16 January 1995

PHYSICS LETTERS A

ELSEVIER

Physics Letters A 197 (1995) 83-87

Optimal distinction between two non-orthogonal quantum states

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Received 14 September 1994; revised manuscript received 9 November 1994; accepted for publication 9 November 1994
Communicated by P.R. Holland

Abstract

Two procedures are developed for classifying an individual system as $|p\rangle$ or $|q\rangle$, non-orthogonal, given an ensemble with respective proportions r and $1-r$. One (generalizing Ivanovic, Dieks, and Peres) infallibly classifies some systems, leaving others unclassified. The second is statistically optimum, allowing individual errors.

Ivanovic [1], Dieks [2], and Peres [3] consider the problem of determining what quantum state an individual system was prepared in, when it is given that it was prepared either in $|p\rangle$ or $|q\rangle$, in general nonorthogonal. The problem, thus stated, is vague and can be clarified in (at least) two distinct ways: (1) What procedure yields on the average a maximum number of correct classifications, in an ensemble of such cases, assuming that for each member of the ensemble a definite classification is made? (2) What procedure enables one in a maximum number of cases to infer with certainty whether the system was prepared in $|p\rangle$ or $|q\rangle$, leaving a minimum number of cases undecided?

The three authors mentioned are primarily interested in the problem (2), though Dieks briefly considers problem (1). Under the assumption that half of the ensemble is prepared in $|p\rangle$ and half in $|q\rangle$, all three obtain the same evaluation of the maximum probability of correct classification and minimum probability of no decisions: namely,

P = probability of correct classification

$$=1-|\langle p|q\rangle| , \tag{1a}$$

$1-P$ = probability of no classification

$$=|\langle p|q\rangle| . \tag{1b}$$

The procedures suggested by Dieks and Peres are essentially the same: prepare an auxiliary system in a state $|s_0\rangle$ and choose a unitary evolution (equivalent to a choice of interaction) which yields

$$|p\, s_0\rangle \to \alpha|p_1\, s_1\rangle + \beta|p_2\, s_2\rangle , \tag{2a}$$

$$|q\, s_0\rangle \to \gamma|q_1\, s_1\rangle + \delta|q_2\, s_2\rangle , \tag{2b}$$

where $|s_1\rangle$, $|s_2\rangle$, $|p_1\rangle$, $|p_2\rangle$, $|q_1\rangle$, $|q_2\rangle$ are all normalized, and

$$\langle p_1|q_1\rangle = 0 , \tag{2c}$$

$$\langle s_1|s_2\rangle = 0 , \tag{2d}$$

and $|q_2\rangle$ is identical with $|p_2\rangle$ except for a phase factor. With this unitary evolution there is a measurement on the auxiliary system that decides unequivocally between $|s_1\rangle$ and $|s_2\rangle$; if the former occurs, then a measurement between $|p_1\rangle$ and $|q_1\rangle$ decides between $|p\rangle$ and $|q\rangle$, and if the latter occurs the state of the primary system is inevitably $|p_2\rangle$ (or equivalently $|q_2\rangle$), leaving the choice between $|p\rangle$ and $|q\rangle$

0375-9601/95/$09.50
SSDI 0375-9601(94)00919-8

undecided. The procedure of Ivanovic is more cumbersome, requiring possibly an infinite number of steps.

The present note extends the work of Ivanovic, Dieks, and Peres in several ways. The first is to allow a proportion r of systems of the ensemble to be in $|p\rangle$ and $s=1-r$ to be in $|q\rangle$, instead of requiring $r=s=\frac{1}{2}$. With no loss of generality,

$$r \geqslant s . \tag{3}$$

The procedure for solving problem (1) is the same as that of Dieks and Peres, except for the evaluation of the coefficients on the r.h.s. of (2a) and (2b). From Eqs. (2a) and (2b), and the assumed proportions r and s, we have

$$P=r|\alpha|^2+s|\gamma|^2=1-r|\beta|^2-s|\delta|^2 . \tag{4}$$

By unitarity and (2c), (2d)

$$|\langle p|q\rangle|=|\beta|\ |\delta|\ |\langle p_2|q_2\rangle| , \tag{5a}$$

and hence

$$|\beta|\ |\delta| \geqslant |\langle p|q\rangle| . \tag{5b}$$

The maximum of P in Eq. (4), subject to the constraint of Eq. (5b) and to $0\leqslant|\beta|^2\leqslant 1$, $0\leqslant|\delta|^2\leqslant 1$, is achieved when

$$|\beta|^2=\max\{|\langle p|q\rangle|(s/r)^{1/2}, |\langle p|q\rangle|^2\} \tag{6a}$$

and

$$|p_2\rangle=|q_2\rangle . \tag{6b}$$

(Note: $|\beta|^2$ is not permitted to be less than $|\langle p|q\rangle|^2$, for if it were, then $|\delta|$ would exceed unity.) If

$$|\beta|^2=|\langle p|q\rangle|(s/r)^{1/2}\geqslant|\langle p|q\rangle|^2 , \tag{7a}$$

then

$$|\delta|^2=|\langle p|q\rangle|(r/s)^{1/2} \tag{8a}$$

and

$$P=1-2(rs)^{1/2}|\langle p|q\rangle| , \tag{9a}$$

but if

$$|\beta|^2=|\langle p|q\rangle|^2\geqslant|\langle p|q\rangle|(s/r)^{1/2} , \tag{7b}$$

then

$$|\delta|^2=1 \tag{8b}$$

and

$$P=r(1-|\langle p|q\rangle|^2) . \tag{9b}$$

Note that if $r=s=\frac{1}{2}$, then Eq. (7a) holds and hence (9a), which in this case agrees with Eq. (1a).

A second extension of the results of Ivanovic, Dieks, and Peres is to assume that the dimensionality of the Hilbert space of the system of interest is greater than two. It is then possible to parallel the strategy of Dieks and Peres without introducing an auxiliary system. We show that because of the dimensionality of the Hilbert space, we can express $|p\rangle$ and $|q\rangle$ in the form

$$|p\rangle=\alpha|p_1\rangle+\beta|p_2\rangle , \tag{10a}$$

$$|q\rangle=\gamma|q_1\rangle+\delta|p_2\rangle , \tag{10b}$$

where $|p_1\rangle$, $|q_1\rangle$ and $|p_2\rangle$ are orthonormal and β and δ satisfy Eqs. (5b) and (6a), and α and γ are real. To achieve this expression we first write

$$|q\rangle=\mathrm{e}^{\mathrm{i}\theta}N|p\rangle+(1-N^2)^{1/2}|m\rangle , \tag{11a}$$

$$N=|\langle p|q\rangle| , \tag{11b}$$

where $|m\rangle$ is normalized and orthogonal to $|p\rangle$. Then Eq. (10b) is equivalent to

$$\mathrm{e}^{\mathrm{i}\theta}N|p\rangle+(1-N^2)^{1/2}|m\rangle=\gamma|q_1\rangle+\delta|p_2\rangle . \tag{10c}$$

There exists in the Hilbert space a normalized vector $|l\rangle$ orthogonal to $|p\rangle$ and $|m\rangle$, and we shall express $|p_1\rangle$, $|q_1\rangle$ and $|p_2\rangle$ explicitly in terms of $|p\rangle$, $|m\rangle$ and $|l\rangle$. At this juncture we must treat the cases of (7a) and (7b) separately.

If (7a) holds, make the following identifications of β, δ, α, γ, $|p_1\rangle$, $|q_1\rangle$, and $|p_2\rangle$,

$$\beta=N^{1/2}(s/r)^{1/4} , \tag{12a}$$

$$\delta=\mathrm{e}^{\mathrm{i}\theta}N^{1/2}(r/s)^{1/4} , \tag{12b}$$

$$\alpha=[1-(s/r)^{1/2}N]^{1/2} , \tag{12c}$$

$$\gamma=[1-(r/s)^{1/2}N]^{1/2} , \tag{12d}$$

$$\begin{aligned}|p_1\rangle&=[1-(s/r)^{1/2}N]^{1/2}|p\rangle\\&\quad-\frac{\mathrm{e}^{-\mathrm{i}\theta}}{(1-N^2)^{1/2}}N[1-(s/r)^{1/2}N]^{1/2}|m\rangle\\&\quad+\frac{1}{(1-N^2)^{1/2}}N^{1/2}(s/r)^{1/4}[1-(r/s)^{1/2}N]^{1/2}|l\rangle,\end{aligned} \tag{12e}$$

$$|q_1\rangle = \frac{1}{(1-N^2)^{1/2}} [1-(r/s)^{1/2}N]^{1/2}|m\rangle + \frac{e^{i\theta}}{(1-N^2)^{1/2}} N^{1/2}(r/s)^{1/4}[1-(s/r)^{1/2}N]^{1/2}|l\rangle , \tag{12f}$$

$$|p_2\rangle = N^{1/2}(s/r)^{1/4}|p\rangle + \frac{e^{-i\theta}}{(1-N^2)^{1/2}} N^{1/2}(r/s)^{1/4}[1-(s/r)^{1/2}N]|m\rangle - \frac{1}{(1-N^2)^{1/2}} [1-(r/s)^{1/2}N]^{1/2} \times [1-(s/r)^{1/2}N]^{1/2}|l\rangle . \tag{12g}$$

Then eqs. (10a), (10b) are satisfied with the requisite orthonormality of $|p_1\rangle$, $|q_1\rangle$, and $|p_2\rangle$, while β and δ satisfy Eqs. (5b) and (6a) and α and γ are real. Hence we can optimally decide with certainty (but sometimes abstaining from a decision) whether the system was initially in $|p\rangle$ or in $|q\rangle$ by a procedure analogous to that given in Eqs. (4)–(9a). Specifically, measure

$$A = |p_1\rangle\langle p_1| + 2|q_1\rangle\langle q_1| + 3|p_2\rangle\langle p_2| . \tag{13}$$

If the measured value of A is 1, we know with certainty that the system was prepared in $|p\rangle$; if the value is 2, we know with certainty that it was prepared in $|q\rangle$; and if the value is 3, we do not know and abstain from deciding. The probability of a correct classification is given by Eq. (9a). This agreement shows that the essence of the procedure is to increase the dimensionality of the relevant Hilbert space beyond the two dimensions determines by $|p\rangle$ and $|q\rangle$. If the Hilbert space of the system of interest is intrinsically greater than two, then one need not introduce an auxiliary system in the manner of Dieks and Peres.

In the case of Eq. (7b), $|\delta|$ is unity and therefore Eqs. (10a), (10b) reduce to

$$|p\rangle = \alpha|p_1\rangle + \beta|p_2\rangle , \tag{14a}$$

$$|q\rangle = \delta|p_2\rangle , \tag{14b}$$

where

$$|\beta| = N \equiv |\langle p|q\rangle| , \tag{15a}$$

$$|\alpha| = (1-N^2)^{1/2} . \tag{15b}$$

Now $|p\rangle$ can be rewritten

$$|p\rangle = e^{i\theta}N|q\rangle + (1-N^2)^{1/2}|n\rangle , \tag{16}$$

where $|n\rangle$ is normalized and orthogonal to $|q\rangle$. An optimum procedure for classifying a system as being in $|p\rangle$ or $|q\rangle$ with certainty is to measure the observable

$$B = |n\rangle\langle n| + 2|q\rangle\langle q| . \tag{17}$$

If the measured value is 1, then the system can be inferred with certainty to have been prepared in $|p\rangle$; if 2, then there is no decision. The probability of a correct classification is

$$P = r|\alpha|^2 = r(1-|\langle p|q\rangle|^2) , \tag{18}$$

in agreement with Eq. (9b). Note that the procedure can be carried out in a Hilbert space of dimension two and hence is more economical than the procedure in case (7a).

A third extension is previous work is to solve problem (1), by finding a procedure for maximizing on the average a correct classification. Again, we consider the general preparation in which a proportion r is prepared in $|p\rangle$ and $s \equiv 1-r$ in $|q\rangle$. We seek a bivalent procedure, which prescribes choosing $|p\rangle$ upon one outcome and $|q\rangle$ upon the other. There is no loss of generality in restricting the bivalent procedure to measuring a projection operator E on the Hilbert space of the system, with eigenvalues [1] 1 and 0. If the measured value of E is 1 choose $|p\rangle$, if 0 choose $|q\rangle$. The probability of a correct choice is

$$P = r\langle p|E|p\rangle + s(1-\langle q|E|q\rangle) . \tag{19}$$

Decompose $|q\rangle$ into a superposition of orthonormal $|p\rangle$ and $|m\rangle$, as in Eq. (11a), and consider the action of the projection operator E on these two vectors,

$$E|p\rangle = c|p\rangle + c'|m\rangle + c''|l\rangle , \tag{20a}$$

$$E|m\rangle = d|p\rangle + d'|m\rangle + d''|l'\rangle , \tag{20b}$$

where $|l\rangle$ and $|l'\rangle$ are orthonormal to both $|p\rangle$ and $|m\rangle$. (Of course, if the Hilbert space is two-dimensional, there are no terms in $|l\rangle$ and $|l'\rangle$.) Then

$$\langle p|E|p\rangle = c \tag{21a}$$

[1] The procedure used here was presented in Section 2 of Ref. [4] by Jaeger et al. In that paper the procedure was used for prediction, whereas here it is used for retrodiction.

and

$$\langle q|E|q\rangle = N^2c + 2\cos(\theta+\phi)N(1-N^2)^{1/2}|c'| + (1-N^2)d' , \quad (21b)$$

where

$$d^* = c' = |c'|e^{i\theta} . \quad (21c)$$

Since $E^2=E$, $\langle x|E|x\rangle = \langle Ex|Ex\rangle$ for any $|x\rangle$, and therefore c and d'' are real and

$$c = c^2 + |c'|^2 + |c''|^2 , \quad (22a)$$

$$d' = |d|^2 + d'^2 + |d''|^2 . \quad (22b)$$

Hence

$$P = rc + s[1 - N^2c - (1-N^2)d' - 2\cos(\theta+\phi)N(1-N^2)^{1/2}|c'|] . \quad (23)$$

When c, d', $|c'|$ are fixed, P is maximized by $\cos(\theta+\phi) = -1$,

$$P = (r - sN^2)c + s - s(1-N^2)d' + 2sN(1-N^2)^{1/2}|c'| . \quad (24)$$

By (22a), (22b)

$$c = \tfrac{1}{2} \pm \tfrac{1}{2}[1 - 4(|c'|^2 + |c''|^2)]^{1/2} , \quad (25a)$$

$$d' = \tfrac{1}{2} \pm \tfrac{1}{2}[1 - 4(|c'|^2 + |d''|^2)]^{1/2} \quad (25b)$$

and since $r \geqslant s$ we shall take the + for c, − for d'. Moreover, when $|c'|$ is fixed, c is maximized and d' is minimized when $c'' = d'' = 0$, in other words, when E is a projection operator on the two-dimensional Hilbert space spanned by $|p\rangle$ and $|q\rangle$. Then

$$c = \tfrac{1}{2}(1+\kappa) \quad (26a)$$

$$d' = \tfrac{1}{2}(1-\kappa) , \quad (26b)$$

where

$$\kappa \equiv (1 - 4|c'|^2)^{1/2} \quad (26c)$$

$$|c'| = \tfrac{1}{2}(1-\kappa^2)^{1/2} . \quad (26d)$$

Hence

$$P = \tfrac{1}{2} + \kappa(\tfrac{1}{2} - sN^2) + sN(1-N^2)^{1/2}(1-\kappa^2)^{1/2} . \quad (27)$$

Maximum P yields

$$\kappa = \frac{\frac{1}{2} - sN^2}{[(\frac{1}{2} - sN^2)^2 + s^2N^2(1-N^2)]^{1/2}} \quad (28)$$

and

$$P = \tfrac{1}{2} + \tfrac{1}{2}(1 - 4rs|\langle p|q\rangle|^2)^{1/2} . \quad (29)$$

The procedure for achieving this optimum probability of a correct classification is to measure the projection operator E determined by the κ of Eq. (28), and choose $|p\rangle$ if the outcome is 1, $|q\rangle$ if the outcome is 0. We emphasize that this procedure works whether the dimension of the Hilbert space of the system of interest is two or greater than two.

For the case of $r = s = \frac{1}{2}$, Eq. (29) implies

$$P = \tfrac{1}{2} + \tfrac{1}{2}(1 - |\langle p|q\rangle|^2)^{1/2} . \quad (30)$$

For this case, Dieks considers (without claiming optimality) a procedure which yields a probability P_D of correct classification,

$$P_D = 1 - \tfrac{1}{2}|\langle p|q\rangle|^2 . \quad (31)$$

But

$$P - P_D = \tfrac{1}{2}[(1 - |\langle p|q\rangle|^2)^{1/2} - (1 - |\langle p|q\rangle|^2)] \geqslant 0 , \quad (32)$$

and therefore our procedure is preferable to the one considered by Dieks.

As a final remark, we consider a class of practical situations in some of which the procedure of problem (1) is appropriate, and in others the procedure of problem (2). Suppose that a wager can be made about each system of the ensemble, with a gain $g > 0$ if the classification is correct, a loss $l > 0$ if the classification is incorrect, and neither gain nor loss if the subject refrains from betting. The average gain using the optimum procedure of problem (2) is

$$G_2 = g[1 - 2(rs)^{1/2}|\langle p|q\rangle|] \quad (33a)$$

if Eq. (7a) holds, and

$$G_2 = gr(1 - |\langle p|q\rangle|^2) \quad (33b)$$

if Eq. (7b) holds. The average gain using the optimum procedure of problem (1) is

$$G_1 = g[\tfrac{1}{2} + \tfrac{1}{2}(1 - 4rs|\langle p|q\rangle|^2)^{1/2}] - l[\tfrac{1}{2} - \tfrac{1}{2}(1 - 4rs|\langle p|q\rangle|^2)^{1/2}] . \quad (34)$$

If $2(rs)^{1/2}|\langle p|q\rangle|$ is less than unity, then clearly for

l sufficiently larger than g we have $G_1 < G_2$. In other words, if the penalty for an incorrect choice is sufficiently severe, then practical reason dictates a procedure which makes no incorrect classifications and as many correct classifications as possible. Again if $2(rs)^{1/2}|\langle p|q\rangle|$ is less than unity but $g=l$, then $G_2 \leqslant G_1$, as we can check for the two cases of Eq. (7a) and (7b). If Eq. (7a) holds, then

$$(G_1^2 - G_2^2)/g^2 = (1 - 4rs|\langle p|q\rangle|^2)$$
$$- [1 - 4(rs)^{1/2}|\langle p|q\rangle| + 4rs|\langle p|q\rangle|^2]$$
$$= 4(rs)^{1/2}|\langle p|q\rangle| - 8rs|\langle p|q\rangle|^2$$
$$= 4(x - 2x^2) \geqslant 1 ,$$

where $x = (rs)^{1/2}|\langle p|q\rangle| \leqslant \frac{1}{2}$.

If Eq. (7b) holds, then

$$(G_1^2 - G_2^2)/g^2 = (1 - 4rs|\langle p|q\rangle|^2$$
$$- (r^2 - 2r^2|\langle p|q\rangle|^2 + r^2 \int \langle p|q\rangle|^4)$$
$$\geqslant (1 - 4r + 4r^2)|\langle p|q\rangle|^2$$
$$+ r^2(|\langle p|q\rangle|^2 - |\langle p|q\langle|^4)$$
$$\geqslant 1 \quad \text{for} \quad \tfrac{1}{2} \leqslant r \leqslant 1 .$$

In general, there exist real numbers ρ_a and ρ_b, both greater than unity, such that procedure 2 is preferable to procedure 1 if and only if

$$1/g \geqslant \rho_a \quad \text{(if Eq. (7a) holds)} ,$$
$$1/g \geqslant \rho_b \quad \text{(if Eq. (7b) holds)} .$$

The research for this paper was supported by the National Science Foundation, grant No. PHY-9321992.

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