Validated computation of spectral stability via conjugate points

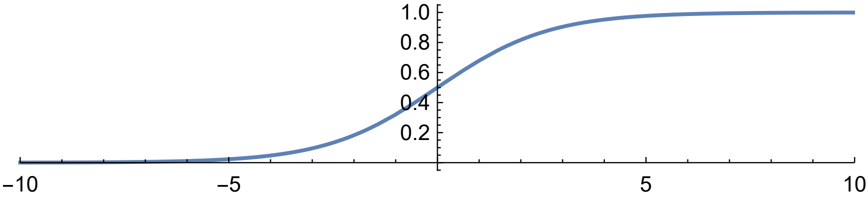
Margaret Beck, Jonathan Jaquette**

Boston University

The Twelfth IMACS International Conference on Nonlinear Evolution Equations and Wave Phenomena:
March 30, 2022

Nonlinear Waves

Coherent structures (eg pulse solutions and traveling waves) are an essential part of understanding nonlinear PDEs

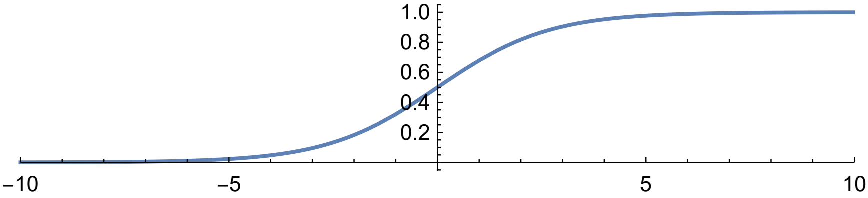


We prove existence and spectral stability of standing waves

- Novel approach based on Maslov index (cf Evans functions)
- This approach produces efficient numerics
- . . . and computer assisted proofs!

Nonlinear Waves

Coherent structures (eg pulse solutions and traveling waves) are an essential part of understanding nonlinear PDEs



We prove existence and spectral stability of standing waves

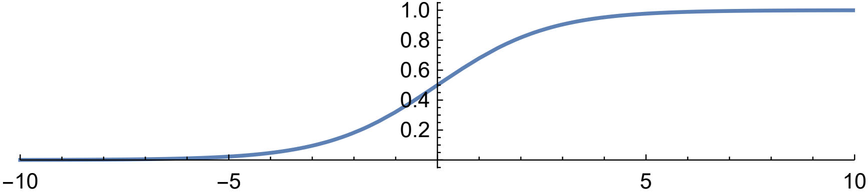
- Novel approach based on Maslov index (cf Evans functions)
- This approach produces efficient numerics
- . . . and computer assisted proofs!

Reaction Diffusion System

Consider a reaction diffusion equation

$$u_t = Du_{xx} + \nabla G(u), \qquad u \in \mathbb{R}^n \qquad x \in \mathbb{R}$$

for diffusion matrix D and nonlinearity $G \in C^2(\mathbb{R}^n, \mathbb{R}^n)$



Assume $\varphi(x) : \mathbb{R} \to \mathbb{R}^n$ is a stationary solution

For symmetric $\mathcal{M}(x) = \nabla^2 G(\varphi(x))$, one obtains the eigenvalue problem

$$\lambda u = Du_{xx} + \mathcal{M}(x)u =: \mathcal{L}u, \qquad u \in \mathbb{R}^n, \qquad x \in \mathbb{R}$$

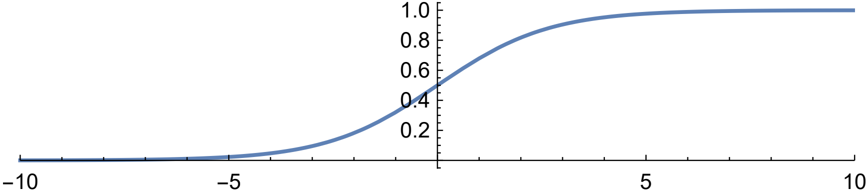
Note $\sigma(\mathcal{L}) = \sigma_{ess}(\mathcal{L}) \cup \sigma_{pt}(\mathcal{L})$. $\quad \sigma_{ess}(\mathcal{L})$ is easy $\quad \sigma_{pt}(\mathcal{L})$ is hard

Reaction Diffusion System

Consider a reaction diffusion equation

$$u_t = Du_{xx} + \nabla G(u), \qquad u \in \mathbb{R}^n \qquad x \in \mathbb{R}$$

for diffusion matrix D and nonlinearity $G \in C^2(\mathbb{R}^n, \mathbb{R}^n)$



Assume $\varphi(x) : \mathbb{R} \to \mathbb{R}^n$ is a stationary solution

For symmetric $\mathcal{M}(x) = \nabla^2 G(\varphi(x))$, one obtains the eigenvalue problem

$$\lambda u = Du_{xx} + \mathcal{M}(x)u =: \mathcal{L}u, \qquad u \in \mathbb{R}^n, \qquad x \in \mathbb{R}$$

Note $\sigma(\mathcal{L}) = \sigma_{ess}(\mathcal{L}) \cup \sigma_{pt}(\mathcal{L})$. $\quad \sigma_{ess}(\mathcal{L})$ is easy $\quad \sigma_{pt}(\mathcal{L})$ is hard

Gradient Nonlinearity & Symplectic Structure

Rewrite $\mathcal{L}u = Du_{xx} + \mathcal{M}(x)u$ using a symplectic structure

$$U_x = J\mathcal{B}(x;\lambda)U \qquad U \in \mathbb{R}^{2n} \qquad J = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix} \qquad \lambda \in \mathbb{R} \qquad (1)$$

where $J^2 = -I$, and we define $U = (u, Du_x)$ and the symmetric matrix

$$\mathcal{B}(x;\lambda) = \begin{pmatrix} \lambda - \mathcal{M}(x) & 0 \\ 0 & -D^{-1} \end{pmatrix}$$

For $U, W \in \mathbb{R}^{2n}$ we define a symplectic form by

$$\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R} \qquad \omega(U, W) = \langle U, JW \rangle$$

Symplectic Form is Preserved Under the Flow

If $U(x)$ and $W(x)$ solve

$$U_x = J\mathcal{B}(x;\lambda)U$$

then $\frac{d}{dx}\omega(U(x), W(x)) = 0$

Gradient Nonlinearity & Symplectic Structure

Rewrite $\mathcal{L}u = Du_{xx} + \mathcal{M}(x)u$ using a symplectic structure

$$U_x = J\mathcal{B}(x;\lambda)U \qquad U \in \mathbb{R}^{2n} \qquad J = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix} \qquad \lambda \in \mathbb{R} \qquad (1)$$

where $J^2 = -I$, and we define $U = (u, Du_x)$ and the symmetric matrix

$$\mathcal{B}(x;\lambda) = \begin{pmatrix} \lambda - \mathcal{M}(x) & 0 \\ 0 & -D^{-1} \end{pmatrix}$$

For $U, W \in \mathbb{R}^{2n}$ we define a symplectic form by

$$\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R} \qquad\qquad \omega(U, W) = \langle U, JW \rangle$$

Symplectic Form is Preserved Under the Flow

If $U(x)$ and $W(x)$ solve

$$U_x = J\mathcal{B}(x;\lambda)U$$

then $\frac{d}{dx}\omega(U(x), W(x)) = 0$

Eigenvalue Problem: A nonautonomous linear ODE

λ and a solution $U_x = J\mathcal{B}(x;\lambda)U$ is an eigen-pair iff

$$\lim_{x\to\pm\infty} |U(x)| = 0$$

Define $J\mathcal{B}_\pm(\lambda) := \lim_{x\to\pm\infty} J\mathcal{B}(x;\lambda)$

Define $\mathbb{E}^{u/s}_\pm(x;\lambda)$ as the space of solutions which are asymptotic to the unstable (stable) eigenspace of $J\mathcal{B}_\pm(\lambda)$ as $x \to \pm\infty$

Eigenvalue Problem: A nonautonomous linear ODE

λ and a solution $U_x = J\mathcal{B}(x;\lambda)U$ is an eigen-pair iff

$$\lim_{x\to\pm\infty} |U(x)| = 0$$

Define $J\mathcal{B}_\pm(\lambda) := \lim_{x\to\pm\infty} J\mathcal{B}(x;\lambda)$

Define $\mathbb{E}^{u/s}_\pm(x;\lambda)$ as the space of solutions which are asymptotic to the unstable (stable) eigenspace of $J\mathcal{B}_\pm(\lambda)$ as $x \to \pm\infty$

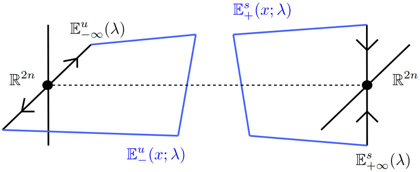
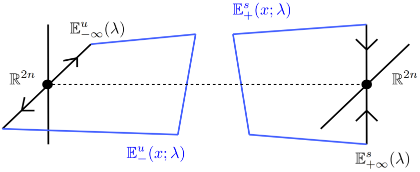


Figure: (MB) The spaces $\mathbb{E}^u_-(x;\lambda)$ and $\mathbb{E}^s_+(x;\lambda)$ consist of solutions which go to 0 as $x \to -\infty$ and $x \to +\infty$ respectively.

λ is an eigenvalue iff

$$\mathbb{E}^u_-(x;\lambda) \cap \mathbb{E}^s_+(x;\lambda) \neq \{0\}$$



Mild Ansatz: $J\mathcal{B}_\pm(0)$ is hyperbolic with n positive & n negative e-val

If φ is a wave, then

$$\begin{pmatrix} \varphi'(x) \\ D\varphi''(x) \end{pmatrix} \in \mathbb{E}^u_-(x;0) \cap \mathbb{E}^s_+(x;0)$$

Sturm Liouville Theory ($n = 1$)

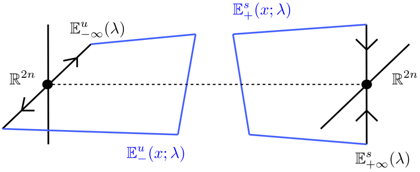
One can count unstable eigenvalues by counting zeros of $\varphi'(x)$

How to generalize to systems?

$\mathbb{E}^u_-(x;\lambda)$ is Lagrangian; $\omega(U,W) = 0 \ \forall U, W \in \mathbb{E}^u_-(x;\lambda)$

λ is an eigenvalue iff

$$\mathbb{E}^u_-(x;\lambda) \cap \mathbb{E}^s_+(x;\lambda) \neq \{0\}$$



Mild Ansatz: $J\mathcal{B}_\pm(0)$ is hyperbolic with n positive & n negative e-val

If φ is a wave, then

$$\begin{pmatrix} \varphi'(x) \\ D\varphi''(x) \end{pmatrix} \in \mathbb{E}^u_-(x;0) \cap \mathbb{E}^s_+(x;0)$$

Sturm Liouville Theory ($n = 1$)

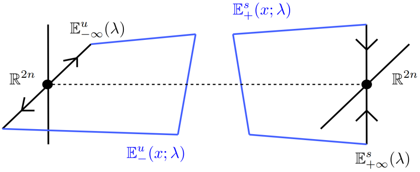
One can count unstable eigenvalues by counting zeros of $\varphi'(x)$

How to generalize to systems?

$\mathbb{E}^u_-(x;\lambda)$ is Lagrangian; $\omega(U,W) = 0 \ \forall U,W \in \mathbb{E}^u_-(x;\lambda)$

λ is an eigenvalue iff

$$\mathbb{E}^u_-(x;\lambda) \cap \mathbb{E}^s_+(x;\lambda) \neq \{0\}$$



Mild Ansatz: $J\mathcal{B}_\pm(0)$ is hyperbolic with n positive & n negative e-val

If φ is a wave, then

$$\begin{pmatrix} \varphi'(x) \\ D\varphi''(x) \end{pmatrix} \in \mathbb{E}^u_-(x;0) \cap \mathbb{E}^s_+(x;0)$$

Sturm Liouville Theory ($n = 1$)

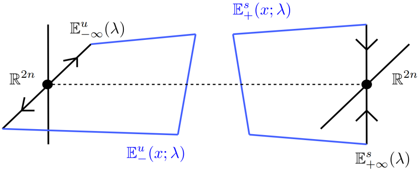
One can count unstable eigenvalues by counting zeros of $\varphi'(x)$

How to generalize to systems?

$\mathbb{E}^u_-(x;\lambda)$ is Lagrangian; $\omega(U,W) = 0 \ \forall U, W \in \mathbb{E}^u_-(x;\lambda)$

λ is an eigenvalue iff

$$\mathbb{E}^u_-(x;\lambda) \cap \mathbb{E}^s_+(x;\lambda) \neq \{0\}$$



Mild Ansatz: $J\mathcal{B}_\pm(0)$ is hyperbolic with n positive & n negative e-val

If φ is a wave, then

$$\begin{pmatrix} \varphi'(x) \\ D\varphi''(x) \end{pmatrix} \in \mathbb{E}^u_-(x;0) \cap \mathbb{E}^s_+(x;0)$$

Sturm Liouville Theory ($n=1$)

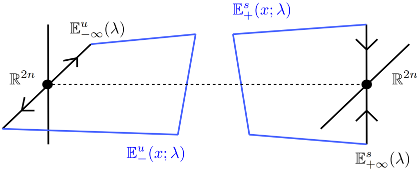
One can count unstable eigenvalues by counting zeros of $\varphi'(x)$

How to generalize to systems?

$\mathbb{E}^u_-(x;\lambda)$ is Lagrangian; $\omega(U,W)=0 \;\; \forall U,W \in \mathbb{E}^u_-(x;\lambda)$

λ is an eigenvalue iff

$$\mathbb{E}^u_-(x;\lambda) \cap \mathbb{E}^s_+(x;\lambda) \neq \{0\}$$



Mild Ansatz: $J\mathcal{B}_\pm(0)$ is hyperbolic with n positive & n negative e-val

If φ is a wave, then

$$\begin{pmatrix} \varphi'(x) \\ D\varphi''(x) \end{pmatrix} \in \mathbb{E}^u_-(x;0) \cap \mathbb{E}^s_+(x;0)$$

Sturm Liouville Theory ($n = 1$)

One can count unstable eigenvalues by counting zeros of $\varphi'(x)$

How to generalize to systems?

$\mathbb{E}^u_-(x;\lambda)$ is Lagrangian; $\omega(U,W) = 0 \;\; \forall U, W \in \mathbb{E}^u_-(x;\lambda)$

The Lagrangian Grassmannian

$$\Lambda(n) := \left\{ \ell \subset \mathbb{R}^{2n} : \dim(\ell) = n, \;\; \omega(U, W) = 0 \;\; \forall U, W \in \ell \right\}$$

$\Lambda(n)$ has dimension $n(n+1)/2$ and has same $\mathbb{Z}_2$ homology as $\mathbb{S}^1 \times \cdots \times \mathbb{S}^n$

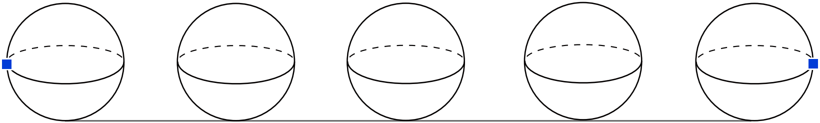


Figure: Cartoon picture of $\Lambda(2)$, which is double covered by $\mathbb{S}^1 \times \mathbb{S}^2$

$\pi_1(\Lambda(n)) = \mathbb{Z}$ and **Maslov Index** measures the homotopy type of paths.

$\Lambda(n) \cong U(n)/O(n)$ and

$$\det{}^2 : U(n)/O(n) \to \mathbb{S}^1 \subseteq \mathbb{C}$$

induces an isomorphism of fundamental groups

The Lagrangian Grassmannian

$$\Lambda(n) := \left\{\ell \subset \mathbb{R}^{2n} : \dim(\ell) = n,\ \ \omega(U,W) = 0\ \ \forall U, W \in \ell\right\}$$

$\Lambda(n)$ has dimension $n(n+1)/2$ and has same $\mathbb{Z}_2$ homology as $\mathbb{S}^1 \times \cdots \times \mathbb{S}^n$

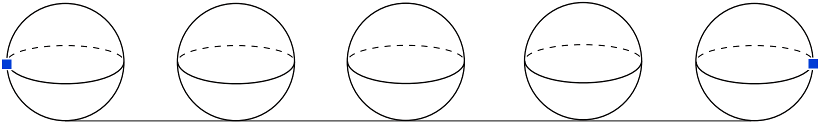


Figure: Cartoon picture of $\Lambda(2)$, which is double covered by $\mathbb{S}^1 \times \mathbb{S}^2$

$\pi_1(\Lambda(n)) = \mathbb{Z}$ and **Maslov Index** measures the homotopy type of paths.

$\Lambda(n) \cong U(n)/O(n)$ and

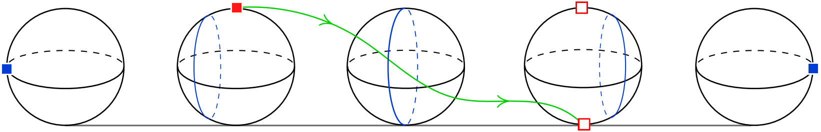
$$\det{}^2 : U(n)/O(n) \to \mathbb{S}^1 \subseteq \mathbb{C}$$

induces an isomorphism of fundamental groups

Conjugate Points

Define the Dirichlet subspace as $\mathcal{D} := \{(u,v) \in \mathbb{R}^{2n} : u = 0\}$

Define the train of $\mathcal{D}$ as $\mathcal{T} := \{\ell \in \Lambda(n) : \ell \cap \mathcal{D} \neq \{0\}\}$



For fixed $\lambda = 0$, define **conjugate points** as values of x s.t. $\mathbb{E}^u_-(x;0) \in \mathcal{T}$

Theorem [BCJ+18]

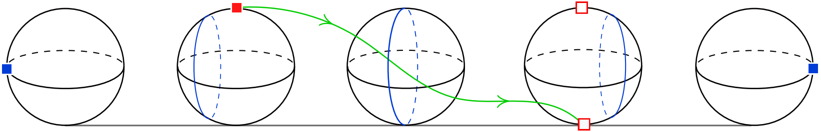
The number of positive eigenvalues of $\mathcal{L}$ is equal to the number of conjugate points

All conjugate points are in $[-L_-, L_+]$ for some $L_-, L_+ \in \mathbb{R}$

Conjugate Points

Define the Dirichlet subspace as $\mathcal{D} := \{(u, v) \in \mathbb{R}^{2n} : u = 0\}$

Define the train of $\mathcal{D}$ as $\mathcal{T} := \{\ell \in \Lambda(n) : \ell \cap \mathcal{D} \neq \{0\}\}$



For fixed $\lambda = 0$, define **conjugate points** as values of x s.t. $\mathbb{E}^u_-(x; 0) \in \mathcal{T}$

Theorem [BCJ+18]

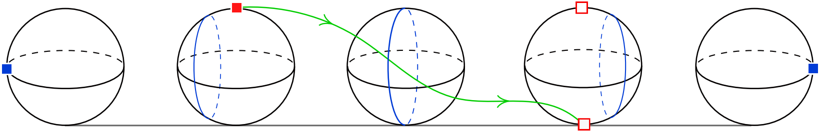
The number of positive eigenvalues of $\mathcal{L}$ is equal to the number of conjugate points

All conjugate points are in $[-L_-, L_+]$ for some $L_-, L_+ \in \mathbb{R}$

Conjugate Points

Define the Dirichlet subspace as $\mathcal{D} := \{(u, v) \in \mathbb{R}^{2n} : u = 0\}$

Define the train of $\mathcal{D}$ as $\mathcal{T} := \{\ell \in \Lambda(n) : \ell \cap \mathcal{D} \neq \{0\}\}$



For fixed $\lambda = 0$, define **conjugate points** as values of x s.t. $\mathbb{E}^u_-(x; 0) \in \mathcal{T}$

Theorem [BCJ+18]

The number of positive eigenvalues of $\mathcal{L}$ is equal to the number of conjugate points

All conjugate points are in $[-L_-, L_+]$ for some $L_-, L_+ \in \mathbb{R}$

How to Find Conjugate Points?

If the columns of $A(x) = \begin{pmatrix} A_1(x) \\ A_2(x) \end{pmatrix}$ span $\mathbb{E}^u_-(x)$ then A is said to be a **frame matrix**

$$\begin{array}{ccc} x \in \mathbb{R} & \xrightarrow{A} & \mathbb{R}^{2n\times n} \\ \downarrow & & \downarrow \\ x \in \mathbb{R} & \xrightarrow{\mathbb{E}^u_-} & \Lambda(n) \end{array}$$

Lemma

$\mathbb{E}^u_-(x;0) \in \mathcal{T} \iff \det A_1(x) = 0$

How to Find Conjugate Points?

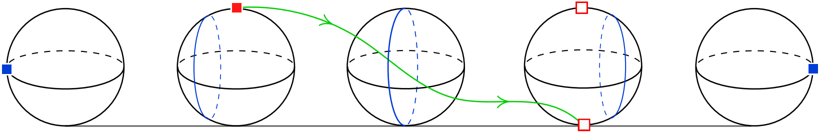
If the columns of $A(x) = \begin{pmatrix} A_1(x) \\ A_2(x) \end{pmatrix}$ span $\mathbb{E}_-^u(x)$ then A is said to be a **frame matrix**

$$\begin{array}{ccc} x \in \mathbb{R} & \xrightarrow{A} & \mathbb{R}^{2n \times n} \\ \downarrow & & \downarrow \\ x \in \mathbb{R} & \xrightarrow{\mathbb{E}_-^u} & \Lambda(n) \end{array}$$

Lemma

$\mathbb{E}_-^u(x; 0) \in \mathcal{T} \iff \det A_1(x) = 0$

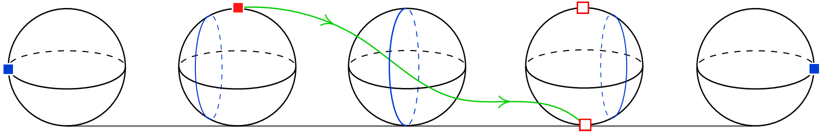
Our Numerical Methodology



1. Compute a stationary solution
2. Fix L_- and prove $\mathbb{E}^u_-(x;0) \notin \mathcal{T}$ for $x \in (-\infty, -L_-]$
3. Fix L_+ and calculate a frame matrix of $\mathbb{E}^u_-(x;0)$ for $x \in [-L_-, L_+]$
4. Prove $\mathbb{E}^u_-(x;0) \notin \mathcal{T}$ for $x \in [L_+, +\infty)$
5. Count all the conjugate points in $[-L_-, L_+]$

Paper [BJ22] & code [BJ21] are available

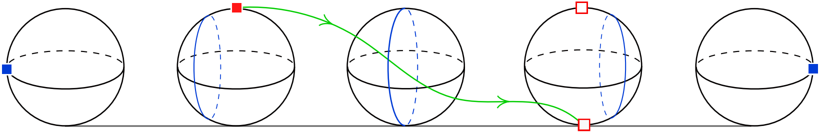
Our Numerical Methodology



1. Compute a stationary solution
2. Fix L_- and prove $\mathbb{E}^u_-(x;0) \notin \mathcal{T}$ for $x \in (-\infty, -L_-]$
3. Fix L_+ and calculate a frame matrix of $\mathbb{E}^u_-(x;0)$ for $x \in [-L_-, L_+]$
4. Prove $\mathbb{E}^u_-(x;0) \notin \mathcal{T}$ for $x \in [L_+, +\infty)$
5. Count all the conjugate points in $[-L_-, L_+]$

Paper [BJ22] & code [BJ21] are available

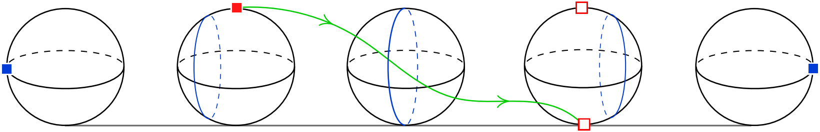
Our Numerical Methodology



1. Compute a stationary solution
2. Fix L_- and prove $\mathbb{E}^u_-(x;0) \notin \mathcal{T}$ for $x \in (-\infty, -L_-]$
3. Fix L_+ and calculate a frame matrix of $\mathbb{E}^u_-(x;0)$ for $x \in [-L_-, L_+]$
4. Prove $\mathbb{E}^u_-(x;0) \notin \mathcal{T}$ for $x \in [L_+, +\infty)$
5. Count all the conjugate points in $[-L_-, L_+]$

Paper [BJ22] & code [BJ21] are available

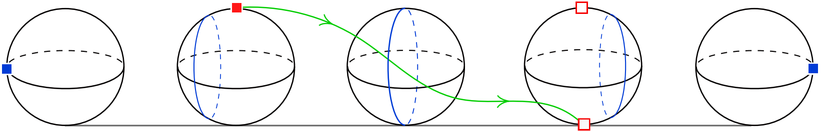
Our Numerical Methodology



1. Compute a stationary solution
2. Fix L_- and prove $\mathbb{E}^u_-(x;0) \notin \mathcal{T}$ for $x \in (-\infty, -L_-]$
3. Fix L_+ and calculate a frame matrix of $\mathbb{E}^u_-(x;0)$ for $x \in [-L_-, L_+]$
4. Prove $\mathbb{E}^u_-(x;0) \notin \mathcal{T}$ for $x \in [L_+, +\infty)$
5. Count all the conjugate points in $[-L_-, L_+]$

Paper [BJ22] & code [BJ21] are available

Our Numerical Methodology



1. Compute a stationary solution
2. Fix L_- and prove $\mathbb{E}^u_-(x;0) \notin \mathcal{T}$ for $x \in (-\infty, -L_-]$
3. Fix L_+ and calculate a frame matrix of $\mathbb{E}^u_-(x;0)$ for $x \in [-L_-, L_+]$
4. Prove $\mathbb{E}^u_-(x;0) \notin \mathcal{T}$ for $x \in [L_+, +\infty)$
5. Count all the conjugate points in $[-L_-, L_+]$

Paper [BJ22] & code [BJ21] are available

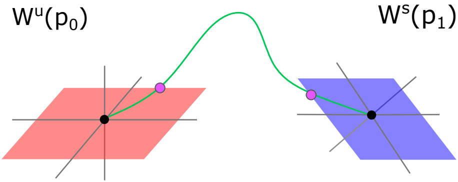
Step 1: Compute a Connecting Orbit

Want to find a stationary solution $\varphi(x) : \mathbb{R} \to \mathbb{R}^n$ to

$$u_t = Du_{xx} + \nabla G(u)$$

Corresponds to a connecting orbit $\phi = (\varphi, D\varphi_x)$ to the Hamiltonian ODE

$$H(v, w) = \frac{1}{2}\|D^{-1}w\|^2 + G(v)$$
$$(v, w)' = -J\nabla H(v, w)$$



Suppose $p_0, p_1 \in \mathbb{R}^{2n}$ are equilibria with equal energy $H(p_0) = H(p_1)$
Each with n-stable and n-unstable real eigenvalues in their linearization.

A lot of prior work on the subject w/ computer assisted proofs.

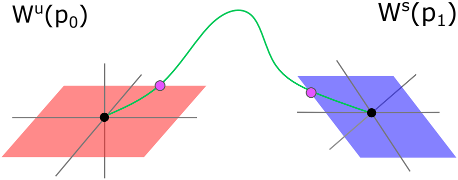
Step 1: Compute a Connecting Orbit

Want to find a stationary solution $\varphi(x) : \mathbb{R} \to \mathbb{R}^n$ to

$$u_t = Du_{xx} + \nabla G(u)$$

Corresponds to a connecting orbit $\phi = (\varphi, D\varphi_x)$ to the Hamiltonian ODE

$$H(v, w) = \frac{1}{2}\|D^{-1}w\|^2 + G(v)$$
$$(v, w)' = -J\nabla H(v, w)$$



Suppose $p_0, p_1 \in \mathbb{R}^{2n}$ are equilibria with equal energy $H(p_0) = H(p_1)$
Each with n-stable and n-unstable real eigenvalues in their linearization.

A lot of prior work on the subject w/ computer assisted proofs.

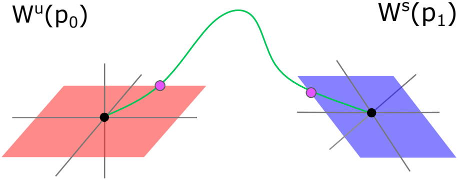
Step 1: Compute a Connecting Orbit

Want to find a stationary solution $\varphi(x) : \mathbb{R} \to \mathbb{R}^n$ to

$$u_t = Du_{xx} + \nabla G(u)$$

Corresponds to a connecting orbit $\phi = (\varphi, D\varphi_x)$ to the Hamiltonian ODE

$$H(v, w) = \frac{1}{2}\|D^{-1}w\|^2 + G(v)$$
$$(v, w)' = -J\nabla H(v, w)$$



Suppose $p_0, p_1 \in \mathbb{R}^{2n}$ are equilibria with equal energy $H(p_0) = H(p_1)$
Each with n-stable and n-unstable real eigenvalues in their linearization.

A lot of prior work on the subject w/ computer assisted proofs.

Step 2: How to define A(x), a frame matrix for $\mathbb{E}^u_-(x;0)$?

Let v_k be the unstable e-vectors of JB_-. Then

$$\lim_{x\to-\infty} \mathbb{E}^u_-(x) = \text{span}\{v_1, \ldots v_n\}$$

There exist eigenfunctions $V_k : (-\infty, -L_-] \to \mathbb{R}^{2n}$ such that

$$V_k' = J\mathcal{B}(x;0)V_k, \qquad \left\| V_k(x)e^{-\lambda_k x} - v_k \right\|_{L^\infty\left((-\infty,-L_-],\mathbb{R}^{2n}\right)} < \epsilon$$

Use exponential dichotomies!

$\mathbb{E}^u_-(x)$ has a frame matrix

$$A(x) = \left[\begin{array}{c|c|c} V_1(x) & \cdots & V_n(x) \end{array} \right]$$

Step 2: How to define A(x), a frame matrix for $\mathbb{E}^u_-(x;0)$?

Let v_k be the unstable e-vectors of JB_-. Then

$$\lim_{x\to-\infty} \mathbb{E}^u_-(x) = \text{span}\{v_1, \dots v_n\}$$

There exist eigenfunctions $V_k : (-\infty, -L_-] \to \mathbb{R}^{2n}$ such that

$$V_k' = J\mathcal{B}(x;0)V_k, \qquad \left\| V_k(x)e^{-\lambda_k x} - v_k \right\|_{L^\infty\left((-\infty,-L_-],\mathbb{R}^{2n}\right)} < \epsilon$$

Use exponential dichotomies!

$\mathbb{E}^u_-(x)$ has a frame matrix

$$A(x) = \left[\begin{array}{c|c|c} V_1(x) & \cdots & V_n(x) \end{array}\right]$$

Step 2: Defining the frame matrix $A(-L_-)$

Let $\mathcal{V}_-^u, \mathcal{V}_-^s$ denote frame matrices for the (un)stable eigenspace of $J\mathcal{B}_-$.

For fixed $L_- \geq 0$, compute error bounds $\mathcal{E}$ for a frame matrix of $\mathbb{E}_-^u(-L_-)$

$$\begin{aligned} A(-L_-) &= \mathcal{V}_-^u + \mathcal{E} \\ &= \mathcal{V}_-^u + \mathcal{V}_-^u \mathcal{E}^u + \mathcal{V}_-^s \mathcal{E}^s \end{aligned}$$

Choose $L_- \gg 1$ s.t. no conjugate pts for $x \in (-\infty, -L_-]$

Error bounds in the unstable eigen-directions will grow exponentially!

We multiply by an invertible matrix to get a new frame matrix

$$\begin{aligned} \tilde{A}(-L_-) &= (\mathcal{V}_-^u + \mathcal{E})(I + \mathcal{E}^u)^{-1} \\ &= \mathcal{V}^u + \mathcal{V}^s \mathcal{E}^s (I + \mathcal{E}^u)^{-1} \end{aligned}$$

Now all the error terms are in the (asymptotically) stable eigen-directions

Step 2: Defining the frame matrix $A(-L_-)$

Let $\mathcal{V}^u_-, \mathcal{V}^s_-$ denote frame matrices for the (un)stable eigenspace of $J\mathcal{B}_-$.

For fixed $L_- \geq 0$, compute error bounds $\mathcal{E}$ for a frame matrix of $\mathbb{E}^u_-(-L_-)$

$$\begin{aligned} A(-L_-) &= \mathcal{V}^u_- + \mathcal{E} \\ &= \mathcal{V}^u_- + \mathcal{V}^u_- \mathcal{E}^u + \mathcal{V}^s_- \mathcal{E}^s \end{aligned}$$

Choose $L_- \gg 1$ s.t. no conjugate pts for $x \in (-\infty, -L_-]$

Error bounds in the unstable eigen-directions will grow exponentially!

We multiply by an invertible matrix to get a new frame matrix

$$\begin{aligned} \tilde{A}(-L_-) &= (\mathcal{V}^u_- + \mathcal{E})(I + \mathcal{E}^u)^{-1} \\ &= \mathcal{V}^u + \mathcal{V}^s \mathcal{E}^s (I + \mathcal{E}^u)^{-1} \end{aligned}$$

Now all the error terms are in the (asymptotically) stable eigen-directions

Step 2: Defining the frame matrix $A(-L_-)$

Let $\mathcal{V}_-^u, \mathcal{V}_-^s$ denote frame matrices for the (un)stable eigenspace of $J\mathcal{B}_-$.

For fixed $L_- \geq 0$, compute error bounds $\mathcal{E}$ for a frame matrix of $\mathbb{E}_-^u(-L_-)$

$$\begin{aligned} A(-L_-) &= \mathcal{V}_-^u + \mathcal{E} \\ &= \mathcal{V}_-^u + \mathcal{V}_-^u \mathcal{E}^u + \mathcal{V}_-^s \mathcal{E}^s \end{aligned}$$

Choose $L_- \gg 1$ s.t. no conjugate pts for $x \in (-\infty, -L_-]$

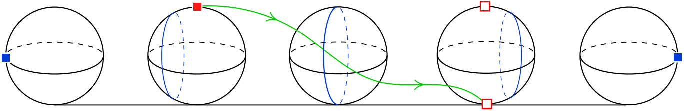
Error bounds in the unstable eigen-directions will grow exponentially!

We multiply by an invertible matrix to get a new frame matrix

$$\begin{aligned} \tilde{A}(-L_-) &= (\mathcal{V}_-^u + \mathcal{E})(I + \mathcal{E}^u)^{-1} \\ &= \mathcal{V}^u + \mathcal{V}^s \mathcal{E}^s (I + \mathcal{E}^u)^{-1} \end{aligned}$$

Now all the error terms are in the (asymptotically) stable eigen-directions

Step 3: Defining $A(x)$ for $x \in [-L_-, L_+]$



Fix L_-, L_+ and initial frame matrix

$$A(-L_-) = \mathcal{V}^u + \mathcal{V}^s \mathcal{E}^s (I + \mathcal{E}^u)^{-1}$$

Define $A(x)$ by integrating the columns forward according to

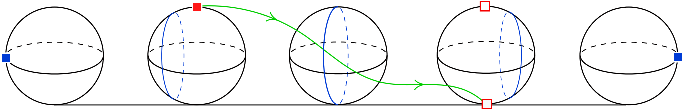
$$U_x = J\mathcal{B}(x;\lambda)U, \qquad U \in \mathbb{R}^{2n}$$

where $\varphi(x)$ is a connecting orbit &

$$\mathcal{B}(x;\lambda) = \begin{pmatrix} -\nabla^2 G(\varphi(x)) & 0 \\ 0 & -D^{-1} \end{pmatrix}$$

- $\Lambda(n)$ is compact, but frame matrices are not; solutions grow exponentially large
- We use the CAPD library [KMWZ20] for rigorous integration

Step 3: Defining $A(x)$ for $x \in [-L_-, L_+]$



Fix L_-, L_+ and initial frame matrix

$$A(-L_-) = \mathcal{V}^u + \mathcal{V}^s \mathcal{E}^s (I + \mathcal{E}^u)^{-1}$$

Define $A(x)$ by integrating the columns forward according to

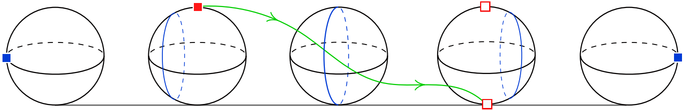
$$U_x = J\mathcal{B}(x;\lambda)U, \qquad U \in \mathbb{R}^{2n}$$

where $\varphi(x)$ is a connecting orbit &

$$\mathcal{B}(x;\lambda) = \begin{pmatrix} -\nabla^2 G(\varphi(x)) & 0 \\ 0 & -D^{-1} \end{pmatrix}$$

- $\Lambda(n)$ is compact, but frame matrices are not; solutions grow exponentially large
- We use the CAPD library [KMWZ20] for rigorous integration

Step 3: Defining $A(x)$ for $x \in [-L_-, L_+]$



Fix L_-, L_+ and initial frame matrix

$$A(-L_-) = \mathcal{V}^u + \mathcal{V}^s \mathcal{E}^s (I + \mathcal{E}^u)^{-1}$$

Define $A(x)$ by integrating the columns forward according to

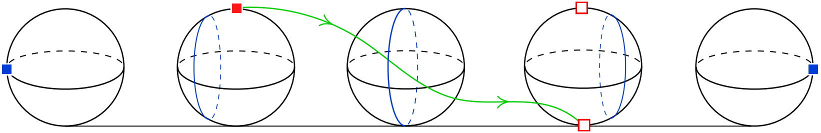
$$U_x = J\mathcal{B}(x;\lambda)U, \qquad U \in \mathbb{R}^{2n}$$

where $\varphi(x)$ is a connecting orbit &

$$\mathcal{B}(x;\lambda) = \begin{pmatrix} -\nabla^2 G(\varphi(x)) & 0 \\ 0 & -D^{-1} \end{pmatrix}$$

- $\Lambda(n)$ is compact, but frame matrices are not; solutions grow exponentially large
- We use the CAPD library [KMWZ20] for rigorous integration

Step 4: Proving there are no conjugate points past L_+



Consider flow on $\Lambda(n)$ induced by $\dot{U} = J\mathcal{B}_+ U$,

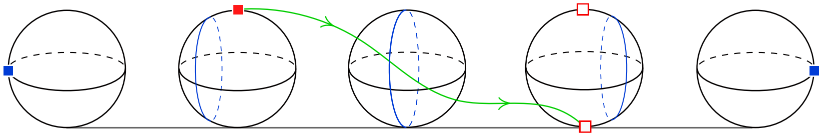
- Fixed points in $\Lambda(n)$ are subspaces spanned by e-vectors of $J\mathcal{B}_+$
- Unstable e-space $\mathbb{E}^u_{+\infty}$ of $J\mathcal{B}_+$ is stable under the flow
- Derivative of the wave is an e-function, so

$$\lim_{x \to +\infty} \mathbb{E}^u_-(x;0) \cap \mathbb{E}^s_{+\infty} \neq \{0\}$$

$\lim_{x\to+\infty} \mathbb{E}^u_-(x;0)$ is unstable under the flow!

- Instead of proving $\mathbb{E}^u_-(L_+;0) \in \Lambda(n)$ is sufficiently close to it's limit point, it suffices to show it is sufficiently far away from $\mathcal{D}$.

Step 4: Proving there are no conjugate points past L_+



Consider flow on $\Lambda(n)$ induced by $\dot{U} = J\mathcal{B}_+ U$,

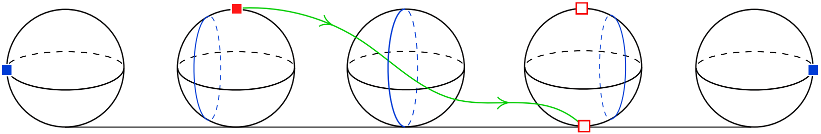
- Fixed points in $\Lambda(n)$ are subspaces spanned by e-vectors of $J\mathcal{B}_+$
- Unstable e-space $\mathbb{E}^u_{+\infty}$ of $J\mathcal{B}_+$ is stable under the flow
- Derivative of the wave is an e-function, so

$$\lim_{x\to+\infty} \mathbb{E}^u_-(x;0) \cap \mathbb{E}^s_{+\infty} \neq \{0\}$$

$\lim_{x\to+\infty} \mathbb{E}^u_-(x;0)$ is unstable under the flow!

- Instead of proving $\mathbb{E}^u_-(L_+;0) \in \Lambda(n)$ is sufficiently close to it's limit point, it suffices to show it is sufficiently far away from $\mathcal{D}$.

Step 4: Proving there are no conjugate points past L_+



Consider flow on $\Lambda(n)$ induced by $\dot{U} = J\mathcal{B}_+ U$,

- Fixed points in $\Lambda(n)$ are subspaces spanned by e-vectors of $J\mathcal{B}_+$
- Unstable e-space $\mathbb{E}^u_{+\infty}$ of $J\mathcal{B}_+$ is stable under the flow
- Derivative of the wave is an e-function, so

$$\lim_{x\to+\infty} \mathbb{E}^u_-(x;0) \cap \mathbb{E}^s_{+\infty} \neq \{0\}$$

$\lim_{x\to+\infty} \mathbb{E}^u_-(x;0)$ is unstable under the flow!

- Instead of proving $\mathbb{E}^u_-(L_+;0) \in \Lambda(n)$ is sufficiently close to it's limit point, it suffices to show it is sufficiently far away from $\mathcal{D}$.

Number of conjugate points equals number of positive e-values

All conjugate points are in $[-L_-, L_+]$

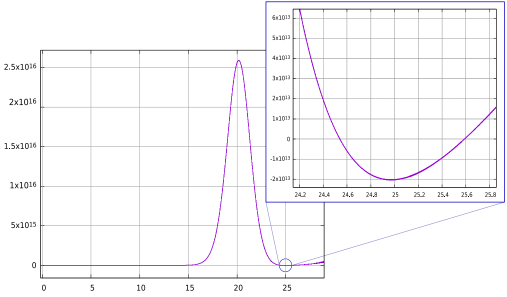


Figure: Graph of $\det A_1(x)$ with two conjugate points

- Consider frame matrix of $\mathbb{E}^u_-(x)$,

$$A(x) = \begin{pmatrix} A_1(x) \\ A_2(x) \end{pmatrix}$$

- Define

$$F(x) = \det A_1(x)$$

- Find all $x \in [-L_-, L_+]$ such that $F(x) = 0$

Number of conjugate points equals number of positive e-values

All conjugate points are in $[-L_-, L_+]$

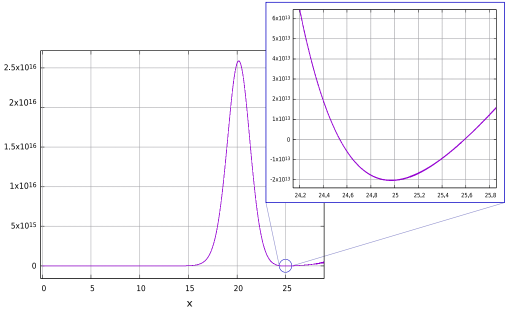


Figure: Graph of $\det A_1(x)$ with two conjugate points

- Consider frame matrix of $\mathbb{E}^u_-(x)$,

$$A(x) = \begin{pmatrix} A_1(x) \\ A_2(x) \end{pmatrix}$$

- Define

$$F(x) = \det A_1(x)$$

- Find all $x \in [-L_-, L_+]$ such that $F(x) = 0$

Consider the (uncoupled) scalar reaction diffusion equation

$$u_t = u_{xx} + \nabla G(u), \qquad u \in \mathbb{R}^1 \qquad x \in \mathbb{R}$$

where

$$G(u) = -\frac{1}{4} b u^2 (1-u)^2, \qquad\qquad b \in \mathbb{R}$$

The resulting PDE is

$$\partial_t u = \partial_x^2 u + b\, f(u)$$

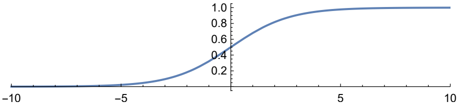
where f is given by

$$f(u) = u(u - \tfrac{1}{2})(1-u)$$

Equilibria 0 and 1 have equal energy for all b

This has an explicit stable front

$$\phi_0(x; b) := \frac{1}{1 + e^{-x\sqrt{b/2}}}$$



Results: Coupled Bistable Equations

Consider the (uncoupled) scalar reaction diffusion equation

$$u_t = u_{xx} + \nabla G(u), \qquad u \in \mathbb{R}^1 \qquad x \in \mathbb{R}$$

where

$$G(u) = -\frac{1}{4} b u^2 (1-u)^2, \qquad b \in \mathbb{R}$$

The resulting PDE is

$$\partial_t u = \partial_x^2 u + b\, f(u)$$

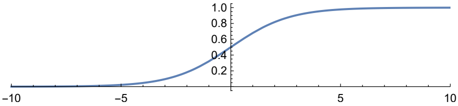
where f is given by

$$f(u) = u(u - \tfrac{1}{2})(1-u)$$

Equilibria 0 and 1 have equal energy for all b

This has an explicit stable front

$$\phi_0(x; b) := \frac{1}{1 + e^{-x\sqrt{b/2}}}$$



Consider $u_t = u_{xx} + \nabla G(u)$, $u \in \mathbb{R}^n$ $x \in \mathbb{R}$ where

$$G(u_1, \dots, u_n) = -\frac{1}{4} \sum_{1 \le i \le n} b_i u_i^2 (1-u_i)^2 - \frac{1}{2} \sum_{1 \le i \le n-1} c_{i,i+1} u_i (1-u_i) u_{i+1} (1-u_{i+1}).$$

with parameters $b \in \mathbb{R}^n$, $c = \{c_{i,i+1}\}_{i=1}^{n-1} \in \mathbb{R}^{n-1}$.

For $n = 3$, this becomes

$$\begin{aligned} \partial_t u_1 &= \partial_x^2 u_1 + b_1 f(u_1) + c_{12} g(u_1, u_2) \\ \partial_t u_2 &= \partial_x^2 u_2 + b_2 f(u_2) + c_{12} g(u_2, u_1) + c_{23} g(u_2, u_3) \\ \partial_t u_3 &= \partial_x^2 u_3 + b_3 f(u_3) + c_{23} g(u_3, u_2), \end{aligned} \quad (2)$$

where f, g are given by

$$f(u) = u(u - \tfrac{1}{2})(1-u) \qquad g(u, w) = w(1-w)(u - \tfrac{1}{2}).$$

Equilibria $(0,0,0)$ and $(1,1,1)$ have equal energy for all b, c

For the uncoupled case $c = 0$ this has an explicit front w/ triple 0 e-val

$$\Phi_0(x) = (\phi_0(x; b_1), \phi_0(x; b_2), \phi_0(x; b_3))$$

Consider $u_t = u_{xx} + \nabla G(u),\ u \in \mathbb{R}^n\ x \in \mathbb{R}$ where

$$G(u_1, \dots, u_n) = -\tfrac{1}{4} \sum_{1 \le i \le n} b_i u_i^2 (1-u_i)^2 - \tfrac{1}{2} \sum_{1 \le i \le n-1} c_{i,i+1} u_i (1-u_i) u_{i+1} (1-u_{i+1}).$$

with parameters $b \in \mathbb{R}^n$, $c = \{c_{i,i+1}\}_{i=1}^{n-1} \in \mathbb{R}^{n-1}$.
For $n = 3$, this becomes

$$\begin{aligned} \partial_t u_1 &= \partial_x^2 u_1 + b_1 f(u_1) + c_{12} g(u_1, u_2) \\ \partial_t u_2 &= \partial_x^2 u_2 + b_2 f(u_2) + c_{12} g(u_2, u_1) + c_{23} g(u_2, u_3) \\ \partial_t u_3 &= \partial_x^2 u_3 + b_3 f(u_3) + c_{23} g(u_3, u_2), \end{aligned} \tag{2}$$

where f, g are given by

$$f(u) = u(u - \tfrac{1}{2})(1-u) \qquad\qquad g(u,w) = w(1-w)(u - \tfrac{1}{2}).$$

Equilibria $(0,0,0)$ and $(1,1,1)$ have equal energy for all b, c

For the uncoupled case $c = 0$ this has an explicit front w/ triple 0 e-val

$$\Phi_0(x) = (\phi_0(x; b_1), \phi_0(x; b_2), \phi_0(x; b_3))$$

Consider $u_t = u_{xx} + \nabla G(u),\ u \in \mathbb{R}^n\ x \in \mathbb{R}$ where

$$G(u_1, \dots, u_n) = -\tfrac{1}{4} \sum_{1 \le i \le n} b_i u_i^2 (1-u_i)^2 - \tfrac{1}{2} \sum_{1 \le i \le n-1} c_{i,i+1} u_i (1-u_i) u_{i+1} (1-u_{i+1}).$$

with parameters $b \in \mathbb{R}^n$, $c = \{c_{i,i+1}\}_{i=1}^{n-1} \in \mathbb{R}^{n-1}$.
For $n = 3$, this becomes

$$\begin{aligned} \partial_t u_1 &= \partial_x^2 u_1 + b_1 f(u_1) + c_{12} g(u_1, u_2) \\ \partial_t u_2 &= \partial_x^2 u_2 + b_2 f(u_2) + c_{12} g(u_2, u_1) + c_{23} g(u_2, u_3) \\ \partial_t u_3 &= \partial_x^2 u_3 + b_3 f(u_3) + c_{23} g(u_3, u_2), \end{aligned} \tag{2}$$

where f, g are given by

$$f(u) = u(u - \tfrac{1}{2})(1-u) \qquad g(u,w) = w(1-w)(u - \tfrac{1}{2}).$$

Equilibria $(0,0,0)$ and $(1,1,1)$ have equal energy for all b, c
For the uncoupled case $c = 0$ this has an explicit front w/ triple 0 e-val

$$\Phi_0(x) = (\phi_0(x; b_1), \phi_0(x; b_2), \phi_0(x; b_3))$$

Results: Coupled Bistable Equations

Theorem with computer assisted proof [BJ22]

Fix the parameter $b = (b_1, b_2, b_3) = (1, .98, .96)$
At each of the four parameter combinations

$$c_{\pm,\pm} = (\pm c_{12}, \pm c_{23}) = (\pm .04, \pm .02)$$

there exists a standing wave solution $\varphi_{\pm,\pm}$ to (2) such that

$$\lim_{x \to -\infty} \varphi_{\pm,\pm}(x) = (0,0,0), \qquad \lim_{x \to +\infty} \varphi_{\pm,\pm}(x) = (1,1,1).$$

Furthermore, there are exactly

- 0 positive eigenvalues in the point spectrum of $\varphi_{-,-}$.
- 1 positive eigenvalue in the point spectrum of both $\varphi_{+,-}$ and $\varphi_{-,+}$.
- 2 positive eigenvalues in the point spectrum of $\varphi_{+,+}$.

Results: Coupled Bistable Equations

Theorem with computer assisted proof [BJ22]

Fix the parameter $b = (b_1, b_2, b_3) = (1, .98, .96)$
At each of the four parameter combinations

$$c_{\pm,\pm} = (\pm c_{12}, \pm c_{23}) = (\pm .04, \pm .02)$$

there exists a standing wave solution $\varphi_{\pm,\pm}$ to (2) such that

$$\lim_{x \to -\infty} \varphi_{\pm,\pm}(x) = (0,0,0), \qquad \lim_{x \to +\infty} \varphi_{\pm,\pm}(x) = (1,1,1).$$

Furthermore, there are exactly

- 0 positive eigenvalues in the point spectrum of $\varphi_{-,-}$.
- 1 positive eigenvalue in the point spectrum of both $\varphi_{+,-}$ and $\varphi_{-,+}$.
- 2 positive eigenvalues in the point spectrum of $\varphi_{+,+}$.

Conclusion

Validated computation of spectral stability via conjugate points

- Less computation than Evans function
  - With the Evans function one computes $\mathbb{E}^u_-(x;\lambda)$ for an entire contour of $\lambda \in \mathbb{C}$
- Less information than Evans function
  - Gives total number (not value) of positive e-values
- Future Work
  - Extending the technique for the Swift-Hohenberg equation w/ Margaret Beck & Hannah Pieper

Conclusion

Validated computation of spectral stability via conjugate points

- Less computation than Evans function
  - With the Evans function one computes $\mathbb{E}^u_-(x;\lambda)$ for an entire contour of $\lambda \in \mathbb{C}$
- Less information than Evans function
  - Gives total number (not value) of positive e-values
- Future Work
  - Extending the technique for the Swift-Hohenberg equation w/ Margaret Beck & Hannah Pieper

Acknowledgments

Thanks to:

- Margaret Beck
- Mathematical Sciences Research Institute 2018 semester on: Hamiltonian systems, from topology to applications through analysis
- Maciej Capiński
- You the audience!

References I

M. Beck, G. Cox, C. Jones, Y. Latushkin, K. McQuighan, and A. Sukhtayev.
Instability of pulses in gradient reaction-diffusion systems: a symplectic approach.
Philos. Trans. Roy. Soc. A, 376(2117):20170187, 20, 2018.

Margaret Beck and Jonathan Jaquette.
Codes of "validated spectral stability via conjugate points".
https://github.com/JCJaquette/Computing-Conjugate-Points, 2021.

Margaret Beck and Jonathan Jaquette.
Validated spectral stability via conjugate points.
SIAM Journal on Applied Dynamical Systems, 21(1):366–404, 2022.

Tomasz Kapela, Marian Mrozek, Daniel Wilczak, and Piotr Zgliczyński.
Capd:: Dynsys: a flexible c++ toolbox for rigorous numerical analysis of dynamical systems.
Communications in Nonlinear Science and Numerical Simulation, page 105578, 2020.