1 Introduction

My research broadly focuses on dynamics and differential equations, and in particular bridging the gap between what can be proven mathematically and what can be computed numerically.

For example, standard numerical methods can solve an initial value problem for an ODE and provide local error bounds at each step. However a global error bound on the final solution requires the cumulative error be quantified. This quickly becomes a nontrivial problem in chaotic systems, where arbitrarily close initial conditions will inevitably diverge, and the difficulties compound in partial differential equations where the phase space is infinite dimensional.

To that end, validated numerics have been developed to keep track of all the sources of error inherent to numerical calculations. To bridge the gap between numerics and a computer assisted proof, a problem must be translated into a list of the conditions that the computer can check. Most famously used to solve the four color theorem [1], computer assisted proofs have been employed to great effect in dynamics, proving results such as the universality of the Feigenbaum constants [2] and Smale’s 14th problem on the nature of the Lorenz attractor [3].

I am particularly interested in developing constructive methods to study invariant sets and their stability in infinite dimensional dynamical systems. Analytically, this involves proving theorems with explicitly verifiable hypotheses (e.g. rather than assuming “there exists some $\epsilon > 0$“, concretely quantifying how small $\epsilon$ must be). Computationally, this involves variety of numerical techniques from fields such as dynamical systems, partial differential equations, global optimization, and algebraic topology.

2 Wright’s Conjecture on a Nonlinear Delay Differential Equation

In my thesis I proved two half-century old conjectures concerning the delay differential equation known as Wright’s equation:

$$ x'(t) = - \alpha (e^{x(t-1)} - 1) \tag{1} $$

First studied in 1955 as a heuristic model of the distribution of primes [4], Wright’s equation has come to be known as a canonical example of a nonlinear scalar delay differential equation (DDE). As with partial differential equations, the initial data for DDEs are functions. In Wright’s seminal work he showed that if $\alpha \leq \frac{3}{2}$ then the equilibrium solution $x \equiv 0$ is the global attractor, and made the following conjecture:

**Theorem 1** (Wright’s Conjecture, 1955). For every $0 < \alpha \leq \frac{\pi}{2}$ the equilibrium solution $x \equiv 0$ to (1) is globally attractive.

In 1962 Jones [5] proved that for $\alpha > \frac{\pi}{2}$ there exists at least one slowly oscillating periodic solution (SOPS) to Wright’s equation. That is, a periodic solution $x : \mathbb{R} \to \mathbb{R}$ such that it is positive for at least the length of the time delay, and then negative for at least the length of the time delay. Based on numerical simulations Jones made the following conjecture:

**Theorem 2** (Jones’ Conjecture, 1962). For every $\alpha > \frac{\pi}{2}$ there is a unique slowly oscillating periodic solution to (1).

I proved both of these conjectures in a trio of papers [6–8]. Prior work had proved Wright’s conjecture for $\alpha \leq \frac{3}{2} - 2 \times 10^{-4}$ via a computer assisted proof which took months of CPU time, and as authors mention, “substantial improvement of the theoretical part of the present proof is needed to prove Wright’s conjecture fully” [9]. Hopf bifurcations are canonically analyzed with the method of normal forms, which transforms a given equation into a simpler expression having the same qualitative behavior as the original equation. By an implicit-function-theorem type argument, this transformation is valid in some neighborhood of the bifurcation. However, the proof does not offer any insight into the size of this neighborhood. In [6] with JB van den Berg (VU Amsterdam)
we develop an explicit description of a neighborhood wherein the only periodic solutions are those originating from the Hopf bifurcation. The main result of this analysis is the resolution of Wright’s conjecture.

In 1991 Xie [10] proved Jones’ conjecture for $\alpha \geq 5.67$. He accomplished this by first showing that there is a unique slowly oscillating periodic solution to (1) if and only if every SOPS is asymptotically stable. By using asymptotic estimates of SOPS for large $\alpha$, Xie was able to estimate their Floquet multipliers and prove that all SOPS had to be stable. However, at $\alpha = 5.67$ these asymptotic estimates break down.

In [7] with JP Lessard (McGill University) and K Mischaikow (Rutgers University) we used the same basic method as Xie, however we replace the asymptotic estimates with rigorous numerics. We use a branch and bound algorithm to develop pointwise estimates on all the possible SOPS to Wright’s equation and then bound their Floquet multipliers. Using these two main steps, we generate a computer assisted proof for $\alpha \in [1.9, 6.0]$ that all SOPS to Wright’s equation are asymptotically stable, and thereby unique up to translation.

I finished the proof to Jones’ conjecture in [8], proving there is a unique SOPS for $\alpha \in (\frac{\pi}{2}, 1.9]$. While previous work [6] showed that there are no folds in the principal branch of periodic orbits this did not rule out the possibility of isolas, that is SOPS far away from the principal branch. To rule out the existence of these isolas, we recast the problem of studying periodic solutions to (1) as the problem of finding zeros of a functional defined on a space of Fourier coefficients, and again employed an infinite dimensional branch and bound algorithm. In future work, I plan extend these Fourier-spectral techniques to produce computer-assisted-proofs for an exact count of the number of equilibria to nonlinear parabolic PDEs (e.g. Swift-Hohenberg) on finite intervals, and eventually multidimensional domains.

## 3 Patterns and Stability in Partial Differential Equations

Connecting orbits provide a road map for how a dynamical system transitions between its various fixed points and periodic orbits. Certain kinds of connecting orbits, such as homoclinics from a periodic orbit to itself, can be used to prove the existence of mathematical chaos. In the spatial dynamics of a PDE, a connecting orbit corresponding to a standing wave describes how two homogeneous steady states can coexist with a transition zone in between. In the temporal dynamics of a PDE, a connecting orbit between two nonhomogeneous equilibria describes how perturbations to an unstable equilibrium unfold, and to which stable equilibrium the perturbed state will be
attracted. A long term goal of my research program is studying connecting orbits in the temporal
dynamics of PDEs and other infinite dimensional dynamical systems.

In general, connecting orbits are calculated by solving a boundary value problem between an
Amsterdam) and J Mireles James (Florida Atlantic University), we present a rigorous computa-
tional method for approximating infinite dimensional stable manifolds of non-trivial equilibria
for parabolic PDEs. Our approach combines the parameterization method – which can provide
high order approximations of finite dimensional manifolds with validated error bounds – together
with the Lyapunov-Perron method – which is a powerful technique for proving the existence of
(potentially infinite dimensional) invariant manifolds. As an example, we apply this technique
to approximate the stable manifold associated with unstable nonhomogeneous equilibria for the
Swift-Hohenberg equation on a finite interval. In future work we plan to rigorously compute saddle
to saddle connecting orbits in the Swift-Hohenberg equation, and additionally to develop methods
to rigorously approximate stable manifolds of periodic orbits in parabolic PDEs (e.g. Kuramoto
Sivashinsky). Longer term goals include constructing a computer assisted proof of chaos in a PDE
via a homoclinic tangle, and computing connecting orbits in strongly indefinite problems motivated
from Floer homology.

Fig. 3: A connecting orbit to the nonlinear Schrödinger equation $-i u_t = u_{xx} + u^2$ with $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$.
(a) Real and imaginary components of a nontrivial equilibrium, with two trajectories on its unstable manifold;
(b) Rigorous integration of an endpoint from the unstable manifold; (c) A trapping cone in Fourier space of
points converging to the center-type 0-equilibrium.
In submitted work [12] with A Takayasu (University of Tsukuba), JP Lessard (McGill University), and H Okamoto (Gakushuin University), we study finite time blow-up profiles in the PDE $u_t = u_{xx} + u^2$ for $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$. By solving the equation, with a rigorous numerical integrator we developed, along a contour in the complex plane of time we are able to prove the existence of a branching singularity. Namely, for a contour defined as $\Gamma_\theta = \{ z \in \mathbb{C} : z = te^{i\theta} \ t \geq 0 \}$ with $\theta \in (-\pi/2, \pi/2)$ this PDE can be written as

$$u_t = e^{i\theta}(u_{xx} + u^2), \quad (2)$$

a PDE analogous to the complex Ginzburg-Landau equation with a quadratic nonlinearity. Solving this equation along contours in the complex plane of time allows us to continue the solution past the blowup point. By showing that contours in the upper and and lower halves of the complex plane yield different solutions, we are able to achieve a computer assisted proof that a branching singularity occurs.

When $\theta = \pi/2$ the PDE in (2) becomes the nonlinear Schrödinger $-iu_t = u_{xx} + u^2$. This is neither a heat-like equation, which would admit a compact attractor, and nor is this equation gauge-invariant, whereby our analysis might be aided by techniques for Hamiltonian PDEs. Together with JP Lessard and A Takayasu, we establish heteroclinic orbits to nontrivial equilibria in this NLS via computer assisted proofs [13], see Figure 3. Furthermore, we prove a general class of nonlinear Schrödinger equations are nonconservative:

**Theorem 3.** For $d \geq 1$, $p \geq 2$, and a real-analytic function $f : \mathbb{C} \to \mathbb{C}$ satisfying $f(0) \neq 0$, consider

$$iu_t = \Delta u + u^p f(u) \quad (3)$$

with analytic initial data $u(0) \in C^\omega(\mathbb{T}^d, \mathbb{C})$.

- There exists an open set of homoclinic solutions limiting to the 0 equilibrium.
- If $X \subset \{ u : \mathbb{T}^d \to \mathbb{C} \}$ is a Banach space and $C^\omega(\mathbb{T}^d, \mathbb{C}) \hookrightarrow X$ is a dense, continuous embedding, then the only real-analytic functionals $F : X \to \mathbb{R}$ conserved under (3) are constant.

Additional difficulties arise when developing rigorous numerics for PDEs on unbounded domains. Recent work on the Maslov index has extended classical results from Sturm-Liouville theory to a much more general setting, thus allowing for spectral stability of nonlinear waves in a variety of contexts to be determined by counting conjugate points. With M Beck (Boston University) we are developing a framework for the computation of conjugate points using rigorous numerics [14]. We apply our method to a parameter-dependent system of bistable equations and show that there exist both stable and unstable standing fronts. In comparison with rigorous numerical methods to compute stability using the Evans function [15], our preliminary results suggest that counting conjugate points is much more efficient. With PhD student Hannah Pieper, we are looking to extend this methodology to 4th order systems such as the Swift-Hohenberg equation.

## 4 Computational Algebraic Topology

Partial differential equations can be extremely useful in describing patterns in biological and physical systems. However, these patterns can be quite complicated, exhibiting distinct structures at different spatial/temporal scales, and it is usually impossible to completely understand them using analytical methods. Often a coarser but computationally tractable description is needed. In recent years computational topology has become widely recognized as an important tool for quantifying complex structures.
Fig. 4: Top: $\epsilon$-neighborhoods about a sample of 100 points from the Sierpinski triangle. Bottom: Persistence diagrams for (a) 100 point sample from the Sierpinski triangle, (b) 1000 points from the Sierpinski triangle and (c) the Sierpinski triangle itself (where the area of a dot is proportional to the number of persistent homology intervals with corresponding birth and death time).

Persistent homology is an algebraic tool that provides a mathematical framework for analyzing the multi-scale structures frequently observed in nature. More specifically, it tracks how the homology groups of a filtration of topological spaces $X_1 \subseteq X_2 \subseteq \ldots \subseteq X_N$ are mapped as one space is included into the next. Similar to classical Morse theory, a filtration can be generated from the sublevel sets of a continuous function $f : X \to \mathbb{R}$ by defining $X_t = f^{-1}((-\infty, t])$. For example, the sublevel set filtration of the distance function to a point cloud in $\{x_i\} \subseteq \mathbb{R}^n$ corresponds to growing $\epsilon$-balls about each point, see Figure 4. The 0-dimensional persistent homology tracks when connected components first appear in the filtration, and later merge together; the 1-dimensional persistent homology tracks when loops in the space appear and disappear. Namely, if a specific homology generator first appears in $H_j(X_b)$, and is first mapped to zero by the inclusion induced map $i_* : H_j(X_b) \to H_j(X_d)$, then $[b, d]$ is referred to as the corresponding $j$-dimensional persistence interval. Long persistence intervals are generally considered to correspond to important topological features, whereas short intervals are considered to be noise.

If a point cloud is sampled from a $d$-dimensional Lebesgue measure, the important persistence intervals will stabilize as the number of points $n$ increases, and the average length of the noisy intervals will decrease. However, the summed-length of all 0-dimensional persistence intervals will grow in proportion with $n^{\frac{d+1}{d}}$. In fact, a fractional dimension can be defined for a measure in terms of the asymptotic growth of the totaled persistence intervals of point samples [16].

In [17] with B Schweinhart (SUNY Albany) we implement an algorithm to estimate the $i$-dimensional persistent homology dimension ($i = 0, 1, 2$) to study self-similar fractals, chaotic attractors, and an empirical dataset of earthquake hypocenters. We compare the performance of these persistent homology dimensions to classical methods to compute the correlation and box-counting dimensions. In summary, the performance of the 0-dimensional persistent homology dimension is comparable to that of the correlation dimension, and generally better than box-counting, whereas
the higher persistent homology dimensions are worse.

When studying multiscale behavior a suitable discretization needs to be chosen. Fine structures may require a fine discretization to accurately describe, whereas a coarser discretization may be sufficient for large regions of space. With M Kramar (University of Oklahoma) we developed a theoretical framework for the algorithmic computation of an arbitrarily good approximation of sublevel-set persistent homology [18]. We implement a rigorous numerical method to compute the persistent homology in the case $f : [0, 1]^2 \to \mathbb{R}$ and provide a posteriori bounds of the approximation error introduced.

5 Chaos in Fast-Slow Systems

In ongoing work with E Sander (George Mason University) and J Touboul (Brandeis University), we are studying how biological function can be maintained or disrupted in chaotic in fast-slow systems. One particular example is the noninvertible 2D Rulkov map, a phenomenological model of a bursting neuron

$$x \mapsto \frac{\alpha}{1 + x^2} + y + I,$$
$$y \mapsto y - \mu (x - \sigma)$$

(4)

taking parameters $\alpha, \mu > 0$ and $\sigma = -1, I = 0$.

For $0 < \mu \ll 1$ the attractor is best thought of as a relaxation cycle – slowly traversing down the bifurcation diagram of a 1D unimodal map on the right, rapidly transitioning to a fixed point of the fast subsystem on left, and then repeating. Two types of such relaxation cycles are possible cf Figure 5 and despite both being chaotic, the periodicity of these cycles are generally quite well behaved. However there is a crisis bifurcation near $\alpha = 4$ and for a range of parameters both types of relaxation cycles can occur, which we argue is the most destructive type of chaos for the proper biological functioning of a neuron. In addition to the 2D Rulkov map, we are also studying this type of chaotic behavior in networks models of neurons, a realistic Hodgkin-Huxley-based ODE model, and empirical data from the Marder lab (Brandeis University).

Fig. 5: Attractors and time series for the Rulkov map at parameters $\mu = 0.001, \sigma = -1, I = 0$, and, from left to right, $\alpha = 3.94, 4.00, 4.05$. 
References


