcal{I}_N u\|_{L^2[-1,1]}^2 \leq C N^{-p} \|u\|_{H^p[-1,1]},
$$

where $\mathcal{I}_N u$ is constructed using ultraspherical polynomials $P_n^{(\alpha)}$ with $|\alpha| \leq 1$.

For higher derivative error estimates, we specialize to the Legendre and Chebyshev expansions.

**Theorem 1.5.** For the discrete Legendre expansion, we have for $u \in H^p[-1,1]$ with $p > 1/2$ and $0 \leq q \leq p$, there exists $C$ independent of $N$ such that

$$
\|u - \mathcal{I}_N u\|_{H^q[-1,1]}^2 \leq C N^{2q-p+1/2} \|u\|_{H^p[-1,1]},
$$

**Theorem 1.6.** For the discrete Chebyshev expansion, we have for $u \in H^p_w[-1,1]$ with $p > 1/2$ and $0 \leq q \leq p$, there exists $C$ independent of $N$ such that

$$
\|u - \mathcal{I}_N u\|_{H^q[-1,1]}^2 \leq C N^{2q-p} \|u\|_{H^p_w[-1,1]}.
$$

We thus see that, as compared to the convergence rate established by Theorem 1.2 for the continuous expansion, we lose a factor $N$ in the discrete convergence rate for Legendre expansion, but only lose $N^{1/2}$ for the Chebyshev expansion. We note, however, that in the $L^\infty$ error for the discrete Chebyshev expansion, we lose another factor of $N^{1/2}$. This result is summarized by the following:

**Theorem 1.7.** For the discrete Chebyshev expansion, we have for $u \in H^p_w[-1,1]$ with $p > 1/2$, there exists $C$ independent of $N$ such that

$$
\|u - \mathcal{I}_N u\|_{L^\infty[-1,1]}^2 \leq C N^{1/2-p} \|u\|_{H^p_w[-1,1]}.
$$

For more details, see [2] and [1].
2 Stability of Galerkin and Collocation Polynomial Spectral Methods

Reminder: We are interested in proving stability of approximations because of the following theorem.

**Theorem 2.1. (Lax–Richtmyer equivalence theorem):** A consistent approximation to a linear wellposed partial differential equation is convergent if and only if it is stable.

**Proof.** See Chapter 4 of [7] or Hannah’s notes. \qed

The Galerkin approach

For some problems, it is convenient to make use of the fact that for polynomial methods, the equation

\[ u_t = Lu \]  \hspace{1cm} (6)

is stable as long as \( L \) is semi-bounded in the norm under which the underlying approximating basis is orthogonal.

**Theorem 2.2.** If \( L \) satisfies 

\[ L + L^* \leq \gamma I, \]

meaning \( ((L + L^*)u, u)_w \leq \gamma(u, u)_w \) for fixed \( \gamma \in \mathbb{R} \) and all \( u \) in the domain of the operator, then the Galerkin method is stable.

**Proof.** Letting

\[ B_N = \text{span } \left\{ \phi_n \in \text{span } \{P^{(n)}_k\}_{k=0}^n : \phi_n(x) \text{ satisfies the B.C. for Equation } 6 \right\} \]

we seek a solution \( u_N \in B_N \) such that the residual is orthogonal to \( B_N \) under the relevant weight function \( w(x) \). Thus we have

\[ \left( \frac{\partial u_N}{\partial t}, u_N \right)_w = 0 \]

so

\[ \left( \frac{\partial u_N}{\partial t}, u_N \right)_w = (Lu_N, u_N)_w. \]

We have

\[ \left( \frac{\partial u_N}{\partial t}, u_N \right)_w = \frac{1}{2} \frac{d}{dt} (u_N, u_N)_w. \]

and

\[ (Lu_N, u_N)_w = \frac{1}{2} [(Lu_N, u_N)_w + (u_N, Lu_N)_w] = \frac{1}{2} [(Lu_N, u_N)_w + (L^* u_N, Lu_N)_w] = \frac{1}{2} ((L + L^*)u_N, u_N)_w \leq \gamma (u_N, u_N)_w \]

so

\[ \frac{d}{dt} (u_N, u_N)_w \leq \gamma (u_N, u_N)_w \]

and by Gronwall’s inequality,

\[ (u_N, u_N)_w \leq e^{\gamma t} (u_N(0), u_N(0))_w. \] \qed
Now let's consider a few examples of stability of certain PDEs under different classes of polynomial methods:

**Example 2.3.** The heat equation

\[ u_t = \mathcal{L} u \]
\[ u(\pm 1, t) = 0 \]

where \( \mathcal{L} = \frac{\partial^2}{\partial x^2} \) is stable under ultraspherical Galerkin methods with \(-1/2 \leq \alpha \leq 1\).

**Proof.** To prove this, we prove the semi-boundedness of \( \mathcal{L} \) under the weight \( w(x) = (1 - x^2)^\alpha \). In this cases, \( \mathcal{L} = \mathcal{L}^* \), and as we will see \( \mathcal{L} + \mathcal{L}^* \leq 0 \).

First we look at values \(-1/2 \leq \alpha \leq 0\) (in particular Legendre and Chebyshev methods both fall under this regime). We note that

\[
(\mathcal{L}u_N, u_N)_w = \int_{-1}^{1} w uu_{xx} dx = w(x) u(x) u_{x}(x)|_{x=-1}^{1} - \int_{-1}^{1} (wu)_{x} u_{x} dx = - \int_{-1}^{1} (wu)_{x} u_{x} dx.
\]

Letting \( v = wu \), we have

\[
u_x = \frac{(wu)_{x}}{w} - \frac{w_x u}{w} = \frac{v_{x}}{w} + v \left( \frac{1}{w} \right)_{x},
\]

so

\[
(\mathcal{L}u_N, u_N)_w = - \int_{-1}^{1} \left( \frac{v_{x}^{2}}{w} \right) dx - \int_{-1}^{1} v_{x} \left( \frac{1}{w} \right)_{x} dx
\]

\[
= - \int_{-1}^{1} \left( \frac{v_{x}^{2}}{w} \right) dx - \frac{1}{2} \int_{-1}^{1} v^{2}(x) \left( \frac{1}{w} \right)_{x} dx
\]

\[
= - \int_{-1}^{1} \left( \frac{v_{x}^{2}}{w} \right) dx - v^{2}(x) \left( \frac{1}{w} \right)_{x} |_{x=-1}^{1} + \frac{1}{2} \int_{-1}^{1} v^{2} \left( \frac{1}{w} \right)_{xx} dx
\]

\[
= - \int_{-1}^{1} \left( \frac{v_{x}^{2}}{w} \right) dx + \frac{1}{2} \int_{-1}^{1} v^{2} \left( \frac{1}{w} \right)_{xx} dx.
\]

Now observing that

\[
\left( \frac{1}{w} \right)_{x} = \alpha(1 - x^2)^{-\alpha-2}(x^2(4\alpha + 2) + 2) \leq 0
\]

we have \( (\mathcal{L}u_N, u_N)_w \leq 0 \).

Now we look at the case where \( 0 \leq \alpha \leq 1 \). We have

\[
(\mathcal{L}u_N, u_N)_w = \int_{-1}^{1} w uu_{xx} dx
\]

\[
= - \int_{-1}^{1} (wu)_{x} u_{x} dx
\]

\[
= - \int_{-1}^{1} w_x u u_{x} dx - \int_{-1}^{1} w(u_{x})^2 dx
\]

\[
= - \frac{1}{2} \int_{-1}^{1} w_x (u^2)_{x} dx - \int_{-1}^{1} w(u_{x})^2 dx
\]

\[
= - \frac{1}{2} u^2(x) w_{x}(x)|_{x=-1}^{1} + \frac{1}{2} \int_{-1}^{1} u^2 w_{xx} dx - \int_{-1}^{1} w(u_{x})^2 dx
\]

\[
= \frac{1}{2} \int_{-1}^{1} u^2 w_{xx} dx - \int_{-1}^{1} w(u_{x})^2 dx.
\]
Seeing that
\[ w_{xx} = \alpha(1 - x^2)^{\alpha-2}(x^2(4\alpha - 2) - 2) \leq 0, \]
we thus have \((\mathcal{L}u_N, u_N)_w \leq 0\).

Now we consider a hyperbolic example.

Example 2.4. Consider the Legendre Galerkin approximation to the linear hyperbolic problem
\[
\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \\
u(1, t) = 0 \\
u(x, 0) = g(x).
\]
Then the approximation is stable under the weighted \(L^2\) norm corresponding to \(w(x) = 1\).

Proof. We have
\[
\left( \frac{\partial u_N}{\partial t}, u_N \right)_w = \left( u_N, \frac{\partial u_N}{\partial x} \right)_w.
\]
Then noting
\[
\left( \frac{\partial u_N}{\partial t}, u_N \right)_w = \frac{1}{2} \frac{d}{dt} \left( u_N, u_N \right)_w
\]
and
\[
\left( u_N, \frac{\partial u_N}{\partial x} \right)_w = \int_{-1}^{1} u_N(x) \frac{\partial u_N}{\partial x}(x) dx = u_N^2(x) \bigg|_{x=-1}^{1} - \int_{-1}^{1} u_N(x) \frac{\partial u_N}{\partial x}(x) dx = -[u_N(-1)]^2 - \int_{-1}^{1} u_N(x) \frac{\partial u_N}{\partial x}(x) dx
\]
we get
\[
\frac{d}{dt} \left( u_N, u_N \right)_w = -[u_N(-1)]^2 \leq 0
\]
so \((u_N, u_N)_w \leq (u_N(0), u_N(0))_w\) and we are done.

In Example 2.5, we extend the result to a more general set of Jacobi Polynomials.

Example 2.5. Consider the Jacobi Galerkin approximation to the linear hyperbolic problem
\[
\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \\
u(1, t) = 0 \\
u(x, 0) = g(x).
\]
Then the approximation is stable under the weighted \(L^2\) norm corresponding to \(w(x) = (1 + x)^\alpha(1 - x)^\beta\) with \(\alpha > 0\) and \(\beta < 0\).
Proof. We have
\[
\frac{d}{dt}(u_N, u_N) = 2(u_N, \frac{\partial}{\partial x} u_N)_w
\]
\[
= 2 \int_{-1}^{1} u_N(x) \frac{\partial}{\partial x} u_N(x) w(x) dx
\]
\[
= \int_{-1}^{1} \frac{\partial}{\partial x} ([u_N(x)]^2) w(x) dx
\]
\[
= w(x)u_N(x)^2 \bigg|_{x=-1}^{1} - \int_{-1}^{1} [u_N(x)]^2 w'(x) dx
\]
\[
= -\int_{-1}^{1} [u_N(x)]^2 w'(x) dx < 0
\]
since
\[w'(x) = (1 + x)^{\alpha - 1}(1 - x)^{\beta - 1}(\alpha (1 - x) - \beta (1 + x)) \geq 0.\]

With Example 2.6 we again extend the class of Jacobi polynomials for this PDE by introducing a new norm.

Example 2.6. Consider the Jacobi Galerkin approximation to the linear hyperbolic problem with \(\alpha \geq -1\) and \(\beta \leq 0\).

Then the approximation is stable in the \(L^2\) norm corresponding to \(\tilde{w}(x) = (1 + x)^{\alpha}(1 - x)^{\beta}\). (In particular this includes Chebyshev and Legendre polynomials.)

Proof. We use the polynomial basis \(\{P_n(x) - P_n(1)\}_{n \in \mathbb{N}}\) where \(P_n = P_n^{(\alpha, \beta)}\) are the Jacobi Polynomials for our approximation, so that the boundary condition is satisfied. Hence

\[u_N = \sum_{n=0}^{N} a_n(t)(P_n(x) - P_n(1))\]

where

\[\frac{\partial u_N}{\partial t} - \frac{\partial u_N}{\partial x} = \tau(t)R_N(x)\] (7)

and \(\tau(t)R_N(x)\) is the residual. Multiplying by \((1 + x)w(x)u_N\) and integrating over \(x\), we get

\[
\text{LHS of 7} = \int_{-1}^{1} (1 + x)w(x)u_N(\frac{\partial u_N}{\partial t} - \frac{\partial u_N}{\partial x}) dx
\]
\[
= \frac{1}{2} \frac{\partial}{\partial t} \int_{-1}^{1} [u_N]^2(1 + x)w(x) dx - \int_{-1}^{1} (1 + x)w(x)u_N \frac{\partial u_N}{\partial x} dx
\]
\[
= \frac{1}{2} \frac{\partial}{\partial t} \int_{-1}^{1} [u_N]^2(1 + x)w(x) dx - \frac{1}{2} \int_{-1}^{1} (1 + x)w(x) \frac{\partial}{\partial x} ([u_N]^2) dx.
\]
Then letting $\tilde{w}(x) = (1 + x)w(x) = (1 + x)^{\alpha+1}(1 - x)^{\beta}$, we have

$$
\text{LHS of 7} = \frac{1}{2} \left[ \frac{d}{dt} \|u_N\|_w^2 - \int_{-1}^{1} \frac{\partial}{\partial x} \left( [u_N]^2 \right) dx \right] \\
= \frac{1}{2} \left[ \frac{d}{dt} \|u_N\|_w^2 - \tilde{w}(x)[u_N]^2(x) \right]_{x=-1}^{1} + \int_{-1}^{1} \tilde{w}'(x)[u_N]^2 dx \\
= \frac{1}{2} \left[ \frac{d}{dt} \|u_N\|_w^2 + \int_{-1}^{1} \tilde{w}'(x)[u_N]^2 dx \right].
$$

Now looking at the RHS of Equation 7, we see

$$
\text{RHS of 7} = \tau(t) \int_{-1}^{1} (1 + x)w(x)u_NR_N dx \\
= \tau(t) \int_{-1}^{1} w(x)u_NR_N dx + \tau(t) \int_{-1}^{1} xw(x)u_NR_N dx \\
= \tau(t) \int_{-1}^{1} xw(x)u_NR_N dx
$$

since $R_N$ is orthogonal to $B_N$ under $(\cdot, \cdot)_w$. Looking at this integral, we see

$$
\int_{-1}^{1} xw(x)u_NR_N dx = \int_{-1}^{1} xw(x)u \left[ \sum_{n=0}^{N} a_n(t)(P_n(x) - P_{n+1}(1)) \right] R_N dx.
$$

By Theorem 4.2 in [6] we have

$$
xP_n(x) = A_nP_{n-1}(x) + B_nP_n(x) + C_nP_{n+1}(x)
$$

where

$$
A_n = \frac{2(n + \alpha)(n + \beta)}{(2n + \alpha + \beta + 1)(2n + \alpha + \beta)}, \\
B_n = -\frac{\alpha^2 - \beta^2}{(2n + \alpha + \beta + 2)(2n + \alpha + \beta)}, \\
C_n = \frac{2(n + 1)(n + \alpha + \beta + 1)}{(2n + \alpha + \beta + 2)(2n + \alpha + \beta + 1)}.
$$

Plugging this into the above expression for $\int_{-1}^{1} xw(x)u_NR_N dx$, we have all terms are orthogonal to $R_N$ other than the $N + 1$ terms, so

$$
\int_{-1}^{1} xw(x)u_NR_N dx = \int_{-1}^{1} w(x)a_n(t)C_n(P_{N+1}(x) - P_{N+1}(1)) R_N dx.
$$

Writing

$$
R_N(x) = \sum_{k=0}^{N} r_kP_k(x)
$$

(noting where that $R_N$ is a polynomial of degree $N$), again we have all terms are orthogonal, so

$$
\text{RHS} = -a_N(t)C_HC(t) \int_{-1}^{1} w(x)P_{N+1}(1)P_0 dx.
$$
To observe that the RHS is negative, we look at each term. \( P_0(x) = 1 \geq 1 \) and

\[
P_{N+1}(1) = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)} \geq 0
\]
since \( \Gamma(x) \geq 0 \) for \( x \geq 0 \). We see also that \( C_N \geq 0 \) as long as \( N + \alpha + 1 \geq 0 \) (can always choose larger \( N \) if need be). Now looking at \( a_N(t)\tau(t)r_0(t) \), we have to observe from the definition of the residual that

\[
\frac{\partial u_N}{\partial t} = \frac{\partial u_N}{\partial x} + \tau(t) \sum_{k=0}^{N} r_k(t) P_k(x).
\]

Since \( \frac{\partial u_N}{\partial x} \) is a polynomial of degree \( N - 1 \), we have that the highest degree polynomial occurs only in the first and third terms. Equating these terms, we then have

\[
d\frac{d a_N}{d t} = \tau r_N.
\]

Now using that \( R_N \) is orthogonal to the space of \( u_N \),

\[
0 = \left( \sum_{k=0}^{N} r_k(t) P_k(x), P_n(x) - P_n(1) \right)_w
\]

\[
= r_n(P_n, P_n)_w - r_0(P_0, P_n(1))_w
\]

so that \( r_n(P_n, P_n)_w = r_0(P_0, P_n(1))_w \) and since \( (P_n, P_n)_w \geq 0 \) and \( P_n(1), P_0 \geq 0 \), we have \( r_n \) and \( r_0 \) have the same sign for all \( n = 1, ..., N \).

Thus equating the LHS and RHS of Equation 7, we have

\[
\frac{d}{dt} \| u_N \|_{\tilde{w}}^2 + \int_{-1}^{1} \tilde{w}'(x) u_N^2 dx = -K \frac{d}{dt}(a_N^2)
\]

where

\[
K = \frac{r_0}{r_N} C_N \int_{-1}^{1} w(x) P_{N+1}(1) P_0(x) dx \geq 0.
\]

So

\[
\frac{d}{dt} \| u_N \|_{\tilde{w}}^2 + Ka_N^2(t) = -\int_{-1}^{1} \tilde{w}'(x)[u_N]^2 dx \leq 0
\]

since we already showed in Example 2.5 that \( \tilde{w}'(x) \geq 0 \) for \( \alpha + 1 \geq 0 \) and \( \beta \leq 0 \). Then

\[
\frac{d}{dt} \| u_N \|_{\tilde{w}}^2 \leq \frac{d}{dt} \| u_N \|_{\tilde{w}}^2 + Ka_N^2(t) \leq 0
\]

so we have stability.

\[\square\]

2.1 The collocation approach

Here we give some examples of stability for Chebyshev collocation methods using Gauss-Lobatto quadrature rules. The properties of this quadrature rule, namely that it is exact for all polynomials in \( B_{2N-1} \) (see Section 5.2 of [6]), allows us to pass from summation to integration.
Example 2.7. The Chebyshev collocation method for the equation

\[
\begin{align*}
  u_t &= \sigma(x)u_{xx}, & \sigma(x) > 0 \\
  u(\pm 1, t) &= 0.
\end{align*}
\]  

(8)  

(9)

with Gauss-Lobatto quadrature is stable under the norm induced by \([\cdot, \cdot]_w\), where \(w_0 = w_N = \frac{\pi}{2N}, \ w_j = \frac{\pi}{N}, j = 1, ..., N - 1\).

Proof. Recall from Example 2.3 we have that this problem is(strongly) well-posed in the Chebyshev norm (the bound we found holds in the limit as \(N \to \infty\) as well). We seek \(u_N \in B_N\) such that the equation

\[
\frac{\partial u_N}{\partial t} = \sigma(x) \frac{\partial^2 u_N}{\partial x^2}
\]

is satisfied at the points

\[
x_j = -\cos\left(\frac{\pi j}{N}\right), \quad j = 1, ..., N - 1.
\]

and \(u_N(\pm 1) = 0\) (refer to [6] Section 5.2.2). Multiplying by \(u_N(x_j, t)w_j\), where \(w_j\) are the Chebyshev Gauss-Lobotto weights, and summing, we get

\[
\frac{1}{\sigma(x_j)} u_N(x_j) \frac{\partial u_N(x_j)}{\partial t} w_j = \frac{1}{\sigma(x_j)} u_N(x_j) \frac{\partial^2 u_N(x_j)}{\partial x^2} w_j.
\]

Since \(u_N \frac{\partial^2 u_N}{\partial x^2}\) is a polynomial of degree \(2N - 2\), the quadrature formula is exact, and

\[
\text{RHS of 8} = \int_{-1}^{1} w(x) u_N \frac{\partial^2 u_N}{\partial x^2} \, dx \leq 0
\]

by the proof of Example 2.3. This gives that

\[
\text{RHS of 8} = \text{LHS of 8} = \frac{\partial}{\partial t} \sum_{j=0}^{N} \frac{1}{\sigma(x_j)} [u_N(x_j, t)]^2 w_j \leq 0
\]

so

\[
\sum_{j=0}^{N} \frac{1}{\sigma(x_j)} [u_N(x_j, t)]^2 w_j \leq \sum_{j=0}^{N} \frac{1}{\sigma(x_j)} [u_N(x_j, 0)]^2 w_j
\]

and since

\[
\frac{1}{\max_j \sigma(x_j)} \sum_{j=0}^{N} \frac{1}{\sigma(x_j)} [u_N(x_j, t)]^2 w_j \leq \sum_{j=0}^{N} \frac{1}{\sigma(x_j)} [u_N(x_j, t)]^2 w_j \leq \frac{1}{\min_j \sigma(x_j)} \sum_{j=0}^{N} \frac{1}{\sigma(x_j)} [u_N(x_j, t)]^2 w_j,
\]

we have

\[
\sum_{j=0}^{N} \frac{1}{\sigma(x_j)} [u_N(x_j, t)]^2 w_j \leq \frac{\max_j \sigma(x_j)}{\min_j \sigma(x_j)} \sum_{j=0}^{N} \frac{1}{\sigma(x_j)} [u_N(x_j, 0)]^2 w_j.
\]

Thus, the method is stable. \(\Box\)
Now we use a similar trick to the continuous case to prove stability for the same hyperbolic linear PDE used in the previous examples.

**Example 2.8.** The Chebyshev collocation method for the equation

\[ u_t = u_x \]

\[ u(1, t) = 0. \]

with Gauss-Lobatto quadrature is stable under the norm induced by \([\cdot, \cdot] \tilde{w}\), where the weight is given by \(\tilde{w}_j = (1 + x_j)w_j\) and where \(w_0 = w_N = \frac{\pi}{2N}, w_j = \frac{\pi}{N}, j = 1, ..., N - 1\).

**Proof.** We seek a polynomial of degree \(N - 1\), \(u_N(x, t)\), which satisfies the equation

\[ \frac{\partial u_N}{\partial t} = \frac{\partial u_N}{\partial x} \]

at \(x_j, j = 1, ..., N\) where

\[ x_j = -\cos\left(\frac{j\pi}{N}\right), \quad j = 1, ..., N. \]

Multiplying this equation by \((1 + x_j)w_ju_N(x_j)\) and summing over \(x_j\), we get

\[ \sum_{j=0}^{N}(1 + x_j)u_N \frac{\partial u_N}{\partial t} w_j = \sum_{j=0}^{N}(1 + x_j)u_N \frac{\partial u_N}{\partial x} w_j. \quad (10) \]

Since \(u_N \frac{\partial u_N}{\partial x}\) is a polynomial of degree \(2N - 3\), the quadrature rule is exact, and

\[ \text{RHS of } 10 = \int_{-1}^{1} (1 + x)^{1/2}(1 - x)^{-1/2}u_N \frac{\partial u_N}{\partial x} dx \leq 0 \]

by the proof of Example 2.5. So

\[ \text{LHS of } 10 = \frac{\partial}{\partial t} \sum_{j=0}^{N}(1 + x_j)[u_N(x_j, t)]^2 w_j \leq 0 \]

and thus

\[ \sum_{j=0}^{N}(1 + x_j)w_j[u_N(x_j, t)]^2 \leq \sum_{j=0}^{N}(1 + x_j)w_j[u_N(x_j, 0)]^2 \leq 2 \sum_{j=0}^{N}w_j[u_N(x_j, 0)]^2 \]

and stability is established.

Note: The stability for the Chebyshev method based on Gauss-Lobatto points (see [5] and [3] Section 12.1.2) and Gauss-Radau points (see [8] and [3] Section 10.2 Example 8) have also been proved, but these proofs are more complicated and have hence been omitted.

**References**


