Complementarity and Path Distinguishability: some recent results concerning photon pairs.

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Abstract

Two results concerning photon pairs, one previously reported and one new, are summarized. It was previously shown that if the two photons are prepared in a quantum state formed from $|A\rangle$ and $|A'\rangle$ for photon 1 and $|B\rangle$ and $|B'\rangle$ for photon 2, then both one- and two-particle interferometry can be studied. If v_i is the visibility of one-photon interference fringes ($i = 1, 2$) and v_{12} is the visibility of two-photon fringes (a concept which we explicitly define), then

$$v_i^2 + v_{12}^2 \leq 1.$$

The second result concerns the distinguishability of the paths of photon 2, using the known 2-photon state. A proposed measure E for path distinguishability is based upon finding an optimum strategy for betting on the outcome of a path measurement. Mandel has also proposed a measure of distinguishability P_D, defined in terms of the density operator ρ of photon 2. We show that E is greater than or equal to P_D and that $v_2 = (1 - E^2)^{1/2}$.

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1 Introduction.

The idea of an entangled quantum state of a composite system – i.e., a state not factorizable into a product of one-particle states – was discovered by Schrödinger in 1926, and has been intensively studied as a result of analyses by Einstein-Podolsky-Rosen and Bell. A very convenient method for preparing entangled photon pairs by parametric down-conversion in laser-pumped nonlinear crystals was discovered by Burnham and Weinberg in 1970. Their discovery permitted the development of two-photon interferometry by Mandel and his school, Alley and Shih, Franson, Rarity and Tapster, Chiao and his school, and others.[1]

For subsequent discussion, it will be useful to refer to a schematic two-photon apparatus (Fig. 1), in which a pair of photons emerges from a source S, one of which propagates in beams A and/or A′, and the other in beams B and/or B′, where the locution "and/or" is a brief way of referring to

With no loss of generality we can assume that

$$\langle\chi_1|\chi_1\rangle \geq \langle\chi_2|\chi_2\rangle \ , \tag{20}$$

which can be achieved, if necessary, by interchanging the labels B and B′ of the two paths of photon 2. Then we can write

$$|\chi_2\rangle = \lambda|\chi_1\rangle + |\chi_3\rangle \ , \tag{21a}$$

where

$$\lambda = \frac{\langle\chi_1|\chi_2\rangle}{\langle\chi_1|\chi_1\rangle} \ , \tag{21b}$$

$$|\lambda| \leq 1 \ , \tag{21c}$$

and

$$\langle\chi_1|\chi_3\rangle = 0 \ . \tag{21d}$$

If we define

$$N_i = \langle\chi_i|\chi_i\rangle \ , \quad i = 1,3 \ , \tag{22a}$$

then the $|\bar{\chi}_i\rangle$, defined by

$$|\bar{\chi}_i\rangle = \frac{|\chi_i\rangle}{\sqrt{N_i}} \ , \quad i = 1,3 \ , \tag{22b}$$

are orthonormal. Furthermore,

$$N_1(1 + |\lambda|^2) + N_3 = 1 \ . \tag{23}$$

Any basis $|\phi_1\rangle, |\phi_2\rangle$ in the space of allowable states of photon 1 can be expressed as

$$|\phi_1\rangle = \mu|\bar{\chi}_1\rangle + \nu|\bar{\chi}_2\rangle \ , \tag{24a}$$

$$|\phi_2\rangle = \nu^*|\bar{\chi}_1\rangle - \mu^*|\bar{\chi}_2\rangle \ , \tag{24b}$$

where

$$|\mu|^2 + |\nu|^2 = 1 \ . \tag{24c}$$

This basis defines the observable $\mathcal{O}$ of Eq.(16). It will also be useful to write

$$\mathcal{B} = |B\rangle\langle B| - |B'\rangle\langle B'| \ , \tag{25}$$

an observable in the allowable space of states of photon 2; clearly $\mathcal{B}$ is observed to have values +1 and −1 according as photon 2 is detected in path B or B′.

If $\mathcal{O}$ is the observable chosen to be measured, then there are four pure strategies for bets on the path of photon 2:

(1) If $\mathcal{O} = +1$, predict $\mathcal{B} = +1$; if $\mathcal{O} = -1$, predict $\mathcal{B} = -1$.
(2) If $\mathcal{O} = +1$, predict $\mathcal{B} = -1$; if $\mathcal{O} = -1$, predict $\mathcal{B} = +1$.
(3) Predict $\mathcal{B} = +1$ regardless of the value of $\mathcal{O}$.
(4) Predict $\mathcal{B} = -1$ regardless of the value of $\mathcal{O}$.

In addition to these pure strategies there are mixed strategies, consisting of following (1), (2), (3), (4) with arbitrary probabilities summing to unity. But since the game is not being played against a rational opponent, the average gain in a mixed strategy cannot exceed the maximum of the average gain $E_{\mathcal{O}}^{(i)}$ of the pure strategies,[6] $i = 1,2,3,4$. These are calculated as follows:

$$E_{\mathcal{O}}^{(1)} = P(\mathcal{O} = 1 \ and \ \mathcal{B} = 1) + P(\mathcal{O} = -1 \ and \ \mathcal{B} = -1)$$

$$-P(\mathcal{O} = 1 \ and \ \mathcal{B} = -1) - P(\mathcal{O} = -1 \ and \ \mathcal{B} = 1)$$

$$= |\langle\Theta|\phi_1\rangle|B\rangle|^2 + |\langle\Theta|\phi_2\rangle|B'\rangle|^2 - |\langle\Theta|\phi_1\rangle|B'\rangle|^2 - |\langle\Theta|\phi_2\rangle|B\rangle|^2$$

$$= S(|\mu|^2 - |\nu|^2) - T|\mu||\nu| \ cos(\theta_\lambda + \theta_\nu - \theta_\mu) \ , \tag{26}$$

where

$$S = N_1(1 - |\lambda|^2) + N_3 \ , \tag{27a}$$

$$T = 4N_1^{1/2}N_2^{1/3}|\lambda| \ , \tag{27b}$$

$$\lambda = |\lambda|e^{i\theta_\lambda}, \ \mu = |\mu|e^{i\theta_\mu}, \ \nu = |\nu|e^{i\theta_\nu} \ ; \tag{27c}$$

$$E_{\mathcal{O}}^{(2)} = -E_{\mathcal{O}}^{(1)} \ ; \tag{28}$$

$$E_{\mathcal{O}}^{(3)} = P(\mathcal{B} = +1) - P(\mathcal{B} = -1) = \langle\chi_1|\chi_1\rangle - \langle\chi_2|\chi_2\rangle$$
$$= N_1(1 - |\lambda|^2) - N_3 = S - 2N_3 \ ; \tag{29}$$

$$E_{\mathcal{O}}^{(4)} = P(\mathcal{B} = -1) - P(\mathcal{B} = +1) = -E_{\mathcal{O}}^{(3)} \ . \tag{30}$$

Note that $E_{\mathcal{O}}^{(3)}$ and $E_{\mathcal{O}}^{(4)}$ are independent of $\mathcal{O}$. Then

$$E_{\mathcal{O}} = max\Big\{|S(|\mu|^2 - |\nu|^2) - T|\mu||\nu| \ cos(\theta_\lambda + \theta_\nu - \theta_\mu)| \ , \ |S - 2N_3|\Big\}. \tag{31}$$

In view of Eqs.(17) and (31) one finds the measure E of path distinguishability by investigating $E_{\mathcal{O}}$ as μ and ν are varied, subject to Eq.(24c). We first note that for any $|\Theta\rangle$ there is an $\mathcal{O}$ such that

$$|E_{\mathcal{O}}^{(1)}| \geq |E_{\mathcal{O}}^{(3)}| \ , \tag{32}$$

so that the second option in Eq.(31) can be neglected when we maximize over all possible $\mathcal{O}$. To prove these statements, it suffices in Eqs.(24a,b), to let $\mu = 1$ and $\nu = 0$, determining an $\mathcal{O}'$ such that Eqs.(26), (27), (28) yield

$$|E_{\mathcal{O}'}^{(1)}| = |N_1(1 - |\lambda|^2) + N_3| \ , \tag{33}$$

and

$$|E_{\mathcal{O}'}^{(3)}| = |N_1(1-|\lambda|^2) - N_3| \ . \tag{34}$$

Since N_1 and N_3 are non-negative, and $(1-|\lambda|^2)$ is non-negative by Eq.(21c), we obtain

$$|E_{\mathcal{O}'}^{(1)}| \geq |E_{\mathcal{O}'}^{(3)}| \ , \tag{35}$$

the rhs being the same as $|E_{\mathcal{O}}^{(3)}|$ for all $\mathcal{O}$. E is therefore obtained by maximizing the first option of Eq.(31) for allowable μ and ν, and the result is

$$E = \frac{1}{2}(4S^2 + T^2)^{1/2}. \tag{36}$$

By Eqs. (27a), (27b), and (23) E can be rewritten as

$$E = (1 - 4N_1^2|\lambda|^2)^{1/2}. \tag{37}$$

We can now make a comparison with Mandel's[4] measure of path distinguishability P_D. Mandel notes that in a two-dimensional Hilbert space, any density operator ρ can be expressed uniquely in the form

$$\rho = P_{ID}\ \rho_{ID} + P_D\ \rho_D \ , \tag{38}$$

where ρ_D is diagonal in the $|B\rangle$, $|B'\rangle$ basis, *i.e.*

$$\rho_D = c_{11}|B\rangle\langle B| + c_{22}|B'\rangle\langle B'| \ , \tag{39}$$

(after adaptation to our notation),

$$tr\ \rho_{ID} = tr\ \rho_D = 1 \ , \tag{40}$$

and

$$P_{ID} \geq 0, \ \ P_D \geq 0 \ . \tag{41}$$

Since ρ_D is a diagonal density operator in the specified basis, one can prepare an ensemble with a definite proportion c_{11} in the state $|B\rangle$ and a definite proportion c_{22} in the state $|B'\rangle$ such that this ensemble is represented by ρ_D. It is this consideration that leads Mandel to identify P_D as the degree of path distinguishability when ρ is given. Mandel also shows that

$$P_D = 1 - \frac{|\rho_{12}|}{(\rho_{11}\rho_{22})^{1/2}} \ , \tag{42}$$

where ρ_{ij} is the ij^{th} matrix element of ρ in the $|B\rangle, |B'\rangle$ basis.

Now let us consider the $|\Theta\rangle$ of Eq.(18), which we can rewrite as

$$|\Theta\rangle = N_1^{1/2}|\bar{\chi}_1\rangle|B\rangle + \lambda\ N_1^{1/2}|\bar{\chi}_1\rangle|B'\rangle + N_3^{1/2}|\bar{\chi}_3\rangle|B'\rangle\ . \tag{43}$$

By the standard procedure for writing the density matrix of particle 2 of a two-particle system,[5] we obtain (with the help of Eq.(23)),

$$\begin{aligned} \rho_{11} &= N_1\ , \\ \rho_{12} = N_1\lambda\ ,\ \rho_{21} &= N_1\lambda^*\ , \\ \rho_{22} = N_1|\lambda|^2 + N_3 &= 1 - N_1\ . \end{aligned} \tag{44}$$

Hence, Eq.(37) can be rewritten as

$$E = (1 - 4|\rho_{12}|^2)^{1/2}\ , \tag{45}$$

which can be shown as follows to be greater than or equal to P_D of Eq.(42).

Proof: First note that if x and y are real numbers in the interval $[0,1]$ which sum to unity, then

$$xy \le \frac{1}{4}\ , \tag{46}$$

from which it follows that

$$(\rho_{11})^{1/2}(\rho_{22})^{1/2} \le \frac{1}{2}\ . \tag{47}$$

Furthermore, since, by Eq.(23)

$$N_1^2\ |\lambda|^2 \le N_1(1 - N_1 - N_3) \le N_1(1 - N_1)\ ,$$

we have

$$|\rho_{12}| = N_1|\lambda| \le \frac{1}{2}\ . \tag{48}$$

From Eqs.(47) and (48) we obtain

$$1 - 4|\rho_{12}|^2 \ge 1 - 2|\rho_{12}| \ge 1 - \frac{|\rho_{12}|}{(\rho_{11}\rho_{22})^{1/2}}\ , \tag{49}$$

where the lhs of this inequality is E^2 and the rhs is P_D^2. Since both E and P_D are non-negative, it follows that

$$E \ge P_D\ . \tag{50}$$

We note that when E is unity, so is P_D: that is, perfect distinguishability (in our sense) on the basis of the two-photon state $|\Theta\rangle$ implies perfect distinguishability (in Mandel's sense) on the basis of the density operator. There is an intuitive reason for this agreement: $E = 1$ implies that there is perfect correlation between the behavior of photon 1 and the entrance of photon 2 into $|B\rangle$ or $|B'\rangle$, but perfect correlation requires the orthogonality of $|\chi_1\rangle$ and $|\chi_2\rangle$ in Eq.(18). This orthogonality, in turn, guarantees that the density operator of photon 2 is diagonal in the $|B\rangle, |B'\rangle$ basis.

If we look at the other extreme, however, we find that $P_D = 0$ does not imply that $E = 0$. Again there is an intuitive reason. When $P_D = 0$, then ρ is a pure case, derived from a quantum state of the form

$$|\psi\rangle = c|B\rangle + c'|B'\rangle \ , \tag{51}$$

so that

$$\begin{aligned} \rho_{11} &= |c|^2 \ , \\ \rho_{12} = cc'^*, \ \rho_{21} &= c^*c' \ , \\ \rho_{22} &= |c'|^2 \ , \end{aligned} \tag{52}$$

Then

$$\begin{aligned} E - P_D &= -4|\rho_{12}|^2 + \frac{|\rho_{12}|}{(\rho_{11}\rho_{22})^{1/2}} \ , \\ &= -4|c|^2|c'|^2 + 1 \ , \end{aligned} \tag{53}$$

and this vanishes if and only if $|c|^2 = |c'|^2 = \frac{1}{2}$. But when the amplitudes of $|B\rangle$ and $|B'\rangle$ in the pure state $|\psi\rangle$ are equal, there is no strategy for betting on the path that will yield a net gain on the average. On the other hand, when $|c|^2$ and $|c'|^2$ are unequal, the strategy of betting on the path associated with the larger coefficient will yield a net gain on the average. The advantage of our E over P_D is the ability of the former to take advantage of inequalities in the amplitudes associated with the two paths.

Mandel also relates path distinguishability to the visibility v_2 of the interference pattern, where

$$v_2 = 2|\rho_{12}| \ . \tag{54}$$

He obtains the inequality

$$v_2 \leq P_{ID} = 1 - P_D \ , \tag{55}$$

with equality holding only when $\rho_{11} = \rho_{22}$. We obtain from the expressions for E and v_2 in Eqs.(45) and (54) the equation

$$v_2 = (1 - E^2)^{1/2} \ , \tag{56}$$

which holds for any preparation of an ensemble of photons in states $|B\rangle$ and $|B'\rangle$ derived from a two-photon state of the form $|\Theta\rangle$. Hence, for the preparation of photon 2 that we have been studying, the visibility v_2 is a natural measure of path indistinguishibility.

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