Quantum Lorentz-group invariants of *n*-qubit systems

Mihail Teodorescu-Frumosu[1] and Gregg Jaeger[2,*]
[1]*Department of Mathematics, Boston University, Boston, Massachusetts 02215*
[2]*Quantum Imaging Laboratory, Department of Electrical and Computer Engineering, Boston University, Boston 02215*
(Received 25 July 2002; published 21 May 2003)

We consider the behavior of quantum states under stochastic local quantum operations and classical communication (SLOCC) for an arbitrary fixed number of qubits. We use a real (Lorentz) group to describe the action of SLOCC operations on *n*-qubit states. We discuss the natural quantum Lorentz-group group-invariant length for an arbitrary number of qubits. We relate this approach to that based on local operations and classical communication and provide an example of how the invariant length can be used to describe entanglement. We also note that this invariant length is the Minkowskian analog to the quantum state purity, which is the corresponding Euclidean length.

DOI: 10.1103/PhysRevA.67.052305 PACS number(s): 03.67.Hk, 03.65.Ta

I. INTRODUCTION

It is known that the state of a single classical spin has an associated invariant length under transformations of the proper Lorentz group $O_0(1,3)$, of classical state transformations [1]. In the study of quantum information, one is most often concerned with the effect on quantum spins (i.e., qubits) of transformations corresponding to local operations and classical communications (LOCC) [2,3] and stochastic local operations and classical communication (SLOCC) [4,5]. The latter are most often described by the group SL(2,C) acting on the complex, density-matrix description of the quantum state of these qubits, which is homomorphic to the Lorentz group, of which it is a double cover [6]. Here, we discuss the real description of the quantum state and examine the corresponding Lorentz-group invariant length for every possible finite number of qubits. This length is the Minkowskian analog of the quantum state purity, which is the corresponding Euclidean length. It is a new tool for describing the behavior of states of any finite number of qubits under SLOCC, the invariants of which had until its introduction [14] been studied for only two qubits (and two-qubit reduced states from three-qubit pure states) using matrix methods but with encouraging results [5].

II. SINGLE QUBITS

In classical physics, one can use the expectation values of the Pauli spin matrices to fully characterize a state of spin, and to visualize it geometrically via a Poincaré sphere. As Han *et al.* [7] have pointed out, these classical parameters form a Minkowskian four-vector under the group of transformations corresponding to ordinary and hyperbolic state rotations. In particular, the elements of the group of proper Lorentz transformations $O_0(1,3)$ acting on the classical Stokes vector can be represented as products of real and hyperbolic rotations, for example,

$$R(\alpha)=\begin{pmatrix}1&0&0&0\\0&\cos\alpha&-\sin\alpha&0\\0&\sin\alpha&\cos\alpha&0\\0&0&0&1\end{pmatrix}, \tag{1a}$$

$$B(\chi)=\begin{pmatrix}\cosh\chi&\sinh\chi&0&0\\\sinh\chi&\cosh\chi&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \tag{1b}$$

that preserve an associated invariant length (cf. [6]). To similarly investigate the properties of qubit states, it is illuminating first to consider the Lorentz group transformations in correspondence to transformations on elements of $H(2)$, the vector space of all 2×2 complex Hermitian matrices that includes the density matrices describing states of single qubits.

The state of a quantum ensemble of independent qubits can be completely described by the set of expectation values

$$x_\mu=\mathrm{Tr}(\rho\sigma_\mu)\quad(\mu=0,1,2,3), \tag{2}$$

where $\sigma_0=1_{2\times2}$ and σ_i, $i=1,2,3$ are the Pauli matrices. Likewise, one can write the density matrix as

$$\rho=\frac{1}{2}\sum_{\mu=0}^{3}x_\mu\sigma_\mu, \tag{3}$$

and the vector space for one-qubit state vectors is $\mathcal{C}^2$. Since $\sigma_\mu^2=1$ and $\frac{1}{2}\mathrm{Tr}(\sigma_\mu\sigma_\nu)=\delta_{\mu\nu}$, the four Pauli matrices form a basis for $H(2)$ of which the density matrices ρ are the positive-definite, elements of unit trace (i.e., those for which $x_0\equiv1$), that capture the general qubit state, pure or mixed.

Now consider these expectation-value vectors in the Minkowskian real vector space $R^4_{1,3}$ the four-dimensional real vector space R^4 endowed with the Minkowski metric $(+,-,-,-)$, i.e., together with a metric tensor $g^{\mu\nu}$ possessing, as nonzero elements, the diagonal entries $+1$, -1, -1, and -1. The length of a four-vector x_μ in $R^4_{1,3}$ is given

*Corresponding author. Email address: jaeger@photon.bu.edu

by $\langle x,x\rangle = g^{\mu\nu}x_\mu x_\nu$. More explicitly, in $R^4_{1,3}$, the length of a vector $x=(x_0,x_1,x_2,x_3)$ is given by

$$\|x\|^2_{R^4_{1,3}} = x_0^2 - x_1^2 - x_2^2 - x_3^2. \tag{4}$$

Using the standard vector basis for R^4, $e_0=(1,0,0,0)$, $e_1=(0,1,0,0)$, $e_2=(0,0,1,0)$, $e_3=(0,0,0,1)$, there exists a natural vector-space isomorphism, $\nu: R^4_{1,3}\to H(2)$, relating the space $R^4_{1,3}$ of these vectors and the space of state matrices $H(2)$ defined by

$$\nu(x_0,x_1,x_2,x_3) = x_0\sigma_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3. \tag{5}$$

Moreover, this isomorphism straightforwardly relates the corresponding basis elements for the space of expectation-value vectors to those for the space of density matrices, namely, $\nu(e_i)=\sigma_i$ [6]. If we then define the norm on the space of density matrices, $H(2)$ to be

$$\|X\|^2_{H(2)} = \det X \quad \forall \quad X\in H(2), \tag{6}$$

then the isomorphism ν between the spaces of these real vectors and the Hermitian matrices becomes a length-preserving mapping, i.e., an isometry, since we have the following simple relationship between lengths in the two spaces:

$$\|\nu(x_0,x_1,x_2,x_3)\|^2_{H(2)} \equiv \det X = x_0^2 - x_1^2 - x_2^2 - x_3^2 = \|x\|^2_{R^4_{1,3}}. \tag{7}$$

The associated mathematical structures are described in detail in Appendix A. We see immediately from Eq. (4) that the Minkowskian length l^2 of the vector of expectation values is

$$l^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2, \tag{8}$$

similar to its analog in the classical realm and invariant under the Lorentz group of transformations represented by the basic forms R and B. This group of transformations goes beyond the limited context of unitary transformations of density matrices (for which $x_0\equiv 1$) (LUTs), to include nonunitary transformations corresponding in the real description to transformations such as B. The loci of constant l^2 are three-dimensional hyperboloids lying within what is the analog of the "forward light cone" of special relativity and the "null cone." These matrices belong to equivalence classes, each represented by a set of pure state density matrices but possessing a range of ensemble relative sizes, x_0. When the corresponding transformation of the density matrix is an element of the SU(2) subgroup of SL(2,C), corresponding to a unitary transformation of density matrices into density matrices (LUT) and x_0 is strictly unity, the states lie within a locus of a fixed distance from the x_0 axis. However, when this transformation is one such as B, probability is lost or gained, so that this constraint is no longer obeyed and x_0 can take other values, x_0', and move to other locations within the hyperboloid corresponding to the same value of the invariant l^2. Expectation-value vectors thus contain information beyond the state information provided by density matrices alone, and can be used to study the behavior of states of a quantum ensemble under both unitary and nonunitary state transformations of the density matrix.

III. TWO OR MORE QUBITS

Let us now consider the effect of SLOCC (the Lorentz group) on two-qubit systems, and find the corresponding invariant length. It is valuable to introduce the joint observables $x_{\mu\nu} = \mathrm{Tr}(\rho\sigma_\mu\otimes\sigma_\nu)$, where $\mu,\nu=0,1,2,3$, and express the matrix of the general state of a two-qubit ensemble [6,7]:

$$\rho = \frac{1}{4}\sum_{\mu,\nu=0}^{3} x_{\mu\nu}\sigma_\mu\otimes\sigma_\nu, \tag{9}$$

where $\sigma_\mu\otimes\sigma_\nu$ ($\mu,\nu=0,1,2,3$) are simply tensor products of the identity and Pauli matrices, and the state-vector space for pure states of two qubits is $\mathcal{C}^2\otimes\mathcal{C}^2$. The four-vector, x_μ, must then be generalized to a 16-element tensor $x_{\mu\nu}$ in order to capture all the quantum correlations potentially present in a two-qubit state.

The two-qubit density matrices ρ are positive, unit-trace elements of the 16-dimensional complex vector space of Hermitian 4×4 matrices $H(4)$. The tensors $\sigma_\mu\otimes\sigma_\nu\equiv\sigma_{\mu\nu}$ provide a basis for $H(4)$, which is isomorphic to the tensor-product space $H(2)\otimes H(2)$ of the same dimension, since $\frac{1}{4}\mathrm{Tr}(\sigma_{\mu\nu}\sigma_{\alpha\beta}) = \delta_{\mu\alpha}\delta_{\nu\beta}$ and $\sigma^2_{\mu\nu} = 1_{4\times 4}$, in analogy to the single-qubit case. We can write the two-qubit expectation values as [8,9]

$$x_{\mu\nu} = \mathrm{Tr}(\rho\sigma_\mu\otimes\sigma_\nu). \tag{10}$$

A density matrix for the general state of a two-qubit system is thus an element of $H(4)\simeq H(2)\otimes H(2)$ of the form

$$\rho = \frac{1}{4}\left(\sigma_0\otimes\sigma_0 + \sum_{i=1}^{3} x_{i0}\sigma_i\otimes\sigma_0 + \sum_{j=1}^{3} x_{0j}\sigma_0\otimes\sigma_j + \sum_{i,j=1}^{3} x_{ij}\sigma_i\otimes\sigma_j\right), \tag{11}$$

an element of the Hilbert-Schmidt space [7] that corresponds to

$$x = e_0\otimes e_0 + \sum_{i=1}^{3} x_{i0}e_i\otimes e_0 + \sum_{j=1}^{3} x_{0j}e_0\otimes e_j + \sum_{i,j=1}^{3} x_{ij}e_i\otimes e_j \tag{12}$$

in $R^4_{1,3}\otimes R^4_{1,3}$, expressed in terms of the elements of standard vector basis for R^4, $e_0=(1,0,0,0)$, $e_1=(0,1,0,0)$, $e_2=(0,0,1,0)$, $e_3=(0,0,0,1)$. The density matrices describing each qubit, 1 and 2, are given by partial tracing over the states of the other qubit: $\rho_1 = \mathrm{Tr}_{H(2)}\rho = \frac{1}{2}(1+\Sigma_{i=1}^3 x_{i0}\sigma_i)$, $\rho_1 = \mathrm{Tr}_{H(2)}\rho = \frac{1}{2}(1+\Sigma_{i=1}^3 x_{0i}\sigma_i)$. In Appendix B, we describe the associated mathematical structures in detail.

Again the length given by the tensor norm

$$l_{12}^2 \equiv \|x\|^2_{R^4_{1,3}\otimes R^4_{1,3}} = \langle x,x\rangle = (x_{00})^2 - \sum_{i=1}^{3} (x_{i0})^2 - \sum_{j=1}^{3} (x_{0j})^2 + \sum_{i=1}^{3}\sum_{j=1}^{3} (x_{ij})^2 \tag{13}$$

is invariant under Lorentz-group transformations $(A,B) \in \mathrm{O}_0(1,3)\times \mathrm{O}_0(1,3)$.

The generalization of the above methods to the case of n-qubits is straightforward, and allows us to find the invariant length for any finite number of qubits. Unlike previous approaches to applying the Lorentz group to quantum states [5], which used matrix methods and are therefore limited to transformations representable in simple matrix form, something problematic for an arbitrary number of qubits, the present tensorial treatment is entirely general [10]. Specifically, one can write the n-qubit state matrix in the form

$$\rho = \left(\frac{1}{2}\right)^n \sum_{i_1,\ldots,i_n=0}^{3} x_{i_1\ldots i_n}\sigma_{i_1}\otimes\cdots\otimes\sigma_{i_n}. \tag{14}$$

Since $\dim H(2^n) = \dim \otimes_{k=1}^n H(2) = \dim \otimes_{k=1}^n R^4_{1,3} = 4^n$, all these spaces are isomorphic, where $H(2^n)$ is the vector space of $2^n\times 2^n$ Hermitian matrices. The n-qubit ensemble expectation values are then

$$x_{i_1\cdots i_n} = \mathrm{Tr}(\rho\sigma_{i_1}\otimes\cdots\otimes\sigma_{i_n}). \tag{15}$$

Under the Lorentz group, the norm defined in $\otimes_{k=1}^n R^4_{1,3}$ remains invariant, i.e., under transformations of the form $\theta(A_1)\theta(A_2)\cdots\theta(A_n)$ with $A_k\in \mathrm{SL}(2,C)$, we have

$$l^2_{12\cdots n} = \|x\|^2 = x^2_{0\cdots 0} - \sum_{k=1}^{n}\sum_{i_k=1}^{3} (x_{0\cdots i_k\cdots 0})^2 + \sum_{k,l=1}^{n}\sum_{i_k,i_l=1}^{3} (x_{0\cdots i_k\cdots i_l\cdots 0})^2 + \cdots + (-1)^n \sum_{i_1\cdots i_n=1}^{3} (x_{i_1\cdots i_n})^2, \tag{16}$$

the invariant length associated with the general n-qubit state.

The n-qubit tensor $x_{i_1\cdots i_n}$ transforms under the group $\mathrm{O}_0(1,3)$ as

$$x'_{i_1\cdots i_n} = \sum_{j_1,\ldots,j_n=0}^{3} L^{j_1}_{i_1}\cdots L^{j_n}_{i_n} x_{j_1\cdots j_n}, \tag{17}$$

where the L^j_i are the Lorentz-group transformations acting in the spaces of qubits $1,\ldots,n$. Again, each Lorentz group of transformation $x_{\mu_1\mu_2\cdots\mu_n} \rightarrow x'_{\mu_1\mu_2\cdots\mu_n}$ of a given n-qubit expectation-value tensor will give rise to a new Hermitian state matrix ρ'. After transformation, the tensor element $x'_{0\cdots 0}$ is the new n-qubit ensemble relative size. Again, renormalizing ρ' provides the resulting density matrix for the ensemble: $\rho'' = \rho'/\mathrm{Tr}(\rho')$.

We note also that the quantum state purity, $\mathrm{Tr}\ \rho^2$, for a general n-photon state,

$$\mathrm{Tr}\ \rho^2 = \mathrm{Tr}\left[\left(\frac{1}{2}\right)^n \sum_{i_1,\ldots,i_n=0}^{3} x_{i_1\ldots i_n}\sigma_{i_1}\otimes\cdots\otimes\sigma_{i_n}\left(\frac{1}{2}\right)^n \sum_{j_1,\ldots,j_n=0}^{3} x_{j_1\ldots j_n}\sigma_{j_1}\otimes\cdots\otimes\sigma_{j_n}\right], \tag{18}$$

also has a particularly simple form in terms of the elements n-qubit four-tensor; since $(\sigma_{i_1}\otimes\sigma_{j_1})\otimes\cdots\otimes(\sigma_{i_n}\otimes\sigma_{j_n}) = \sigma_0 \otimes\cdots\otimes\sigma_0$ if and only if $i_k = j_k$, for all $k = 1,2,\ldots,n$, only the coefficient of the term $\sigma_0\otimes\cdots\otimes\sigma_0$ contributes to the trace, and we have

$$\mathrm{Tr}\ \rho^2 = \frac{1}{2^n}\sum_{i_1,\ldots,i_n=0}^{3} x^2_{i_1\ldots i_n}. \tag{19}$$

The state purity is thus seen to be the Euclidean analog of the Minkowskian invariant length introduced here.

IV. RELATION TO LOCC INVARIANTS

We now discuss our Lorentz-group (SLOCC) invariants in relation to known local invariants under LOCC in the context of studies of entanglement monotones as explored by Nielsen [2], Vidal [3], and others with an emphasis on its extension to nontrivial case of three qubits. The standard approach to LOCC invariance of multiple-qubit systems begins with the pure states $|\psi\rangle$ in $H_1\otimes H_2\otimes\cdots\otimes H_n$, Kraus operators $A_k^{(i)}: H_i \rightarrow H'_i$ acting in the Hilbert spaces H_i with $\Sigma_k A_k^{(i)\dagger}A_k^{(i)} = I_i$ (I_i being the identity on H_i), the investigation of entanglement monotones and the search for invariant quantities; an entanglement monotone is a real-valued function $E(|\psi\rangle)$ such that

$$E(|\psi\rangle) \geq \sum_k p_k E\left(\frac{(I_1\otimes\cdots\otimes A_k^{(i)}\otimes\cdots\otimes I_n)|\psi\rangle}{\sqrt{p_k}}\right) \tag{20}$$

for any state $|\psi\rangle$, operation $A_k^{(i)}$ and space i, where p_k

$=\|(I_1\otimes\cdots\otimes A_k^{(i)}\otimes\cdots\otimes I_n)|\psi\rangle\|^2$, which is nonincreasing under LOCC and possesses other valuable properties.

One approach for finding local invariants has been that of seeking them as polynomials. In the three-qubit case, we now point out that several LOCC polynomial invariants [11,12] are simply related to our Lorentz-group invariant lengths. To see this, first note that, given a wave function $|\psi\rangle$, one can write the equivalent density matrix, $\rho=|\psi\rangle\langle\psi|$ and define a corresponding "spin-flipped" density matrix [13] as $\tilde{\rho}\equiv(\sigma_2\otimes\cdots\otimes\sigma_2)\rho^*(\sigma_2\otimes\cdots\otimes\sigma_2)$, where $*$ denotes complex conjugation. Note also that one generally obtains a state only representable as a density matrix (as opposed to a state-vector) when one traces out properties associated with some particles of a multiple-qubit system by partial tracing over them in order to describe only those properties of other subsystems.

It is readily verified that one can write our quantum Lorentz-group invariant lengths in terms of density operators, that is,

$$l_{12\cdots n}^2=2^n\mathrm{Tr}(\rho_{12\cdots n}\tilde{\rho}_{12\cdots n}). \tag{21}$$

With this in mind, one can then see that three of the five polynomial LOCC invariants of the three-qubit pure state polynomial invariants [11], namely, I_1, I_2, and I_3 are readily expressible in terms of such traces over two-qubit reduced density matrices. In particular, one finds that

$$I_i=1-\frac{1}{2}[\mathrm{Tr}(\rho_{ij}\tilde{\rho}_{ij})+\mathrm{Tr}(\rho_{ik}\tilde{\rho}_{ik})], \tag{22}$$

that is,

$$I_i=1-\frac{1}{2^3}(l_{ij}^2+l_{ik}^2), \tag{23}$$

for $\{i,j,k\}=\{1,2,3\},\{2,1,3\},\{3,1,2\}$.

Finally, we present a specific simple example of how the Lorentz-group invariant length is useful for the study of quantum information (see Ref. [14] for a discussion of this length in a practical, optical context). This is particularly simple for a pair of entangled qubits. Namely, in the particular case of pure states of two qubits, this invariant length coincides with the concurrence squared, i.e., the tangle τ [13],

$$\tau=4l_{12}^2, \tag{24}$$

which has been a useful tool for measuring pure state entanglement that is directly related to the entanglement of formation, $h(\frac{1}{2}[1+\sqrt{1-4l_{12}^2}])$, where $h(x)\equiv-x\log_2 x-(1-x)\log_2(1-x)$, the well-known bipartite system entanglement monotone.

V. CONCLUSION

We have considered the application of the Lorentz group beyond the level of two-qubit states to the quantum state of an n-qubit quantum system. We showed that the multiple-qubit state expectation values form a Minkowskian tensor and give rise to invariant lengths, first introduced in Ref. [14] for each possible number n of qubits under the action of the Lorentz group. This length is the Minkowskian analog of the quantum state purity, which is the corresponding Euclidean length. We showed how in the case of three-qubit pure states, the known LOCC invariants can be expressed in terms of this length for $n=2$ for appropriate mixed states. We noted also that the quantum Lorentz-invariant length for $n=2$ is identical to the tangle in the case of pure states. This length for general n [14] is thus a natural, geometrical tool for describing the behavior of states of any finite number of qubits under SLOCC, which have thus far been studied, with positive results but for only two-qubit states and two-qubit reduced states of three-qubit pure states [3]. Our approach to SLOCC also has the advantage of being founded on a clear geometrical basis.

ACKNOWLEDGMENTS

G.J. acknowledges support from the DARPA QuIST program and the National Science Foundation (NSF) and encouragement from A. V. Sergienko.

APPENDIX A

Since the Pauli matrices are traceless and $\sigma_\mu^2=1_{2\times2}(\mu=0,1,2,3)$, we obtain the following expression for the inverse, $\nu^{-1}:H(2)\to R_{1,3}^4$, of the vector-space isomorphism of Eq. (7):

$$\nu^{-1}(X)=\frac{1}{2}(\mathrm{Tr}(X),\mathrm{Tr}(X\sigma_1),\mathrm{Tr}(X\sigma_2),\mathrm{Tr}(X\sigma_3))\ \forall\ X\in H(2), \tag{A1}$$

which maps the space of 2×2 Hermitian matrices containing the density matrices into the space, $R_{1,3}^4$, containing the quantum four-vectors. In particular, the density matrices of quantum mechanics are identified within the space of Hermitian matrices $H(2)$ as those having trace one, a condition guaranteeing that the sum of probabilities of all the possible events for the quantum state is unity.

We now define the contraction map $\lambda:H(2)\to H(2)$,

$$\lambda(X)=\frac{1}{2}X\ \forall\ X\in H(2)\ ,$$

which allows us to define the directly applicable isomorphism $\omega(x)=\lambda\circ\nu:R_{1,3}^4\to H(2)$ of the space containing expectation-value vectors to that containing the density matrices:

$$\omega(x_0,x_1,x_2,x_3)=\frac{1}{2}\sum_{i=0}^{3}x_\mu\sigma_\mu,\quad \forall x=(x_0,x_1,x_2,x_3)\in R_{1,3}^4. \tag{A2}$$

The corresponding inverse map $\omega^{-1}:H(2)\to R_{1,3}^4$ is

$$\omega^{-1}(X)=[\mathrm{Tr}(X),\mathrm{Tr}(X\sigma_1),\mathrm{Tr}(X\sigma_2),\mathrm{Tr}(X\sigma_3)]. \tag{A3}$$

As with the initial isomorphism ν, ω becomes an isometry if we define $\|\omega(x_0,x_1,x_2,x_3)\|^2_{H(2)} \equiv \det(2X) = \|x\|^2_{R^4_{1,3}}$.

The mapping ω^{-1} now directly returns the vector of expectation values, $x_\mu = \mathrm{Tr}(\rho\sigma_\mu)$ $(\mu=0,1,2,3)$, as desired.

Consider the group action α: $\mathrm{SL}(2,\mathbb{C})\times H(2)\rightarrow H(2)$, defined by

$$\alpha(A,X)=AXA^* \;\forall\; A\in \mathrm{SL}(2,C) \quad \text{and} \quad \forall X\in H(2), \tag{A4}$$

involving the density matrices. We see that the norm induced by the isomorphism ω is preserved under the mapping α, since

$$\|AXA^*\|^2_{H(2)} = \det(AXA^*) = |\det A|^2 \det X = \det X = \|X\|^2_{H(2)}. \tag{A5}$$

The natural group action, $\beta{:}\,\mathrm{O}_0(1,3)\times R^4_{1,3}\rightarrow R^4_{1,3}$, of the Lorentz group $\mathrm{O}_0(1,3)$ on the quantum observables, the elements of $R^4_{1,3}$ including the vectors describing this ensemble is defined by

$$\beta(x)=Bx \quad \forall \;\; B\in \mathrm{O}_0(1,3) \quad \forall \;\; x\in R^4_{1,3}, \tag{A6}$$

and is norm preserving (by definition), i.e., $\|Bx\|^2_{R^4_{1,3}} = \|x\|^2_{R^4_{1,3}}$.

Since the isomorphism ω of the expectation value space to the space containing the quantum states is an isometry, we can also define a map, $\theta{:}\,\mathrm{SL}(2,C)\rightarrow \mathrm{O}_0(1,3)$, between the transformations on elements of $H(2)$, including the density matrices, to those transformations of elements of $R^4_{1,3}$. The action of a matrix A on the matrices $X\in H(2)$ induces a corresponding Lorentz transformation $\theta(A)$ of vectors in $R^4_{1,3}$, such that $\|\omega^{-1}(AXA^*)\|_{R^4_{1,3}} = \|\theta(A)\omega^{-1}(X)\|_{R^4_{1,3}} = \|\omega^{-1}(X)\|_{R^4_{1,3}}$.

Note that θ has the property of being a group homomorphism, i.e., given $X\in H(2)$ and $A,B\in$ SL(2,C), θ has the following properties:

(i) $\theta(AB)\omega^{-1}(X) = \omega^{-1}(ABXB^*A^*) = \theta(A)\omega^{-1}(BXB^*) = \theta(A)\theta(B)\omega^{-1}(X)$, i.e., $\theta(AB)=\theta(A)\theta(B)$, and

(ii) $\omega^{-1}(X)1_2 = \omega^{-1}(X) = \omega^{-1}(1_2 X 1_2) = \theta(1_2)\omega^{-1}(X)$, i.e., $\theta(1_2)=1_2$.

Finally, by defining a map γ of the quantum state transformations into the corresponding transformations of the qubit expectation values, $\gamma{:}\,\mathrm{SL}(2,C)\times H(2)\rightarrow \mathrm{O}_0(1,3)\times R^4_{1,3}$,

$$\gamma(A,X)=[\theta(A),\omega^{-1}(X)], \tag{A7}$$

we obtain a commuting diagram, i.e., a set of mathematical objects and mappings such that any two mappings between any pair of objects obtained by composition of mappings are equal.

$$\begin{tikzcd}
\mathrm{SL}(2,C)\times H(2) \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & H(2) \\
\mathrm{O}_0(1,3)\times R^4_{1,3} \arrow[r, "\beta"] & R^4_{1,3} \arrow[u, "\omega"']
\end{tikzcd}$$

APPENDIX B

The isomorphism between the space of two-qubit expectation values and two-qubit density matrices, $\omega\otimes\omega$: $R^4_{1,3}\otimes R^4_{1,3}\rightarrow H(2)\otimes H(2)\simeq H(4)$ is defined as

$$(\omega\otimes\omega)(v\otimes w)\equiv \omega(v)\otimes\omega(w), \tag{B1}$$

for all $v,w\in R^4_{1,3}$. $\sigma_\mu\otimes\sigma_\nu$ form a basis for the required space of two-qubit Hermitian matrices $H(2)\otimes H(2)\simeq H(4)$ and $(\omega\otimes\omega)(e_\mu\otimes e_\nu)=\omega(e_\mu)\otimes\omega(e_\nu)=\sigma_\mu\otimes\sigma_\nu$ $(\mu,\nu=0,1,2,3)$. Furthermore, the inverse map taking density matrices to two-qubit tensors, $(\omega\otimes\omega)^{-1}$, $H(4)\rightarrow R^4_{1,3}\times R^4_{1,3}$ is given by

$$(\omega\otimes\omega)^{-1}(X)=\mathrm{Tr}(X\sigma_\mu\otimes\sigma_\nu), \tag{B2}$$

for all $X\in H(4)$. To describe the effect of the full set of group transformations, we use the map $\alpha\otimes\alpha{:}\,\mathrm{SL}(2,C)\times\mathrm{SL}(2,C)\times H(2)\otimes H(2)\rightarrow H(2)\otimes H(2)$, since for each qubit the group of transformations $\mathrm{SL}(2,C)$ acts via the action α on the vector space $H(2)$ that includes the density matrices. The action on the two-qubit Hermitian matrices is defined as

$$(\alpha\otimes\alpha)[(A,B),X\otimes Y]=(AXA^*)\otimes(BYB^*) \tag{B3}$$

for all $(A,B)\in \mathrm{SL}(2,C)\times\mathrm{SL}(2,C)$, and $\forall X,Y\in H(2)$. The action $\alpha\otimes\alpha$ is norm preserving on the tensor-product space, since

$$\|AXA^*\|^2_{H(2)}=\|X\|^2_{H(2)}, \tag{B4a}$$

$$\|BYB^*\|^2_{H(2)}=\|Y\|^2_{H(2)}. \tag{B4b}$$

Similarly, the action β of the Lorentz group $\mathrm{O}_0(1,3)$ on the space of expectation values, $R^4_{1,3}$, generalizes in the two-qubit case to $\beta\otimes\beta{:}\,\mathrm{O}_0(1,3)\times\mathrm{O}_0(1,3)\times R^4_{1,3}\otimes R^4_{1,3}\rightarrow R^4_{1,3}\otimes R^4_{1,3}$,

$$(\beta\otimes\beta)[(C,D),v\otimes w]=(Cv)\otimes(Dw) \tag{B5}$$

for all $(C,D)\in \mathrm{O}_0(1,3)\times\mathrm{O}_0(1,3)$ and $\forall v\otimes w\in R^4_{1,3}\otimes R^4_{1,3}$.

Since the isomorphism $\omega\otimes\omega$ is an isometry, as in the one-qubit case, we define the group homomorphism $\theta\times\theta{:}\,\mathrm{SL}(2,C)\times\mathrm{SL}(2,C)\rightarrow\mathrm{O}_0(1,3)\times\mathrm{O}_0(1,3)$. The action of the transformations $A\times B\in\mathrm{SL}(2,C)\times\mathrm{SL}(2,C)$ on the matrices $X\otimes Y\in H(2)\otimes H(2)$, which include the density matrices, induces a corresponding Lorentz-group transformation $\theta(A)\times\theta(B)$ on the space of expectation-value tensors $R^4_{1,3}\otimes R^4_{1,3}$,

$$\begin{aligned}&(\omega\otimes\omega)^{-1}[(AXA^*)\otimes(BYB^*)]\\ &\quad=\omega^{-1}(AXA^*)\otimes\omega^{-1}(BYB^*)\\ &\quad=\theta(A)\omega^{-1}(X)\otimes\theta(B)\omega^{-1}(Y).\end{aligned}\tag{B6}$$

The $\theta(A)\times\theta(B)$ are well-defined Lorentz-group transformations since, as before,

$$\|\omega^{-1}(AXA^*)\|^2_{R^4_{1,3}}=\|\theta(A)\omega^{-1}(X)\|^2_{R^4_{1,3}}=\|\omega^{-1}(X)\|^2_{R^4_{1,3}}$$

and

$$\|\omega^{-1}(BYB^*)\|^2_{R^4_{1,3}}=\|\theta(B)\omega^{-1}(Y)\|^2_{R^4_{1,3}}=\|\omega^{-1}(Y)\|^2_{R^4_{1,3}}.$$

Hence, defining the map acting on the space $H(2)\otimes H(2)$ including the density matrices, $\gamma\otimes\gamma$: $\mathrm{SL}(2,C)\times\mathrm{SL}(2,C)\times H(2)\otimes H(2)\rightarrow \mathrm{O}_0(1,3)\times\mathrm{O}_0(1,3)\times R^4_{1,3}\otimes R^4_{1,3}$, by

$$\begin{aligned}&(\gamma\otimes\gamma)[(A,B),(X\otimes Y)]\\ &\quad=([(\theta\times\theta)(A,B)],\ (\omega\otimes\omega)^{-1}(X\otimes Y))\\ &\quad=([\theta(A),\theta(B)],\ \omega^{-1}(X)\otimes\omega^{-1}(Y)),\end{aligned}\tag{B7}$$

for all $(A,B)\in\mathrm{SL}(2,C)\times\mathrm{SL}(2,C)$ and for all $X\otimes Y\in H(2)\otimes H(2)$, the following diagram is seen to commute:

$$\begin{tikzcd}
\mathrm{SL}(2,C)\times\mathrm{SL}(2,C)\times H(2)\otimes H(2) \arrow[r, "\alpha\otimes\alpha"] \arrow[d, "\gamma\otimes\gamma"'] & H(2)\otimes H(2) \\
\mathrm{O}_0(1,3)\times\mathrm{O}_0(1,3)\times R^4_{1,3}\otimes R^4_{1,3} \arrow[r, "\beta\otimes\beta"] & R^4_{1,3}\otimes R^4_{1,3} \arrow[u, "\omega\otimes\omega"']
\end{tikzcd}$$

demonstrating the welldefinedness of the mathematical structures used in our analysis in the case of two qubits. This construction generalizes in the now obvious way for the case of any finite number of qubits.

[1] See, for example, W.E. Baylis, J. Bonefant, J. Derbyshire, and J. Huschilt, Am. J. Phys. **61**, 534 (1993).

[2] M.A. Nielsen, Phys. Rev. Lett. **83**, 436 (1999).

[3] G. Vidal, Phys. Rev. Lett. **83**, 1046 (1999).

[4] C.H. Bennett, S. Popescu, D. Rohrlich, J.A. Smolin, and A.V. Thapliyal, Phys. Rev. A **63**, 012307 (2001); W. Dur, G. Vidal, and J.I. Cirac, *ibid.* **62**, 062314 (2000).

[5] F. Verstraete, J. Dehaene, and B. De Moor, Phys. Rev. A **65**, 032308 (2002); **64**, 010101(R) (2001).

[6] S. Sternberg, *Group Theory and Physics* (Cambridge University Press, Cambridge, 1994), Sec. 1.2.

[7] D. Han, Y.S. Kim, and M.E. Noz, Phys. Rev. E **56**, 6065 (1997); D. Han, Y.S. Kim, and M.E. Noz, *ibid.* **60**, 1036 (1999).

[8] U. Fano, J. Opt. Sci Am. **39**, 859 (1949); D.L. Falkoff and J.E. Macdonald, J. Opt Sci. Am. **41**, 862 (1951); U. Fano, Phys. Rev. **93**, 121 (1954); U. Fano, Rev. Mod. Phys. **29**, 74 (1957); U. Fano, *ibid.* **55**, 855 (1983).

[9] R. Horodecki and M. Horodecki, Phys. Rev. A **54**, 1838 (1996).

[10] In Ref. [5], the authors applied the Lorentz group to rank-two density operators of two qubits (in some cases arrived at by partial tracing out two qubits of a three-qubit pure state) and used the Lorentz singular-value decomposition of a matrix to arrive at local invariants.

[11] R.M. Gingrich, Phys. Rev. A **65**, 052302 (2002).

[12] H.A. Carteret, A. Higuchi, and A. Sudbery, J. Math. Phys. **41**, 7932 (2000).

[13] W.K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[14] G.S. Jaeger, M. Teodorescu-Frumosu, A.V. Sergienko, B.E.A. Saleh, and M.C. Teich, Phys. Rev. A **67**, 032307 (2003).