Math 876: PDE Seminar 2022
Week 10: Exponential Dichotomies

April 5 & 7, 2022

General:

Exponential dichotomies are an indispensable tool in the study of dynamics and differential equations. At first glace, this may seem very similar to what we've seen before: write out the variation of constants formula, and then hit it with $\|e^{-At}\| \le Me^{-at}$ type estimates until we obtain a mild/strong/classical solution. But there's more to it than just that.

The purpose using exponential dichotomies is separate the stable trajectories from the unstable trajectories. The clever trick for getting this to work is that we always want to flow the STABLE orbits FORWARD, and the UNSTABLE orbits BACKWARDS. This way we can avoid trying to solve a backwards heat equation. It is somewhat like how when you pet a cat, you want to go with the fur, not against it. (That said, the cat may still swipe at you.)

So what do we get out of exponential dichotomies? Since we always set up our estimates so things are bounded by negative exponentials, we are able to characterize the behavior of solutions on an infinite time intervals, and characterize stable/unstable sets!

Primary Reading: §4.5.1, §4.5.2 Exponential Dichotomies

Secondary Reading: §4.9. The Linearized Equation.

Important Concepts: All of the bolded words in the §4.5, there are many; Lemma 45.2; Theorem 45.7.

Reading Questions: Email me at least 3 questions on the reading at least an hour before class on Tuesday.

Presentations:

Marcus-Yamabe Example (Tu):

The Marcus-Yamabe example is the non-autonomous linear ODE given by $u_t = M(t)u$, where

$$M(t) = \begin{pmatrix} -1 + \frac{3}{2}\cos^2(t) & 1 - \frac{3}{2}\sin(t)\cos(t) \\ -1 - \frac{3}{2}\sin(t)\cos(t) & -1 + \frac{3}{2}\sin^2(t) \end{pmatrix}$$

Note, this matrix has eigenvalues $\frac{1}{4}(-1 \pm i\sqrt{7})$ for all t; we might suspect that since the real parts of the eigenvalues are negative, then solutions to the system decay exponentially. However $u(t) = e^{\frac{t}{2}} \begin{pmatrix} -\cos(t) \\ \sin(t) \end{pmatrix}$ and $w(t) = e^{-t} \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$ are solutions.

Work out this example and show that it has an exponential dichotomy. This chapter introduces a lot of new bolded definition word. Try to hightlight as many as you can in this example.

Mild solutions in the presence of an exponential dichotomy (Tu):

Present Theorem 45.7. As always, try to impart an intuitive understanding of the theorem in your presentation, and judicially omit parts of the theorem for conciseness. For example, you could focus on parts (1), (4), and perhaps (3) too, briefly commenting on any differences for the other parts. Also, while shifted semiflows are important for studying the stability of stationary solutions/traveling waves, given the time limitations I'd recommend just covering the $\lambda = 0$ case.

Problems:

#1. Constructing an Exponential Dichotomy

On page 199 the authors say: *The characteristic K, which is dependent on the norm on $\mathbb{R}^n$ or $\mathbb{C}^n$ has the property that $K \to \infty$, as the angle between any two of the spaces $\mathcal{R}(Q)$ $\mathcal{R}(R)$ or $\mathcal{R}(P)$ goes to 0.* We explore this phenomena below. For $R, \epsilon > 0$, consider the family of linear ODEs given by

$$\dot{x} = \mathbb{A}x, \qquad \mathbb{A} = \begin{pmatrix} 1 & R \\ 0 & -1/\epsilon \end{pmatrix}. \tag{1}$$

(a) Show that this system has an exponential dichotomy with characteristics

$$K = \sqrt{1 + \frac{R^2\epsilon^2}{(\epsilon+1)^2}}, \qquad \alpha = \min\{1, 1/\epsilon\}.$$

2 Inhomogeneous Equations

Consider the non-autonomous, inhomogeneous equation

$$w_t - Aw = B(t)w \tag{2}$$

for coupling parameter $\alpha \in \mathbb{R}$ and matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \qquad B(t) = \begin{pmatrix} 0 & \alpha e^{-4t} \\ 0 & 0 \end{pmatrix}$$

(a) Show that the solution operator to (2) is given by

$$\Phi(B_\tau, t) = \begin{pmatrix} e^t & \alpha e^{4\tau}\left(e^{2t} - e^t\right) \\ 0 & e^{-2t} \end{pmatrix},$$

and that it satisfies the cocycle identity (44.9).

(b) Show that the map

$$P(t) = \begin{pmatrix} 1 & -\alpha e^{4t} \\ 0 & 0 \end{pmatrix}$$

is a projector (ie $P^2 = P$) and that it is invariant, satisfying (45.2). Go on to show that the system (2) has an exponential dichotomy.

(c) Consider the inhomogeneous equation

$$w_t - Aw = B(t)w + h(t) \tag{3}$$

where we define

$$h(t) = \begin{pmatrix} 0 \\ 10e^t \end{pmatrix}.$$

Show that Theorem 45.7 (1) applies. Moreover, show that for every point $\xi \in \mathcal{R}(P(B))$ there is a mild solution $w = w(t) = w(\xi, t)$ such that $P(B)w(0) = \xi$, and use (45.38) to compute

$$w(t) = \Phi(B,t)\xi + \begin{pmatrix} \frac{10}{3}\alpha e^{5t} - \frac{5}{2}\alpha e^t \left(e^{4t} - 1\right) \\ \frac{10e^t}{3} \end{pmatrix}$$

(d) Show that Theorem 45.7 (4) (Invariance Property 1) holds, and that $\xi(t) = P(B_t)(w(\xi, t))$ satisfies the reduced evolutionary equation.

(e) What can you say about the unstable set $\mathcal{U}$?

#3. Unique Negative Continuations

(Optional) Do book problem #45.4, verifying the strong cocycle identity (45.10).

#4 Marcus-Yamabe continued

Consider the following nonlinear-autonomous version of the Marcus-Yamabe example

$$\begin{pmatrix} x_t \\ y_t \\ \theta_t \end{pmatrix} = \begin{pmatrix} -1 + \frac{3}{2}\cos^2(\theta) & 1 - \frac{3}{2}\sin(\theta)\cos(\theta) & 0 \\ -1 - \frac{3}{2}\sin(\theta)\cos(\theta) & -1 + \frac{3}{2}\sin^2(\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where we take $(x, y, \theta) \in \mathbb{R}^2 \times \mathbb{T}^1$, for $\mathbb{T}^1 \cong \mathbb{R}/2\pi\mathbb{Z}$. This system has a periodic orbit

$$m_t = (0, 0, t)^T \tag{4}$$

with period 2π. If we linearize about this periodic orbit, we get back a linear-non-autonomous ODE

$$u_t - Au = B(t)u \tag{5}$$

for $u = (x, y, p_\theta) \in \mathbb{R}^2 \times \mathbb{R}^1$, where we define

$$A = \begin{pmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B(t) = \frac{3}{2} \begin{pmatrix} \cos^2(t) & -\sin(t)\cos(t) & 0 \\ -\sin(t)\cos(t) & \sin^2(t) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

It is important to note that p_θ, the "angular momentum", lives in a Banach space $\mathbb{R}^1$, whereas the angle θ lives in a manifold $\mathbb{T}^1$.

(a) Show that (5) has an exponential trichotomy. Make sure to define all necessary mathematical objects (eg, the four characteristics, the space $\mathcal{E} = W \times M$, the projectors, etc.)