Invariants of multiple-qubit systems under stochastic local operations

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We investigate the behavior of quantum states under stochastic local quantum operations and classical communication (SLOCC) for fixed numbers of qubits. We explicitly exhibit the homomorphism between complex and real groups for two-qubits, and use the latter to describe the effect of SLOCC operations on two-qubit states. We find an expression for the polarization Lorentz group invariant length, which is the Minkowskian analog of the quantum state purity, the corresponding Euclidean length. The construction presented is immediately generalizable to any finite number of qubits.

1 Introduction.

In quantum information theory, stochastic local operations and classical communication (SLOCC) on single-qubit density matrices [2] are described by the group SL(2,C), which is homomorphic to the proper Lorentz group, $O_o(1,3)$. The state of a single classical spin is known to have an invariant length under transformations of the proper Lorentz group [1]. Here, we consider the Lorentz-group invariant length for every possible finite number of qubits, *i.e.* quantum spins, which are capable of being entangled. This length is seen to be the Minkowskian analog of the quantum state purity, which is the corresponding Euclidean length. This length is a new tool for describing the behavior of states of any finite number of qubits under SLOCC, which have thus far been studied in detail for only two qubits using matrix methods, which are not generalizable to more than two qubits but have produced encouraging results [3]. By constrast, the method and results presented here are generalizable to any fixed number of qubits without difficulty.

2 A single qubit.

In classical physics, one can use the expectation values of the Pauli spin matrices to fully characterize a state of spin, and to visualize it geometrically via a Poincaré sphere. As Han *et al.* [4] have pointed out, these classical parameters form a Minkowskian four-vector under the group of transformations corresponding to ordinary and hyperbolic state rotations. In particular, the elements of the group of proper Lorentz transformations $O_o(1,3)$ acting on the classical Stokes vector can be represented as products of the following six forms of matrix, $M_1, ..., M_6$:

$$M_1(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\alpha & -sin\alpha & 0 \\ 0 & sin\alpha & cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1a}$$

$$M_2(\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\beta & 0 & -sin\beta \\ 0 & 0 & 1 & 0 \\ 0 & sin\beta & 0 & cos\beta \end{pmatrix} \tag{1b}$$

$$M_3(\gamma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & cos\gamma & -sin\gamma \\ 0 & 0 & sin\gamma & cos\gamma \end{pmatrix} \tag{1c}$$

$$M_4(\chi) = \begin{pmatrix} cosh\chi & sinh\chi & 0 & 0 \\ sinh\chi & cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1d}$$

$$M_5(\omega) = \begin{pmatrix} cosh\omega & 0 & sinh\omega & 0 \\ 0 & 1 & 0 & 0 \\ sinh\omega & 0 & cosh\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1e}$$

$$M_6(\zeta) = \begin{pmatrix} cosh\zeta & 0 & 0 & sinh\zeta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ sinh\zeta & 0 & 0 & cosh\zeta \end{pmatrix} , \tag{1f}$$

that preserve an associated invariant length (*cf.* [5]). For the investigation of the properties of qubit states, it is illustrative first to consider Lorentz group transformations in correspondence to transformations on elements of $H(2)$, the vector space of all 2x2 complex Hermitian matrices that includes the density matrices describing states of single qubits.

The state of a quantum ensemble of independent qubits can be completely described by the set of expectation values

$$x_\mu = Tr(\rho\sigma_\mu) \quad (\mu = 0, 1, 2, 3) , \tag{2}$$

where $\sigma_0 = 1_{2\times2}$ and σ_i, $i = 1, 2, 3$, are the Pauli matrices. Likewise, one can write the density matrix as

$$\rho = \frac{1}{2}\sum_{\mu=0}^{3} x_\mu\sigma_\mu, \tag{3}$$

and the vector space for one qubit state-vectors is $\mathcal{C}^2$. Since $\sigma_\mu^2 = 1$ and $\frac{1}{2}\sigma_\mu\sigma_\nu = \delta_{\mu\nu}$, the four Pauli matrices form a basis for $H(2)$ of which the density matrices, ρ, are the positive-definite, elements of unit trace (*i.e.*, those for which $x_0 \equiv 1$), that capture the general qubit state, pure or mixed.

Now consider these expectation-value vectors in the Minkowskian real vector space, $R^4_{1,3}$, the four-dimensional real vector space R^4 endowed with the Minkowski metric $(+, -, -, -)$, *i.e.* together with a metric tensor $g^{\mu\nu}$ possessing, as non-zero elements, the diagonal entries $+1, -1, -1$, and -1 . The length of a four-vector x_μ in $R^4_{1,3}$ is given by $< x, x >= g^{\mu\nu} x_\mu x_\nu$. More explicitly, in $R^4_{1,3}$, the length of a vector $x = (x_0, x_1, x_2, x_3)$ is given by

$$\| x \|^2_{R^4_{1,3}} = x_0^2 - x_1^2 - x_2^2 - x_3^2 \ . \tag{4}$$

Using the standard vector basis for R^4, $e_0 = (1,0,0,0)$, $e_1 = (0,1,0,0)$, $e_2 = (0,0,1,0)$, $e_3 = (0,0,0,1)$, there exists a natural vector-space isomorphism, $\nu : R^4_{1,3} \longrightarrow H(2)$, relating the space, $R^4_{1,3}$, of these vectors and the space of state matrices, $H(2)$, defined by

$$\nu(x_0, x_1, x_2, x_3) = x_0\sigma_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \ . \tag{5}$$

This isomorphism straightforwardly relates the corresponding basis elements for the space of expectation-value vectors to those for the space of density matrices, namely $\nu(e_i) = \sigma_i$ [5]. If we then define the norm on the space of density matrices, $H(2)$ to be

$$\| X \|^2_{H(2)} = detX \ , \ \forall X \in H(2) \ , \tag{6}$$

then the isomorphism ν between the spaces of these real vectors and the Hermitian matrices becomes a length-preserving mapping, *i.e.* an isometry, since we have the following simple relationship between lengths in the two spaces:

$$\| \nu(x_0, x_1, x_2, x_3) \|^2_{H(2)} \equiv detX = x_0^2 - x_1^2 - x_2^2 - x_3^2 = \| x \|^2_{R^4_{1,3}} \ . \tag{7}$$

Since the Pauli matrices are traceless and $\sigma^2_\mu = 1_2 \ \ (\mu = 0,1,2,3)$, we obtain the following expression for the inverse, $\nu^{-1} : H(2) \longrightarrow R^4_{1,3}$, of this vector-space isomorphism:

$$\nu^{-1}(X) = \frac{1}{2}\Big(Tr(X), Tr(X\sigma_1), Tr(X\sigma_2), Tr(X\sigma_3) \Big) \ , \ \forall X \in H(2), \tag{8}$$

which maps the space of 2×2 Hermitian matrices containing the density matrices into the space, $R^4_{1,3}$, containing the quantum four-vectors. In particular, the density matrices of quantum mechanics are identified within the space of Hermitian matrices $H(2)$ as those having trace one, a condition guaranteeing that the sum of probabilities of all the possible events for the quantum state is unity.

Defining the contraction map $\lambda : H(2) \longrightarrow H(2)$:

$$\lambda(X) = \frac{1}{2} X \ , \forall X \in H(2) \ ,$$

allows us to define the isomorphism $\omega(x) = \lambda \circ \nu : R^4_{1,3} \to H(2)$ of the space containing expectation-value vectors to that containing the density matrices:

$$\omega(x_0, x_1, x_2, x_3) = \frac{1}{2}\sum_{i=0}^{3} x_\mu \sigma_\mu \ , \ \forall x = (x_0, x_1, x_2, x_3) \in R^4_{1,3} \ . \qquad (9)$$

The corresponding inverse map, $\omega^{-1} : H(2) \to R^4_{1,3}$ is

$$\omega^{-1}(X) = \Big(Tr(X), Tr(X\sigma_1), Tr(X\sigma_2), Tr(X\sigma_3)\Big) \ . \qquad (10)$$

As with ν, ω becomes an isometry if we define $\| \ \omega(x_0, x_1, x_2, x_3) \ \|^2_{H(2)} \equiv det(2X) = \| \ x \ \|^2_{R^4_{1,3}}$.

ω^{-1} now directly returns the vector of expectation values, $x_\mu = Tr(\rho\sigma_\mu)$ ($\mu = 0, 1, 2, 3$), as desired.

The group action α: SL(2,C)$\times H(2) \to H(2)$, on $H(2)$ is defined by

$$\alpha(A, X) = AXA^*, \ \ \forall A \in SL(2, C) \ \ and \ \ \forall X \in H(2) \ , \qquad (11)$$

involving the density matrices. We see that the norm induced by the isomorphism ω is preserved under α, since

$$\| \ AXA^* \ \|^2_{H(2)} = det(AXA^*) = |detA|^2 detX = detX = \| \ X \ \|^2_{H(2)} \ . \qquad (12)$$

The natural group action, $\beta : O_o(1,3) \times R^4_{1,3} \to R^4_{1,3}$, of the Lorentz group $O_0(1,3)$ on the quantum observables, the elements of $R^4_{1,3}$ including the vectors describing this ensemble is defined by

$$\beta(x) = Bx, \ \forall B \in O_o(1,3), \ \ \forall x \in R^4_{1,3} \ , \qquad (13)$$

and is norm-preserving (by definition), *i.e.* $\| \ Bx \ \|^2_{R^4_{1,3}} = \| \ x \ \|^2_{R^4_{1,3}}$.

Since the isomorphism ω of the expectation value space to the space containing the quantum states is an isometry, we can also define a map, $\theta : SL(2, C) \to O_0(1,3)$, between the transformations on elements of $H(2)$, including the density matrices, to those transformations of elements of $R^4_{1,3}$. The action of a matrix A on the matrices $X \in H(2)$ induces a corresponding Lorentz transformation $\theta(A)$ of vectors in $R^4_{1,3}$, such that $\| \ \omega^{-1}(AXA^*) \ \|_{R^4_{1,3}} = \| \ \theta(A)\omega^{-1}(X) \ \|_{R^4_{1,3}} = \| \ \omega^{-1}(X) \ \|_{R^4_{1,3}}$.

By defining a map, γ, of the quantum state transformations into the corresponding transformations of the qubit expectation values, $\gamma : SL(2, C) \times H(2) \rightarrow O_o(1,3) \times R^4_{1,3}$,

$$\gamma(A, X) = \Big(\theta(A), \omega^{-1}(X)\Big) , \tag{14}$$

we then obtain a commuting diagram, *i.e.* a set of mathematical objects and mappings such that any two mappings between any pair of objects obtained by composition of mappings are equal. This illustrates in full detail the well-known relationship between SL(2,C) and O_0(1,3), but tailored to the quantum mechanical context.

$$\begin{tikzcd}
SL(2,C) \times H(2) \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & H(2) \\
O_o(1,3) \times R^4_{1,3} \arrow[r, "\beta"] & R^4_{1,3} \arrow[u, "\omega"']
\end{tikzcd}$$

The above construction allows one to freely analyze the behavior of the quantum expectation values under Lorentz group transformations and will be generalized below.

The Minkowskian length, l^2, of the vector of expectation values is

$$l^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2, \tag{15}$$

following Eq. (4), being similar to its analog in the classical realm and invariant under the Lorentz group of transformations represented by the basic forms $M_1, ..., M_6$. This group of transformations goes beyond the limited context of unitary transformations of density matrices (for which $x_0 \equiv 1$), to include non-unitary transformations (for example, corresponding to the Lorentz group transformations M_4, M_5, M_6). The loci of constant l^2 are three-dimensional hyperboloids – a range of ensemble relative sizes x_0 and polarization vector states - lying within what is the probability analog of the "forward light cone" of special relativity.

When the corresponding transformation of the density matrix is an element of the SU(2) subgroup of SL(2,C), corresponding to a unitary transformation of density matrices into density matrices and x_0 is strictly unity, the states lie within a locus a fixed distance from the x_0 axis; when this transformation involves one of M_4, M_5, M_6 probability is lost/gained, so that this constraint is no longer obeyed and x_0 can take other values, x'_0, and move to other locations within the hyperboloid represented by the same value of the invariant.

3 More than one qubit.

To show how we can apply in a well-defined way Lorentz transformations to multiple qubit systems, including those that are entangled, consider now the application of the Lorentz group to two-qubit systems. We introduce the joint expectation values $x_{\mu\nu} = Tr(\rho\sigma_\mu \otimes \sigma_\nu)$, where $\mu, \nu = 0, 1, 2, 3$, and express the matrix of the general state of a two qubit ensemble [6,7]:

$$\rho = \frac{1}{4} \sum_{\mu,\nu=0}^{3} x_{\mu\nu}\sigma_\mu \otimes \sigma_\nu \ , \tag{16}$$

where $\sigma_\mu \otimes \sigma_\nu$ $(\mu, \nu = 0, 1, 2, 3)$ are simply tensor products of the identity and Pauli matrices, and the state-vector space for pure states of two qubits is $\mathcal{C}^2 \otimes \mathcal{C}^2$. The four-vector, x_μ, must then be generalized to a 16-element tensor, $x_{\mu\nu}$.

The two-qubit density matrices ρ are positive, unit-trace elements of the 16-dimensional complex vector space of Hermitian 4×4 matrices, $H(4)$. The tensors $\sigma_\mu \otimes \sigma_\nu \equiv \sigma_{\mu\nu}$ provide a basis for $H(4)$, which is isomorphic to the tensor product space $H(2) \otimes H(2)$ of the same dimension, since $\frac{1}{4}Tr(\sigma_{\mu\nu}\sigma_{\alpha\beta}) = \delta_{\mu\alpha}\delta_{\nu\beta}$ and $\sigma^2_{\mu\nu} = 1_{2\times 2}$, in analogy to the single-qubit case. We can write the two-qubit expectation values as

$$x_{\mu\nu} = Tr(\rho\ \sigma_\mu \otimes \sigma_\nu). \tag{17}$$

A density matrix for the general state of a two-qubit system is thus an element of $H(4) \simeq H(2) \otimes H(2)$ of the form

$$\rho = \frac{1}{4}\left(\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} x_{i0}\sigma_i \otimes \sigma_0 + \sum_{j=1}^{3} x_{0j}\sigma_0 \otimes \sigma_j + \sum_{i,j=1}^{3} x_{ij}\sigma_i \otimes \sigma_j\right) , \tag{18}$$

an element of the Hilbert-Schmidt space [7] that corresponds to

$$x = \ e_0 \otimes e_0 + \sum_{i=1}^{3} x_{i0}\ e_i \otimes e_0 + \sum_{j=1}^{3} x_{0j} e_0 \otimes e_j + \sum_{i,j=1}^{3} x_{ij}\ e_i \otimes e_j \tag{19}$$

in $R^4_{1,3} \otimes R^4_{1,3}$, expressed in terms of the elements of standard vector basis for R^4, $e_0 = (1,0,0,0)$, $e_1 = (0,1,0,0)$, $e_2 = (0,0,1,0)$, $e_3 = (0,0,0,1)$.

The isomorphism between the space of two-qubit expectation values and two-qubit density matrices, $\omega \otimes \omega$: $R^4_{1,3} \otimes R^4_{1,3} \to H(2) \otimes H(2) \simeq H(4)$ is defined as

$$(\omega \otimes \omega)(v \otimes w) \equiv \omega(v) \otimes \omega(w) \ , \tag{20}$$

for all $v, w \in R^4_{1,3}$. $\sigma_\mu \otimes \sigma_\nu$ form a basis for the required space of two-qubit Hermitian matrices $H(2) \otimes H(2) \simeq H(4)$, and $(\omega \otimes \omega)(e_\mu \otimes e_\nu) = \omega(e_\mu) \otimes \omega(e_\nu) = \sigma_\mu \otimes \sigma_\nu$ $(\mu, \nu = 0,1,2,3)$. Furthermore, the inverse map taking density matrices to two-qubit tensors, $(\omega \otimes \omega)^{-1} : H(4) \to R^4_{1,3} \times R^4_{1,3}$ is given by

$$(\omega \otimes \omega)^{-1}(X) = Tr(X\sigma_\mu \otimes \sigma_\nu) \ , \tag{21}$$

for all $X \in H(4)$. To describe the effect of the full set of group transformations, we use the map $\alpha \otimes \alpha : SL(2,C) \times SL(2,C) \times H(2) \otimes H(2) \to H(2) \otimes H(2)$, since for each qubit the group of transformations $SL(2,C)$ acts via the action α on the vector space $H(2)$ that includes the density matrices. The action on the two-qubit Hermitian matrices is defined as

$$(\alpha \otimes \alpha)(A, B, X \otimes Y) = (AXA^*) \otimes (BYB^*), \tag{22}$$

for all $(A, B) \in SL(2,C) \times SL(2,C)$, and $\forall X, Y \in H(2)$. The action $\alpha \otimes \alpha$ is norm-preserving on the tensor-product space, since

$$\| AXA^* \|^2_{H(2)} = \| X \|^2_{H(2)} \tag{23a}$$

$$\| BYB^* \|^2_{H(2)} = \| Y \|^2_{H(2)} \ . \tag{23b}$$

The action β of the Lorentz group $O_o(1,3)$ on the space of expectation values, $R^4_{1,3}$, also generalizes in the two-qubit case to $\beta \otimes \beta : O_o(1,3) \times O_o(1,3) \times R^4_{1,3} \otimes R^4_{1,3} \to R^4_{1,3} \otimes R^4_{1,3}$,

$$(\beta \otimes \beta)((C, D), v \otimes w) = (Cv) \otimes (Dw) \tag{24}$$

for all $(C, D) \in O_o(1,3) \times O_o(1,3)$ and $\forall v \otimes w \in R^4_{1,3} \otimes R^4_{1,3}$.

The isomorphism $\omega \otimes \omega$ is an isometry, so we define the group homomorphism $\theta \times \theta : SL(2,C) \times SL(2,C) \to O_o(1,3) \times O_o(1,3)$. The action of the transformations $A \times B \in SL(2,C) \times SL(2,C)$ on the matrices $X \otimes Y \in H(2) \otimes H(2)$, which include the density matrices, induces a corresponding

Lorentz group transformation $\theta(A) \times \theta(B)$ on the space of expectation-value tensors $R^4_{1,3} \otimes R^4_{1,3}$:

$$(\omega \otimes \omega)^{-1}[(AXA^*) \otimes (BYB^*)] = \omega^{-1}(AXA^*) \otimes \omega^{-1}(BYB^*)$$

$$= \theta(A)\omega^{-1}(X) \otimes \theta(B)\omega^{-1}(Y) \ . \qquad (25)$$

The $\theta(A) \times \theta(B)$ are well-defined Lorentz group transformations since, as before,

$$\| \omega^{-1}(AXA^*) \|^2_{R^4_{1,3}} = \| \theta(A)\omega^{-1}(X) \|^2_{R^4_{1,3}} = \| \omega^{-1}(X) \|^2_{R^4_{1,3}}$$

and

$$\| \omega^{-1}(BYB^*) \|^2_{R^4_{1,3}} = \| \theta(B)\omega^{-1}(Y) \|^2_{R^4_{1,3}} = \| \omega^{-1}(Y) \|^2_{R^4_{1,3}} \ .$$

Defining the map acting on the space $H(2) \otimes H(2)$ including the density matrices,

$\gamma \otimes \gamma : SL(2,C) \times SL(2,C) \times H(2) \otimes H(2) \to O_o(1,3) \times O_o(1,3) \times R^4_{1,3} \otimes R^4_{1,3}$, by

$$(\gamma \otimes \gamma)\Big((A,B),(X \otimes Y)\Big) = \Big(\Big((\theta \times \theta)(A,B)\Big), (\omega \otimes \omega)^{-1}(X \otimes Y)\Big)$$

$$= \Big(\Big(\theta(A), \theta(B)\Big), \omega^{-1}(X) \otimes \omega^{-1}(Y)\Big), \qquad (26)$$

for all $(A,B) \in SL(2,C) \times SL(2,C)$ and for all $X \otimes Y \in H(2) \otimes H(2)$, we obtain the following commuting diagram

$$\begin{array}{ccc}
SL(2,C) \times SL(2,C) \times H(2) \otimes H(2) & \xrightarrow{\alpha \otimes \alpha} & H(2) \otimes H(2) \\
\gamma \otimes \gamma \downarrow & & \uparrow \omega \otimes \omega \\
O_o(1,3) \times O_o(1,3) \times R^4_{1,3} \otimes R^4_{1,3} & \xrightarrow{\beta \otimes \beta} & R^4_{1,3} \otimes R^4_{1,3}
\end{array}$$

demonstrating the well-definedness of the construction on a set of two-qubit states, including those that are entangled.

Again the length given by the tensor norm

$$l_{12}^2 \equiv \| x \|_{R_{1,3}^4 \otimes R_{1,3}^4}^2 = < x, x > = (x_{00})^2 - \sum_{i=1}^{3}(x_{i0})^2 - \sum_{j=1}^{3}(x_{0j})^2 + \sum_{i=1}^{3}\sum_{j=1}^{3}(x_{ij})^2 \, . \tag{20}$$

is invariant under Lorentz group transformations $(A, B) \in O_o(1,3) \times O_o(1,3)$. A similar approach can be used to find an expression for this length for an arbitrary number of qubits.

The generalization of the above methods to the case of n-qubits is straightforward, and allows us to find the invariant length for any finite number of qubits. Unlike previous approaches to applying the Lorentz group to quantum states [3], which used matrix methods to arrive at quantities of interest and are therefore limited to transformations representable in simple matrix form, something that cannot be done for an arbitrary number of qubits, the present treatment is entirely general.

The n-qubit tensor $x_{i_1 \ldots i_n}$ transforms under the group $\mathrm{O}_o(1,3)$ as

$$x'_{i_1 \ldots i_n} = \sum_{j_1,\ldots,j_n=0}^{3} L_{i_1}^{j_1} \ldots L_{i_n}^{j_n} x_{j_1 \ldots j_n} \ , \tag{37}$$

where the L_i^j are such transformations acting in the spaces of qubits $1, ..., n$. Again, each such transformation $x_{\mu_1\mu_2\ldots\mu_n} \rightarrow x'_{\mu_1\mu_2\ldots\mu_n}$ of a given n-qubit expectation-value tensor will yield a new Hermitian state matrix ρ'. After transformation, the tensor element $x'_{0\ldots0}$ is the new n-qubit ensemble relative size. Again, the renormalizing of ρ' gives the resulting density matrix for the ensemble: $\rho'' = \rho'/Tr(\rho')$.

Note that the quantum state purity $Tr\rho^2$ for a general n-photon state,

$$Tr\rho^2 = Tr\left[\left(\frac{1}{2}\right)^n \sum_{i_1,\ldots,i_n=0}^{3} x_{i_1\ldots i_n}\sigma_{i_1}\otimes\ldots\otimes\sigma_{i_n} \left(\frac{1}{2}\right)^n \sum_{j_1,\ldots,j_n=0}^{3} x_{j_1\ldots j_n}\sigma_{j_1}\otimes\ldots\otimes\sigma_{j_n}\right] , \tag{38}$$

has a particularly simple form in terms of the elements n-qubit four-tensor; since $(\sigma_{i_1} \otimes \sigma_{j_1}) \otimes \ldots \otimes (\sigma_{i_n} \otimes \sigma_{j_n}) = \sigma_0 \otimes \ldots \otimes \sigma_0$ if and only if $i_k = j_k$, for all $k = 1, 2, ..., n$, only the coefficient of the term $\sigma_0 \otimes \ldots \otimes \sigma_0$ contribute to the trace, and we have

$$Tr\rho^2 = \frac{1}{2^n} \sum_{i_1,\ldots,i_n=0}^{3} x^2_{i_1\ldots i_n} \ . \tag{39}$$

The state purity is thus seen to be the Euclidean analog of the Minkowskian invariant length.

4 Conclusion.

We have considered the application of the Lorentz group to multiple-qubit states. We have exhibited the necessary construction for two-qubit case in detail. We showed that the multiple qubit state expectation values form Minkowskian tensors with a related invariant length under the action of the Lorentz group. This length is the Minkowskian analog of the quantum state purity, which is the corresponding Euclidean length. This length provides a new tool for describing the behavior of states of any finite number of qubits under SLOCC, including those in entangled states, which have thus far been studied with positive results but for only two-qubit states and two-qubit reduced states of three-qubit pure states [3]. We conjecture that the SLOCC invariant length describes entanglement properties of multiple qubit states.

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