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Bell-type equalities for SQUIDs on the assumptions of macroscopic realism and non-invasive measurability

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Abstract

Bell-type *equalities* are derived for the Leggett–Garg proposal for a SQUID experiment on the assumptions of macroscopic realism (MR) and non-invasive measurability (NIM). Equalities of this sort have not previously been presented. It is shown that there are 18 such equalities. These equalities put stronger constraints on the consequences of MR and NIM than the corresponding inequalities.

1. Introduction

Leggett and Garg [1,2] (see also Refs. [3–5]) have proposed the use of a superconducting quantum interference device (SQUID) to test *macroscopic realism* (MR), which they define formally (see below). They derive Bell-type inequalities [17,18] in a form analogous to that of Clauser, Horne, Shimony, and Holt [6] from this hypothesis and a natural correlate assumption, *non-invasive measurability* (NIM). Quantum mechanics predicts a violation of these inequalities through macroscopic quantum coherence. They present a SQUID consisting of a superconducting ring, interrupted by a single Josephson junction, and located in an extended magnetic field, for which quantum mechanics predicts that a trapped magnetic flux can tunnel between two states in a way that violates these inequalities. An experimental test of these inequalities is yet to be carried out.

Macroscopic realism seems to be central to our everyday experience of mesoscopic and macroscopic

objects. Yet, quantum mechanics, allegedly our most fundamental theory of matter, predicts its failure. The most striking clash between macroscopic realism and the predictions of quantum mechanics is Schrödinger's [7] cat, for which quantum mechanics apparently makes the deeply unsatisfactory prediction of being in a superposition of two states, alive and dead. Leggett and Garg [1, p. 857] describe the situation as follows: "Despite sixty years of schooling in quantum mechanics, most physicists have a very non-quantum mechanical notion of reality at the macroscopic level, which implicitly makes two assumptions [(MR) and (NIM)]." They characterize these two assumptions formally and use them to derive an inequality which is violated by quantum mechanics.

Using methods similar to Leggett [5], we show here that the Leggett–Garg assumptions are sufficient to produce *equalities* similar in form to the inequalities (Section 3). Since these equalities are more easily violated than the corresponding inequali-

ties, our results show that macroscopic realism puts even stronger constraints on the behavior of SQUIDS than those which they have presented. We provide a method for obtaining all such equalities: there are exactly 18. We also clarify the relationship of the Leggett–Garg assumptions to alternative proposals designed to demonstrate contradictions between macroscopic realism and quantum mechanics (Section 2).

2. Previous discussions

Leggett and Garg’s [1] characterizations of (MR) and (NIM) are:

Macroscopic realism: “A macroscopic system with two or more macroscopically distinct states available to it will at all times *be* in one or the other of those states” [1, p. 857].

Non-invasive measurability: “It is possible, in principle, to determine the state of the system with arbitrarily small perturbation on its subsequent dynamics” [1, p. 857].

In their model, the external field induces screening currents in the ring, which produce their own magnetic field which remains trapped in the ring. When an appropriate external field is applied, a symmetric double potential well arises for the trapped flux. The two degenerate states, of being in these two wells, correspond to a supercurrent I_0 circulating either in a clockwise or a counterclockwise direction (and having a magnitude of several milliamperes). This is sufficient for the two states to be macroscopically distinct (see Ref. [4, Section 6]). Because of possible tunneling between these states, quantum mechanics makes different predictions about four observables, Q_i , $i = 1, \dots, 4$, which report whether or not the system is in one (or the other) of these states at specified times, t_i .

Leggett and Garg [1,2, p. 1621] argue that NIM is a natural corollary of MR. Ballentine [8] has argued that a violation of the Leggett–Garg inequalities by quantum mechanics does not contradict MR but only NIM. This argument is based on a straightforward quantum mechanical calculation for a SQUID. However, Leggett and Garg [2] point out that this calculation refers to an experimental situation different from theirs. Ballentine takes, *contra* Leggett and Garg,

each member of the analyzed ensemble of systems to be measured at *four* of the successive times under consideration. Leggett and Garg [1] on the other hand, refer only *counterfactually* to measurements at intermediate times, and only require measurements at one initial and one final time. Thus, Ballentine’s experiment is also harder to perform. His experiment was designed to test whether quantum mechanics violates NIM *independently* of violating MR whereas the Leggett–Garg scheme only determines whether it violates these two assumptions taken together.

In the Leggett–Garg analysis, as pointed out by Tesche [9], the only requirement on the measurement system is that it must be capable of determining the probability that the system occupies particular states at *two* moments t_i and t_j without affecting the dynamics of the system during the interim. Tesche [9] outlines such a scheme, where two hysteretic dc-SQUID switches and two dissipative dc-SQUID magnetometers are the analogs of the polarizers and particle detectors, respectively, of the standard two-photon experimental tests of Bell’s inequalities [10].

Home [11] claimed to derive a contradiction between quantum mechanics by assuming only a macrorealism condition. However, as Clifton [12] has pointed out, Home’s condition implies a contradiction with quantum mechanics only for subensembles deliberately selected out from a larger ensemble while his calculations apply only to the full ensemble. Thus Home’s attempt to isolate MR as the source of contradictions with quantum mechanics fails.

Finally, Foster and Elby [13] have also criticized the Leggett–Garg assumptions as being stronger than necessary. Let Q_i be the observable corresponding to the SQUID state at time t_i , and q_i be the value (+1 or –1) taken by Q_i at t_i . Foster and Elby give two alternative assumptions to MR and NIM, namely:

Weak realism (WR): “A SQUID measurement-result probability for the outcome q_i depends only on the macrostate of the SQUID” [13, p. 778].

They call this condition “SQUID completeness,” in analogy to Jarrett [14]. However, they take this assumption to mean (in our notation rather than theirs) that, e.g.,

$$P[q_2 = \pm 1 | \lambda_2, Q_1] = P[q_2 = \pm 1 | \lambda_2], \quad (1)$$

where $P[]$ indicate probabilities and λ_i represents the macrostate of the SQUID at time t_i , which is a form analogous to Jarrett's "locality" rather than his "completeness" condition. (We will not consider the extraneous issue of the analogy between the various realism and locality conditions any further in this paper.)

Generalized non-invasive measurability (GNIM): "The subsequent evolution of a SQUID's macrostate is affected arbitrarily weakly by a sufficiently careful measurement of Q_i " [13, p. 777].

Foster and Elby express this condition by expressions of the following form,

$$P[\lambda_k | \lambda_i, Q_j] \approx P[\lambda_k | \lambda_i], \quad (2)$$

where $i > j > k$ refer to different times. (Elby and Foster [15] present an alternative version of this assumption for the special case of null experiments.) Their assumptions are weaker than their Leggett–Garg counterparts by one condition, which we call:

Macroscopic distinguishability (MD): the states of the macroscopic system are distinct, i.e., $P[q_i = x] \in \{0, 1\}$, where $x = \pm 1$ for all times, t_i .

Thus, $WR \wedge MD \Leftrightarrow MR$ and $GNIM \wedge MD \Leftrightarrow NIM$. Therefore $MR \Rightarrow WR$, $NIM \Rightarrow GNIM$, and $WR \wedge GNIM \wedge MD \Leftrightarrow MR \wedge NIM$.

Foster and Elby find a contradiction between quantum mechanics and

$$P[q_2 = +1 | Q_1] \approx P[q_2 = +1], \quad (3)$$

which follows from the conjunction of their assumptions. They claim, therefore, that they have derived a contradiction with quantum mechanics *without* assuming MR, and need only to assume WR which is not violated by quantum mechanics. Therefore contradictions with quantum mechanics for SQUID systems apparently arise because of assumptions such as NIM and GNIM. Moreover, they use Ballentine's [8] experimental scheme rather than that of Leggett and Garg [1]. This is open to the objection made by Leggett and Garg [2], as noted above. Elby and Foster [15] also apply their analysis to Tesche's [9] experimental scheme. However, they implicitly assume MD when they assume that the states measured are distinguished in the experiment. Therefore, if the logical relationships sketched in the last paragraph

are correct, there is no significant difference between the Leggett–Garg and Foster–Elby assumptions. (Note, moreover, the equalities we present below are in no way formally similar to the Foster–Elby equality (Eq. (3)).)

3. Equalities

We will, therefore, restrict our attention to Leggett and Garg's [1] original assumptions. (This has the advantage of being directly connected with Tesche's [9] proposed experiment.) It is then possible to derive an *equality* of a form similar to the Leggett–Garg inequalities. This is seen as follows. Consider all possible pairs of values q_i, q_j ($i, j = 1, 2, 3, 4$) for the dichotomic observables Q_i corresponding to which of two possible states the system is in at different times. (This style of argument goes back to Wigner [16].) These are shown in Table 1. In Table 1, Q_1, Q_2, Q_3, Q_4 are four (dichotomic) observables which can be in the states "+" and "-". The *value* of any Q_i , represented by q_i , is equal to +1 if Q_i is in "+"; otherwise it is equal to "-1". The sixteen rows represent all the possible sets of values that can be measured, given Leggett and Garg's assumptions. The value of the l.h.s. of Leggett and Garg's inequality [1, p. 858]

$$|\langle Q_1 Q_2 \rangle + \langle Q_2 Q_3 \rangle| + |\langle Q_3 Q_4 \rangle - \langle Q_1 Q_4 \rangle| \leq 2 \quad (4)$$

will be a weighted average over these sixteen cases. (The inequality arises because these weights must lie between 0 and 1.)

However, in each of these cases $|q_1 q_2 + q_2 q_3| + |q_3 q_4 - q_1 q_4|$ is identical, namely 2. This means that any average of them, weighted or not, must also equal 2. Thus the Leggett–Garg assumptions are sufficient to prove not only the inequality (4) but also the following *equality*,

$$\langle |q_1 q_2 + q_2 q_3| \rangle + \langle |q_3 q_4 - q_1 q_4| \rangle = 2, \quad (5a)$$

or, alternatively:

$$\langle (Q_1 Q_2 + Q_2 Q_3)^2 \rangle + \langle (Q_3 Q_4 - Q_1 Q_4)^2 \rangle = 4. \quad (5b)$$

Q_1	Q_2	Q_3	Q_4	$Q_1 Q_2$	$Q_1 Q_3$	$Q_1 Q_4$	$Q_2 Q_3$	$Q_2 Q_4$	$Q_3 Q_4$	$Q_1 Q_2 Q_3$	$Q_1 Q_2 Q_4$	$Q_1 Q_3 Q_4$	$Q_2 Q_3 Q_4$	$Q_1 Q_2 Q_3 Q_4$	$q_1 q_2$	$q_1 q_3$	$q_1 q_4$	$q_2 q_3$	$q_2 q_4$	$q_3 q_4$	$q_1 q_2 q_3$	$q_1 q_2 q_4$	$q_1 q_3 q_4$	$q_2 q_3 q_4$	$q_1 q_2 q_3 q_4$	
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0	0	0	0	0	0	0	0	0	0	0	0
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2	+2

(Following Leggett's [5, p. 241] method of demonstration by exhaustion, the l.h.s. of Eq. (5b) can similarly be written as a linear combination, in this case with coefficients ± 1 , of correlation functions, now four-time correlation functions rather than Leggett's two-time correlation functions, for the Q_i (for example, $K_{ijkl} \equiv \langle Q_i Q_j Q_k Q_l \rangle$ whereas Leggett has $K_{ij} \equiv \langle Q_i Q_j \rangle$). These functions determine the physical interpretation of Eq. (5b).)

A simple combinatoric argument will now be used to show that there are exactly 18 equalities of this form (5). This argument will now be presented in the form of three propositions and their proofs. These proofs will concern Eq. (5a). The same argument, in each case, can give Eq. (5b).

Proposition 1. There are 90 distinct formulae of type $\langle | q_a q_b + q_c q_d | \rangle + \langle | q_e q_f - q_g q_h | \rangle$. (6)

Proof. There are exactly 6 distinct products $q_a q_b$. Thus in the first term of (6) there are $\binom{6}{2} = 15$ different possible arrangements. For each such arrangement, two of the remaining four products can occur in the second term of (6). This can occur $\binom{4}{2} = 6$ different ways. Thus the total number of distinct formulae of type (6) is $15 \times 6 = 90$.

Proposition 2. (a) There are 18 formulae of type (6) with exactly two occurrences of each index. (These will be called the "standard form".) (b) There are 72 formulae of type (6) in which some index occurs three times.

Proof of Proposition 2(a). If an index number occurs twice in the first term of (6), i.e. that term is of the form $q_a q_b + q_c q_d$, then the second term must contain $q_a q_b$ and $q_b q_c$ (order being irrelevant) in order for each index to occur exactly twice. There are twelve ways to obtain the form $q_a q_b + q_b q_c$ as follows. The index b can take any of the four possible index numbers. For each choice of b , the numbers a and c must be distinct, and since there are only 3 possible values they can take, this can occur 6 ways. Thus there are exactly $4 \times 3 = 12$ ways to obtain the form $q_a q_b + q_b q_c$, each of which uniquely determines a formula of type (6).

If the first term does not have the form above (one index number repeated), then all 4 indices must be distinct since a product of q_i 's can occur at most once in a formula of type (6). That is, the first term is of the type $q_a q_b + q_c q_d$. Clearly for any choice of a and b there is exactly one choice for c and d (since $q_i q_j = q_j q_i$). Of the 6 choices for a and b , 3 are redundant because they occur as c and d for some other choice, hence there are 3 distinct cases of this form. Now, for each possibility for the first term, there are 2 possible arrangements in the second term: index a occurs with either c or d in which case b occurs with the other. Thus we have a total of $3 \times 2 = 6$ distinct formulae of the type (6) having this form. Since the two other cases are exhaustive, there are exactly 18 formulae of type (6) in which each index occurs exactly twice.

Proof of Proposition 2(b). First consider the case where two of the three products having a common index number occur in the first term of (6). This can occur $\binom{3}{2} = 3$ ways for a fixed index number. For each such possibility any of the other three products can occur with the third product of common index. So for a fixed index we have $3 \times 3 = 9$ possibilities. Now, since there are four index numbers there are 36 such possibilities. The other case is when two of the three products having a common index number occur in the second term. By the same reasoning we get another 36 possibilities giving a total of 72 formulae of type (6) having an index number that occurs 3 times.

Proposition 3. (a) All 18 standard form formulae give rise to a Bell-type equality. (b) The 72 formulae of type (6) that are not in standard form cannot give rise to these equalities.

Proof of Proposition 3(a). Using Table 1, we handle the six cases in which the first term of (6) has the form $\langle |q_a q_b + q_c q_d| \rangle$ by exhaustion. The reasoning for these six cases is as follows: $|A| + |B| = 2 \Rightarrow \langle |A| + |B| \rangle = 2$ (since when all values are equal their average is just that value). Therefore, $\langle |A| \rangle + \langle |B| \rangle = 2$. From Table 1, we obtain $|q_1 q_2 + q_3 q_4| + |q_1 q_3 - q_2 q_4| = 2$ from which it follows by

our reasoning above that $\langle |q_1 q_2 + q_3 q_4| \rangle + \langle |q_1 q_3 - q_2 q_4| \rangle = 2$. Similarly, from Table 1 and the above reasoning we find that

$$\begin{aligned} \langle |q_1 q_2 + q_3 q_4| \rangle + \langle |q_1 q_4 - q_2 q_3| \rangle &= 2, \\ \langle |q_1 q_3 + q_2 q_4| \rangle + \langle |q_1 q_2 - q_3 q_4| \rangle &= 2, \\ \langle |q_1 q_3 + q_2 q_4| \rangle + \langle |q_1 q_4 - q_2 q_3| \rangle &= 2, \\ \langle |q_1 q_4 + q_2 q_3| \rangle + \langle |q_1 q_2 - q_3 q_4| \rangle &= 2, \\ \langle |q_1 q_4 + q_2 q_3| \rangle + \langle |q_1 q_3 - q_2 q_4| \rangle &= 2. \end{aligned}$$

The remaining 12 cases are all of the form

$$\langle |q_a q_b + q_b q_c| \rangle + \langle |q_a q_d - q_c q_d| \rangle. \quad (7)$$

In order to establish the equality (5a) it is sufficient to show $|q_a q_b + q_b q_c| + |q_a q_d - q_c q_d| = 2$, as we have seen above. Notice, however, that if $q_a = q_c$ then $q_a q_b = q_b q_c \Rightarrow |q_a q_b + q_b q_c| = 2$ and $q_a q_d = q_c q_d \Rightarrow |q_a q_d - q_c q_d| = 0$. On the other hand, if $q_a = -q_c$ then $q_a q_b = -q_b q_c \Rightarrow |q_a q_b + q_b q_c| = 0$ and $q_a q_d = -q_c q_d \Rightarrow |q_a q_d - q_c q_d| = 2$. Thus in either case $|q_a q_b + q_b q_c| + |q_a q_d - q_c q_d| = 2$, from which it follows that $\langle |q_a q_b + q_b q_c| \rangle + \langle |q_a q_d - q_c q_d| \rangle = 2$.

Proof of Proposition 3(b). In all of these cases one index number occurs three times. We consider first the cases in which two of the three products with the common index number are in the first term of (6). It suffices to show that $|q_a q_b + q_a q_c| + |q_a q_d - X| \neq k$, for any constant k and $X \in \{q_b q_c, q_b q_d, q_c q_d\}$, since this precludes an equality of the form $\langle |q_a q_b + q_a q_c| + |q_a q_d - X| \rangle = k$ from being found from Table 1. In the situation $q_a \neq q_b = q_c = q_d$, $|q_a q_b + q_a q_c| + |q_a q_d - X| = 4$, since $q_a q_b = q_a q_c$ and $q_a q_d = -X$. The last equality comes about as follows. Since $q_b = q_c = q_d$, $q_b q_c = q_b q_d = q_c q_d$, so X has the same value in all cases. But $q_a = -q_b$, so $q_a q_d = -q_b q_d = -X$. In the situation $q_a = q_b = q_c = q_d$, clearly $|q_a q_b + q_a q_c| + |q_a q_d - X| = 2$ since the first term is 2 and the second term is 0.

For cases in which two of the three products with the common index number are in the second term of (6), i.e. $|q_a q_d + X| + |q_a q_b - q_a q_c|$, taking $q_a \neq q_b = q_c = q_d$ gives a result of 0, and taking $q_a = q_b = q_c = q_d$ gives 2; so once again there is no strict equality. Since a, b, c, d are arbitrary in the above discussion all 72 cases are covered.

This completes the proof that there are exactly 18 equalities of the form (5a), (5b).

4. Discussion

There are two significant differences between the results (5a), (5b) and previous Bell-type results: (i) it is an equality rather than an inequality, and (ii) the l.h.s. involves the average of absolute values (or squares) rather than the absolute value of averages. Regarding the former, strict equalities of this form are harder to satisfy than the corresponding inequalities (Eq. (4)). Therefore, Eqs. (5a), (5b) should be easier to test experimentally. Regarding the latter, questions might arise as to whether the results (5a), (5b) can be tested experimentally. Tests of ordinary Bell-type expressions measure the statistics arising from joint measurements of observables Q_i, Q_j and calculating linear combinations of the results. The experimental procedures involved in testing (5a), (5b) are not qualitatively different from this; the only difference is, rather than measuring the statistics of joint measurements the experimenter measures the statistics of pairs of “joint measurements”, as this is what is involved in evaluating the l.h.s. of (5a), (5b). These equalities provide a stringent direct test of NIM and MR, quite independent of what quantum mechanics would predict in this situation.

It might be also asked why it is that one cannot derive an equality in the case of the ordinary Bell-test situation, such as that of Aspect et al. [10], in the same manner. To do so would seem to require an assumption similar to our MD or Leggett’s MR which guarantees the value definiteness used in our proofs. Bell-type locality conditions do not provide such value definiteness, and to assume such a condition would be tantamount to an assumption of “*microscopic realism*,” for which there is no physical support.

Finally, Leggett and Garg [1,2] have argued that NIM is a natural corollary of MR. If this is accepted, and taken to mean that $MR \Rightarrow NIM$, then the failure of $MR \wedge NIM$, which the failure of our equalities would show, requires that, at the very least, MR

fails. (This follows from the fact that $MR \Rightarrow NIM$ and $\neg(MR \wedge NIM)$ is consistent with NIM but not with MR.) Thus, in this sense, MR but not NIM is directly implicated as the problematic assumption by our analysis, in contrast to the analyses of Ballentine [8], Foster and Elby [13], and Elby and Foster [15]. Of course, this conclusion was implicit in the work of Leggett and Garg [1,2] – we are only noting it explicitly.

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