Two-coherent-state interferometry

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We examine interference phenomena involving entangled pairs of quantum coherent states. Two-coherent-state interferometry, which can involve macroscopic numbers of photons, is shown to share several characteristics of two-particle quantum interferometry in such a macroscopic limit. These include the complementarity between one-system and two-system interference visibilities in the extreme cases of product and maximally entangled quantum states, and the violation of a Bell-type inequality.

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I. INTRODUCTION

Over the last decade coherent-state interferometry and two-particle interferometry have provided new confirmations of quantum mechanics and greater violations of Bell-type inequalities [1–22]. Two-particle interferometry involves entangled microscopic (photon) systems. Two-coherent-state interferometry involves (coherent-state) systems that can be macroscopic while still behaving similarly to microscopic (photon) pairs [1–6], where a macroscopic state is characterized by having a large mean number of photons. The coherent-state superpositions are often referred to as ''Schrödinger cat states,'' emphasizing that quantum mechanics is being used to describe macroscopic physical systems [7–12]. Almost all of the analyses of quantum optical interferometry have centered on elements of strictly orthogonal Hilbert bases. However, the orthogonality of coherent states is only approximate and is strictly presently only in the large average particle number limit, $|\alpha|\to\infty$.

Here, in addition to showing evidence for the complementarity of one- and two-coherent state interference visibilities that is in accord with what is found in two-particle interferometry, we find a counter-intuitive result for a Bell-type inequality at odds with the correspondence principle. The correspondence principle demands that as a system gets more macroscopic, in the sense of going to large particle numbers, its behavior should become increasingly like that of the corresponding classical mechanical system. Thus, one expects that in the macroscopic limit quantum effects such as the violation of a Bell-type inequality will disappear. A previous analysis of Bell-type inequalities used nonorthogonal coherent states in the Schmidt decomposition. Surprisingly, the violation of a Bell-type inequality was shown to increase as the intensity of the coherent states increases [13]. Here, using a more concrete approach, we similarly find that violation of a Bell-type inequality increases as the system gets more macroscopic. This throws further doubt on the applicability of the correspondence principle here and, by extension, on the reducibility of classical to quantum optics.

II. THE TWO-COHERENT-STATE INTERFEROMETER

Two-coherent-state interferometry is the application of techniques of coherent-state recombination to macroscopic photon-system pairs of the general form

$$|\Psi\rangle=\frac{1}{\sqrt{2}}[|\alpha\rangle_1|\gamma\rangle_2+i|\delta\rangle_1|\beta\rangle_2], \tag{1}$$

where $|\alpha\rangle_1$ and $|\delta\rangle_1$ are near-orthonormal coherent-state vectors in the Hilbert space H_1 of system 1, and $|\beta\rangle_2$ and $|\gamma\rangle_2$ are elements of H_2 of system 2. States of the form (1) are entangled, i.e., they cannot be factorized in any way into the form $|\chi\rangle_1|\xi\rangle_2$, where $|\chi\rangle_1\in H_1$ and $|\xi\rangle_2\in H_2$. The new phenomena studied here arise when the production of entangled coherent-state pairs is combined with interferometric techniques tailored to coherent states. Here, we demonstrate that two-coherent-state interferometry produces results analogous to those achieved in two-particle interferometry. In particular, detection probabilities consistent with a complementarity between one-coherent-state and two-coherent-state visibilities are given and the violation of a Bell-type inequality is demonstrated. The following points are also emphasized: (i) two-coherent-state interferometry depends on the preparation of entangled coherent-state pairs; (ii) entangled states like the $|\Psi\rangle$ of Eq. (1) could be produced via the nonlinear interaction with Hamiltonian [7]

$$\hat{H}_I=\hbar\chi(\hat{a}^\dagger\hat{a})^n \tag{2}$$

for $n>1$ an integer, χ being proportional to the medium's nonlinear susceptibility of order $2n-1$; (iii) the phenomena described here depend on the utilization of a nonlinear version of well-known interferometers, such as the Mach-Zehnder interferometer; and (iv) quantum effects persist in a macroscopic limit.

The general schematic arrangement that we propose for two-coherent state interferometry is shown in Fig. 1. A

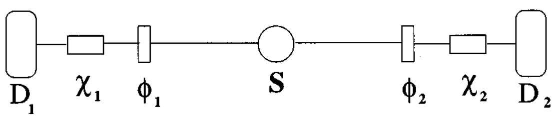

FIG. 1. Schematic experimental arrangement for two-coherent-state interferometry. The source S creates the entangled state given in Eq. (2). The output to wing 1 (2) proceeds to the coherent phase shifter $\phi_1(\phi_2)$, and then to the nonlinear cell $\chi_1(\chi_2)$, and finally to the coherent state detector $D_1(D_2)$.

source simultaneously produces macroscopic photon systems 1 and 2. In the most interesting case, each pair is prepared in the state

$$|\Psi\rangle=\frac{1}{\sqrt{2}}[|\alpha\rangle_1|-\alpha\rangle_2+i|-\alpha\rangle_1|\alpha\rangle_2], \tag{3}$$

a coherent superposition of two distinct pairs of correlated coherent states of systems 1 and 2. This state can be created by injecting coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ into the two input ports of a nonlinear Mach-Zehnder interferometer [11,20–22]. In one of these pairs, system 1 is in coherent state $|\alpha\rangle$ and undergoes a phase shift upon encountering the coherent-state phase shifter ϕ_1 [20] on the way to nonlinear cell 1, from which it enters detector 1; similarly, system 2 is in coherent-state $|-\alpha\rangle$ and encounters the coherent-state phase shifter ϕ_2 on the way to nonlinear cell 2, after which it enters detector 2. In the other pair of correlated coherent-states, system 1 is in coherent-state $|-\alpha\rangle$, encounters coherent-state phase shifter ϕ_1, proceeds to cell 1 and enters detector 1, while system 2, in state $|\alpha\rangle$ encounters ϕ_2, cell 2 and then enters detector 2. Note that in the limit of $|\alpha|\rightarrow\infty$, the state (3) can be converted to a Bell state by local unitary operations. Hence, the violation of Bell's inequality in this limit is expected to be maximal if the appropriate measurements can be made.

The transformation operator

$$\hat{K}=\exp[-i\chi(\hat{a}^\dagger\hat{a})^2] \tag{4}$$

is associated with the optical Kerr nonlinearity [24–26]. For the value $\chi=\pi/2$, this nonlinear operator acting on a coherent state creates a so-called Schrödinger cat state, a superposition of the $|\alpha\rangle$ and $|-\alpha\rangle$ coherent states [11]:

$$\hat{K}|\pm\alpha\rangle=\frac{1}{\sqrt{2}}(e^{-i\pi/4}|\pm\alpha\rangle+e^{i\pi/4}|\mp\alpha\rangle). \tag{5}$$

The unitary operator [20]

$$\hat{\phi}_\alpha=\exp(i\phi|\alpha\rangle\langle\alpha|) \tag{6}$$

denotes the evolution due to the coherent-phase shifters. This transforms a coherent input state $|\alpha\rangle$ with the result

$$\hat{\phi}_\alpha|\alpha\rangle=e^{i\phi}|\alpha\rangle. \tag{7}$$

This transformation is analogous to that for a normal phase-shifter transforming a single photon state, with the phase term *outside* the ket. However, due to the nonorthogonality of the coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ the coherent phase shifter for $|\alpha\rangle$, $\hat{\phi}_\alpha$, does not leave the $|-\alpha\rangle$ state completely unaltered; rather, one has

$$\hat{\phi}_\alpha|-\alpha\rangle=|-\alpha\rangle+(e^{i\phi}-1)\langle\alpha|-\alpha\rangle|\alpha\rangle. \tag{8}$$

Nonetheless, if α is large, so that $|\alpha|\rightarrow\infty$, then $|\langle\alpha|-\alpha\rangle|\rightarrow 0$ and the state $|-\alpha\rangle$ will remain effectively unchanged by the operation of the coherent phase shifter for the $\hat{\phi}_\alpha$. An exact experimental realization of the unitary operator (6) may not be possible, but an approximate realization is possible by exploiting media with higher-order nonlinear susceptibilities. This approximate realization is discussed in Appendix A.

Alternatively, the entangled coherent state with a variable phase shift,

$$|\Psi\rangle=\frac{1}{\sqrt{2}}[e^{i\phi_1}|\alpha\rangle_1|-\alpha\rangle_2+ie^{i\phi_2}|-\alpha\rangle_1|\alpha\rangle_2] \tag{9}$$

can be created directly [23].

In our two-coherent-state interferometer, the initial state (3) first encounters the coherent phase shifters [20] ϕ_1 and ϕ_2, with the effect

$$|\Psi\rangle\rightarrow\frac{1}{\sqrt{2}}[e^{i\phi_1}|\alpha\rangle_1|-\alpha\rangle_2+ie^{i\phi_2}|-\alpha\rangle_1|\alpha\rangle_2+\gamma|\alpha\rangle_1|\alpha\rangle_2], \tag{10}$$

where γ is given by

$$\gamma=e^{-2|\alpha|^2}[e^{i\phi_1}(e^{i\phi_2}-1)+ie^{i\phi_2}(e^{i\phi_1}-1)]. \tag{11}$$

Each nonlinear cell behaves analogously to a beamsplitter in a Mach-Zehnder interferometer, by recombining the $|\alpha\rangle$ and $|-\alpha\rangle$ states. Traversing the nonlinear cells, the normalized quantum state transforms as

$$\begin{aligned}|\Psi\rangle\rightarrow N[&(ie^{i\phi_1}-e^{i\phi_2}+\gamma)|\alpha\rangle_1|\alpha\rangle_2+(ie^{i\phi_1}-e^{i\phi_2}-\gamma)\\&\times|-\alpha\rangle_1|-\alpha\rangle_2+(e^{i\phi_1}-ie^{i\phi_2}+i\gamma)|\alpha\rangle_1|-\alpha\rangle_2\\&+(-e^{i\phi_1}+ie^{i\phi_2}+i\gamma)|-\alpha\rangle_1|\alpha\rangle_2],\end{aligned} \tag{12}$$

where N is the normalization factor,

$$\begin{aligned}N=\frac{1}{\sqrt{2}}\{&4+\exp(-4|\alpha|^2)[-2+\sin(\phi_1-\phi_2)\\&+\sin\phi_1-\sin\phi_2+\cos\phi_1+\cos\phi_2]\}^{-1/2}.\end{aligned} \tag{13}$$

III. BELL-TYPE INEQUALITY VIOLATION

The amplitudes for a coincidence measurement of any combination of the states $|\pm\alpha\rangle_1$ and $|\pm\alpha\rangle_2$ given the state (12) are

$$\begin{aligned}A(\pm\alpha,\pm\alpha|\phi_1,\phi_2)=N[&(1+e^{-4|\alpha|^2})(ie^{i\phi_1}-e^{i\phi_2})\\&+\gamma(\pm1+2ie^{-2|\alpha|^2}\mp e^{-4|\alpha|^2})],\end{aligned} \tag{14}$$

$$A(\pm\alpha,\mp\alpha|\phi_1,\phi_2)=N[\pm(1+e^{-4|\alpha|^2})(e^{i\phi_1}-ie^{i\phi_2})+2e^{-2|\alpha|^2}(ie^{i\phi_1}-e^{i\phi_2})+i\gamma(1+e^{-4|\alpha|^2})]. \quad (15)$$

The probabilities for a coincidence measurement are then calculated by multiplying the relevant amplitude with its complex conjugate. These coincident measurements of specific combinations of the coherent states can be achieved by applying quadrature-phase homodyne measurements [27–30].

The powers of $e^{-|\alpha|^2}$ that appear in Eqs. (14) and (15) are due to the nonorthogonality of the states $|\alpha\rangle$ and $|-\alpha\rangle$, since for an output state $|-\alpha\rangle$ there is a nonzero probability of measuring this system as $|\alpha\rangle$. One result of this nonorthogonality is that in the limit $|\alpha|\rightarrow 0$,

$$P(\alpha,\alpha),P(-\alpha,-\alpha),P(\alpha,-\alpha),P(-\alpha,\alpha)\rightarrow 1. \quad (16)$$

However, in the macroscopic limit where $|\alpha|\rightarrow\infty$,

$$0\leqslant P(\alpha,\alpha),P(-\alpha,-\alpha),P(\alpha,-\alpha),P(-\alpha,\alpha)\leqslant 1/2, \quad (17)$$

as occurs for two-particle interferometry with a pair of particles [1–6]. Give each detection a value: the detection of the $|\alpha\rangle$ state is designated by $D(\alpha)=1$, and the detection of the $|-\alpha\rangle$ state by $D(-\alpha)=-1$. Then, for a single experiment we have

$$E(\alpha,\phi_1,\phi_2)=P(\alpha,\alpha|\phi_1,\phi_2)+P(-\alpha,-\alpha|\phi_1,\phi_2)-P(\alpha,-\alpha|\phi_1,\phi_2)-P(-\alpha,\alpha|\phi_1,\phi_2). \quad (18)$$

As is usual for the CHSH-type Bell inequalities [31], we construct the function

$$B(\alpha,\phi_1,\phi_2,\phi_1',\phi_2')=E(\alpha,\phi_1,\phi_2)+E(\alpha,\phi_1,\phi_2')+E(\alpha,\phi_1',\phi_2)-E(\alpha,\phi_1',\phi_2'). \quad (19)$$

A Bell-locality violation would then be indicated by the result $|B|>2$. The value of $|B|$ is maximized when

$$\phi_1=\frac{3\pi}{4}, \quad \phi_2=0, \quad \phi_1'=\frac{\pi}{4}, \quad \phi_2'=\frac{\pi}{2}. \quad (20)$$

There is then a violation for sufficiently large values of $|\alpha|$. The larger $|\alpha|$, the larger the Bell locality violation, approaching the limit $|B|\rightarrow 2\sqrt{2}$ as $|\alpha|\rightarrow\infty$.

For $|\alpha|\rightarrow\infty$, we find that $e^{-2|\alpha|^2}\rightarrow 0$ and $e^{-4|\alpha|^2}\rightarrow 0$. In this case, the coincidence detection probabilities become, assuming perfect detector efficiencies (see Sec. IV below),

$$P(\alpha,\alpha|\phi_1,\phi_2)\rightarrow\frac{1}{4}[1+\sin(\phi_1-\phi_2)], \quad (21)$$

$$P(-\alpha,-\alpha|\phi_1,\phi_2)\rightarrow\frac{1}{4}[1+\sin(\phi_1-\phi_2)], \quad (22)$$

$$P(\alpha,-\alpha|\phi_1,\phi_2)\rightarrow\frac{1}{4}[1-\sin(\phi_1-\phi_2)], \quad (23)$$

$$P(-\alpha,\alpha|\phi_1,\phi_2)\rightarrow\frac{1}{4}[1-\sin(\phi_1-\phi_2)]. \quad (24)$$

The results in Eqs. (21), (22), (23), and (24) are analogous to those obtained for two-particle interferometry using entangled photon pairs [4,5,32], and a Bell-type inequality is similarly violated.

IV. EVIDENCE OF A VISIBILITY COMPLEMENTARITY

Assuming the detectors to have the efficiency η, the probability for joint detection of subsystems 1 and 2 in the states $|\alpha\rangle_1$ and $|\alpha\rangle_2$, when the phase shifters $i=1,2$ give phase shifts ϕ_i, to $|\alpha\rangle_i$, is η^2 times the square of the total amplitude $A(\alpha,\alpha|\phi_1,\phi_2)$. This amplitude is the superposition of the amplitudes associated with each of the two pairs of correlated coherent states:

$$A(\alpha,\alpha|\phi_1,\phi_2)=\frac{1}{\sqrt{2}}[(2^{-1/2})(2^{-1/2}i)e^{i\phi_1}+(2^{-1/2}i)(2^{-1/2})(e^{i\phi_2})i], \quad (25)$$

where the factors $e^{i\phi_1}$ and $e^{i\phi_2}$ arise from the phase shifters encountered along the respective beams and the factors $2^{-1/2}$ and $2^{-1/2}i$ arise from the encounters with the Kerr cells of states $|\alpha\rangle$ and $|-\alpha\rangle$, respectively. These factors are analogous to encounters of single particles with an ordinary beam splitter from opposite sides. Expressions analogous to Eq. (25) can be given for the amplitudes $A(\alpha,-\alpha|\phi_1,\phi_2)$, $A(-\alpha,\alpha|\phi_1,\phi_2)$, and $A(-\alpha,-\alpha|\phi_1,\phi_2)$ as well.

The probabilities of corresponding joint detections are η^2 times the absolute square of the respective amplitudes:

$$P(\alpha,\alpha|\phi_1,\phi_2)=P(-\alpha,-\alpha|\phi_1,\phi_2)=\eta^2\left[\frac{1}{4}+\frac{1}{4}\sin(\phi_1-\phi_2)\right], \quad (26)$$

$$P(\alpha,-\alpha|\phi_1,\phi_2)=P(-\alpha,\alpha|\phi_1,\phi_2)=\eta^2\left[\frac{1}{4}-\frac{1}{4}\sin(\phi_1-\phi_2)\right]. \quad (27)$$

The sinusoidal dependence of the joint detection probabilities on the phase shifts in (26) and (27) is characteristic of quantum mechanical interference. The modulation of coincidence detection while shifting ϕ_1 and ϕ_2 gives rise to interference fringes but does not give rise to fringes in single coherent-state detection. This can be seen from the following probabilities of single coherent-state detections:

$$\begin{aligned}P(\alpha,-|\phi_1,\phi_2)&=P(-\alpha,-|\phi_1,\phi_2)\\&=P(-,\alpha|\phi_1,\phi_2)\\&=P(-,-\alpha|\phi_1,\phi_2)\\&=\eta/2,\end{aligned}\tag{28}$$

where the ''$-$'' mark by itself indicates that the result of the corresponding detection is ignored. This exhibits the essentially two-coherent-state interference, inexplicable in terms of individual coherent-state behavior, and is directly analogous to what has been shown for two-particle interference [5,6]. Both sorts of phenomena are due to quantum entanglement in two-system states, in this case $|\Psi\rangle$.

When our two-coherent-state system is in a product state, by contrast, essentially single coherent-state behavior arises and no essentially two-coherent-state behavior is observed. For example, consider the detection probabilities when $|\Psi\rangle$ is replaced by $|\Phi\rangle$:

$$|\Phi\rangle=\frac{1}{\sqrt{2}}(|\alpha\rangle_1+i|-\alpha\rangle_1)\frac{1}{\sqrt{2}}(|\alpha\rangle_2+i|-\alpha\rangle_2),\tag{29}$$

a state still involving superpositions of elements of the single-system detection basis but of product form. In this case one obtains, for example, the joint-detection amplitudes

$$A(\alpha,\alpha|\phi_1,\phi_2)=\frac{1}{4}[e^{i(\phi_1+\phi_2)}-e^{i\phi_1}-e^{i\phi_2}+1],\tag{30}$$

$$A(\alpha,-\alpha|\phi_1,\phi_2)=\frac{1}{4}[ie^{i(\phi_1+\phi_2)}+ie^{i\phi_1}-ie^{i\phi_2}-i],\tag{31}$$

the joint-detection probabilities

$$P(\alpha,\alpha|\phi_1,\phi_2)=\frac{1}{4}(1-\cos\phi_1)(1-\cos\phi_2),\tag{32}$$

$$P(\alpha,-\alpha|\phi_1,\phi_2)=\frac{1}{4}(1-\cos\phi_1)(1+\cos\phi_2),\tag{33}$$

and the single-detection probabilities for the first coherent-state system:

$$P(\alpha)=\frac{1}{2}(1-\cos\phi_1),\tag{34}$$

$$P(-\alpha)=\frac{1}{2}(1+\cos\phi_1).\tag{35}$$

These results show that the state $|\Phi\rangle$ gives rise to single coherent-state fringes but no genuine two-coherent-state fringes. Equations (30)–(33) show that the apparent joint-detection fringing is actually explicable entirely in terms of single-detection fringing, in contrast to that of Eqs. (24)–(26).

V. CONCLUSIONS

We have shown how the interferometry of entangled pairs of quantum coherent states has several characteristics in common with two-particle quantum interferometry. These include the complementarity between one-system and two-system interference visibilities in the extreme cases of product and maximally entangled quantum states, and the violation of Bell-type inequalities. This quantum behavior persists even in the limit of macroscopic average particle numbers. Indeed, a Bell-type inequality is maximally violated in this limit.

The correspondence principle demands that as a system gets more macroscopic, its behavior should become increasingly like that of the corresponding classical mechanical system. Thus, the correspondence principle suggests that as coherent states become more macroscopic, the possibility of violating a Bell-type inequality should diminish. The results presented here throw doubt upon the universal validity of the correspondence principle. In these results, as the coherent states become more macroscopic, the results in fact become less classical, which is indicated by the increasing violation of a Bell-type inequality as $|\alpha|\rightarrow\infty$. This demonstrates that the standard quantum mechanics of an isolated system, on its own, does not naturally give rise to classical mechanics at any scale. However, in normal settings, field modes will decohere faster for higher values of α, so that classicality is usually observed.

APPENDIX: REALIZATION OF THE COHERENT-STATE PHASE SHIFTER

An approximate realization of the phase shift unitary operator $e^{i\phi\hat{\pi}_\alpha}$, for $\hat{\pi}_\alpha=|\alpha\rangle\langle\alpha|$ a coherent-state phase shift operator, is possible. Given that $\hat{\pi}_0=:e^{-a^\dagger a}:$, for : : denoting normal ordering [33], the coherent-state projection operator is

$$\begin{aligned}e^{i\phi\hat{\pi}_\alpha}=\hat{\mathbb{1}}+(e^{i\phi}-1)\hat{\pi}_\alpha&=D(\alpha)[\hat{\mathbb{1}}+(e^{i\phi}-1)\hat{\pi}_0]D^\dagger(\alpha)\\&=D(\alpha)\exp[i\phi:e^{-a^\dagger a}:]D^\dagger(\alpha),\end{aligned}\tag{A1}$$

where $D(\alpha)$ is the displacement operator [34]

$$D(\alpha)=e^{\alpha a^\dagger-\alpha^* a}.\tag{A2}$$

In the system we are considering, there are two states of concern: $|\pm\alpha\rangle$. Without loss of generality, we assume that α is real. In order to effect the phase shift operation, we act on either state first with $D^\dagger(\alpha)$. This maps $|\alpha\rangle$ to $|0\rangle$ and $|-\alpha\rangle$ to $|-2\alpha\rangle$. We then follow with the vacuum phase shift operator

$$e^{i\phi:e^{-a^\dagger a}:}\approx\exp\left[i\phi\left(1-a^\dagger a+\frac{1}{2}a^{\dagger 2}a^2-\frac{1}{3!}a^{\dagger 3}a^3\pm\cdots\right)\right]$$
$$=\exp\left[i\phi\left(1-\hat{n}+\frac{1}{2}\hat{n}(\hat{n}-1)-\frac{1}{3!}\hat{n}(\hat{n}-1)(\hat{n}-2)\pm\cdots\right)\right]. \quad (A3)$$

In the Fock state basis $\{|n\rangle\}$,

$$\langle m|e^{i\phi:e^{-a^\dagger a}:}|n\rangle=\exp\left[i\phi\sum_{k=0}^{n}(-1)^k\binom{n}{k}\right]\delta_{mn}. \quad (A4)$$

In order to realize the unitary transformation (A3), an interaction of the type (truncated at third order in powers of the number operator $\hat{n}$)

$$H_I/\hbar\approx\phi\left(1-a^\dagger a+\frac{1}{2}a^{\dagger 2}a^2-\frac{1}{3!}a^{\dagger 3}a^3\right) \quad (A5)$$

must be established. This interaction is a generalization of the Hamiltonian in Eq. (2). In an optical system the interaction Hamiltonian

$$H/\hbar=\phi+\kappa_1 a^\dagger a+\kappa_3 a^{\dagger 2}a^2+\kappa_5 a^{\dagger 3}a^3+\cdots \quad (A6)$$

can be produced, in principle, by choosing an appropriate nonlinear medium. The coefficient κ_n is proportional to the nonlinear susceptibility $\chi^{(n)}$ and the interaction time. In an optical system, it would be challenging to construct a medium satisfying (A5) because subsequent odd-ordered nonlinear susceptibilities do not diminish greatly. However, other realizations of superposition coherent states, for example, in ion traps, offer higher nonlinear susceptibilities and a greater promise of realizing this interaction [35].

The approximate expression (A5) relies on the validity of truncating the Fock-state expansion for the two states under consideration, namely, $|0\rangle$ and $|-2\alpha\rangle$. If a state $|\psi\rangle$ can be represented reasonably accurately in the Fock-state expansion as

$$|\psi\rangle=\sum_{n=0}^{\infty}\psi_n|n\rangle\approx\sum_{n=0}^{N}\psi_n|n\rangle, \quad (A7)$$

for N some non-negative integer, then the expansion of the unitary operator (A3) is valid to some order. If $N\leqslant 3$, then the expansion (A5) is sufficient. For the vacuum state, this condition is trivially satisfied, but the other state must be able to be accurately truncated as

$$|-2\alpha\rangle\approx\frac{|0\rangle-2\alpha|1\rangle+2\alpha^2|2\rangle-\frac{4}{3}\alpha^3|3\rangle}{\sqrt{1+4\alpha^2+4\alpha^4+\frac{16}{9}\alpha^6}}. \quad (A8)$$

Therefore,

$$e^{i\phi:e^{-a^\dagger a}:}|0\rangle=e^{i\phi}|0\rangle \quad (A9)$$

and

$$e^{i\phi:e^{-a^\dagger a}:}|-2\alpha\rangle\approx\frac{e^{i\phi}|0\rangle-2\alpha|1\rangle+2\alpha^2|2\rangle-\frac{4}{3}\alpha^3|3\rangle}{\sqrt{1+4\alpha^2+4\alpha^4+\frac{16}{9}\alpha^6}}$$
$$=|-2\alpha\rangle+\frac{e^{i\phi}-1}{\sqrt{1+4\alpha^2+4\alpha^4+\frac{16}{9}\alpha^6}}|0\rangle. \quad (A10)$$

We can see that the interaction (A5) leads to the desired transformation for sufficiently small α, and the algorithm for extending (A5) for larger α is straightforward.

We now act on either state with $D(\alpha)$, which maps $|0\rangle$ to $|\alpha\rangle$ and $|-2\alpha\rangle$ to $|-\alpha\rangle$. The whole process therefore produces the transformations

$$D(\alpha)e^{i\phi:e^{-a^\dagger a}:}D^\dagger(\alpha)|\alpha\rangle=e^{i\phi}|\alpha\rangle \quad (A11)$$

and

$$D(\alpha)e^{i\phi:e^{-a^\dagger a}:}D^\dagger(\alpha)|-\alpha\rangle$$
$$\approx|-\alpha\rangle+\frac{e^{i\phi}-1}{\sqrt{1+4\alpha^2+4\alpha^4+\frac{16}{9}\alpha^6}}|\alpha\rangle. \quad (A12)$$

This compares favorably with Eqs. (7) and (8).

[1] Z.Y. Ou and L. Mandel, Quantum Opt. **2**, 71 (1990).
[2] C.O. Alley and Y.H. Shih, Phys. Rev. Lett. **61**, 2921 (1988).
[3] X.Y. Zou, L.J. Wang, and L. Mandel, Phys. Rev. Lett. **67**, 318 (1991).
[4] J.G. Rarity and P.R. Tapster, Phys. Rev. Lett. **64**, 2495 (1990).
[5] M.A. Horne, A. Shimony, and A. Zeilinger, Phys. Rev. Lett. **62**, 2209 (1989).
[6] G. Jaeger, A. Shimony, and L. Vaidman, Phys. Rev. A **51**, 54 (1995).
[7] B. Yurke and D. Stoler, Phys. Rev. Lett. **57**, 13 (1986).
[8] N. Imoto, H.A. Haus, and Y. Yamamoto, Phys. Rev. A **32**, 2287 (1985).
[9] M. Shirasaki, H.A. Haus, and D.L. Wong, J. Opt. Soc. Am. B **6**, 82 (1989).
[10] B.C. Sanders and G.J. Milburn, Phys. Rev. A **39**, 694 (1989).
[11] B.C. Sanders, Phys. Rev. A **45**, 6811 (1992); **46**, 2966 (1992).

[12] B. Wielinga and B.C. Sanders, J. Mod. Opt. **40**, 192 (1993).
[13] A. Mann, B.C. Sanders, and W.J. Munro, Phys. Rev. A **51**, 989 (1995).
[14] B.C. Sanders, K.S. Lee, and M.S. Kim, Phys. Rev. A **52**, 735 (1995).
[15] I. Jex, P. Törmä, and S. Stenholm, J. Mod. Opt. **42**, 1377 (1995).
[16] C.C. Gerry, Phys. Rev. A **55**, 2478 (1997).
[17] G-C. Guo and S-B. Zheng, Opt. Commun. **133**, 142 (1997).
[18] J.M. Raimond, M. Brune, and S. Haroche, Phys. Rev. Lett. **79**, 1964 (1997).
[19] S. Bose, K. Jacobs, and P.L. Knight, Phys. Rev. A **56**, 4175 (1997).
[20] D.A. Rice and B.C. Sanders, Quantum Semiclassic. Opt. **10**, L41 (1998).
[21] B.C. Sanders and D.A. Rice, Opt. Quantum Electron. **31**, 525 (1999).
[22] B.C. Sanders and D.A. Rice, Phys. Rev. A **61**, 013805 (2000).
[23] S-B. Zheng, Quantum Semiclassic. Opt. **10**, 699 (1998).
[24] G.J. Milburn, Phys. Rev. A **33**, 674 (1986).
[25] G.J. Milburn and C.A. Holmes, Phys. Rev. Lett. **56**, 2237 (1986).
[26] A cavity with a moving mirror also causes a similar transformation (Ref. [19]).
[27] B. Yurke and D. Stoler, Phys. Rev. Lett. **79**, 4941 (1997).
[28] A. Gilchrist, P. Deuar, and M.D. Reid, Phys. Rev. Lett. **80**, 3169 (1998).
[29] W.J. Munro and G.J. Milburn, Phys. Rev. Lett. **81**, 4285 (1998).
[30] W.J. Munro, Phys. Rev. A **59**, 4197 (1999).
[31] J.F. Clauser, M.A. Horne, A. Shimony, and R.A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[32] M.A. Horne and A. Zeilinger, in *Symposium on the Foundations of Modern Physics*, edited by P. Lahti and P. Mittelstaedt (World Scientific, Singapore, 1985), p. 435.
[33] Hong-Yi Fan and H.R. Zaidi, Opt. Commun. **68**, 143 (1988).
[34] R.J. Glauber, Phys. Rev. **131**, 2766 (1963).
[35] P.T. Cochrane, G.J. Milburn, and W.J. Munro, Phys. Rev. A **59**, 2631 (1999).