Math 876: PDE Seminar 2022
Week 11: Robustness of Exponential Dichotomies

April 12 & 14, 2022

General:

We continue our study of exponential dichotomies, obtaining powerful theorems which show that exponential dichotomies are robust to "small" perturbations $B : \mathbb{R} \to \mathcal{L}(V^{2\beta}, W)$. There is a heavy dose of sarcasm in saying these perturbations are small. We don't require $B \to 0$ as $t \to \pm\infty$ and moreover $B(t)$ may be an unbounded operator if we consider it in $\mathcal{L}(W, W)$! This level of generality is not without motivation: it's what is needed to study the stability of periodic orbits in PDEs like Navier-Stokes.

The key ingredients which allow us to call these maps "small" are the various topologies $\mathcal{T}_{bo}, \mathcal{T}_A$ on the spaces $\mathcal{M}^p, \mathcal{M}^\infty$ we introduced in §4.4. Furthermore the proofs in these sections have quite a different flavor to them; it's not our standard fare of variation of constants and Gronwall's lemma!

It is essential that we understand (linear) exponential dichotomies before moving on to (nonlinear) invariant manifolds. We'll be studying the Burgers-Sivashinsky in the homework for the next few weeks as a toy example to apply all the theory we've been building up. This week we'll work out some foundational estimates.

Primary Reading:

§4.5.3-4.5.4: Robustness of Exponential Dichotomies

Secondary Reading: Read the following material to see how the stage is being set to tackle the proofs of stable/unstable/center manifolds.

- §7.1.1 Up to 466: Local dynamics near an equilibrium
- Preparation Lemma 47.10

Important Concepts: Strong boundedness properties; relation between discrete and continuous exponential dichotomies; robustness of exponential dichotomies; prepared nonlinearities; local coordinates and the error term E.

Reading Questions: Email me at least 3 questions on the reading at least an hour before class on Tuesday.

Presentations:

Graph Transforms and Strong Boundedness (Tu):

Present Theorem 45.8, starting by explaining what this operator G is what it is used for. (Can you compute what G is in the Marcus Yamabe example for the time-2π map?). Then try to sketch out the intuition behind the proof without getting caught up in the smaller details.

Robustness of Exponential Dichotomies (Tu):

Tell us about the Robustness of Exponential Dichotomies (Theorem 45.11) and explain what is going on in Figure 4.3. There are a lot of different spaces floating around with different geometries/topologies. Try to give a big picture presentation of what is going on without getting caught up in the smaller details.

Problems:

#1 Strong Boundedness Property.

Do book problem # 45.13 (Prove Lemma 45.9).

#2 Partially Coupled Systems.

Do book problem # 45.17 (1), (2), (4) studying partially coupled systems. *Note: One place partially coupled systems come up is in the Galerkin approximation* $W = W_N \times W_\infty$ *where* $W_N \cong \mathbb{R}^N$.

#3. Burgers-Sivashinsky Equation

Fix the Hilbert space $H = L^2_{per,sym}(-\pi,\pi)$ subject to periodic and odd boundary conditions:

$$u(t,x) = -u(t,-x), \qquad u(t,x) = u(t,x+2\pi). \tag{1}$$

Remark 1. *Alternatively, we can work with a weighted Wiener algebra; for a number* $s \geq 0$ *and* $u \in H$ *we define a norm*

$$\|u\|_{A_s} = \sum_{n\in\mathbb{Z}} (1+|n|)^s |\hat{u}(n)|, \qquad \hat{u}(n) = \frac{1}{2\pi}\int_{[-\pi,\pi]} u(x)e^{-inx},$$

for $s \geq 0$. *We define the weighted Wiener algebra* $A_s \subseteq H$ *to be the set of functions* u *with* $\|u\|_{A_s} < \infty$.

For $s \in \mathbb{N}$ we have $C^{s,0} \subseteq A_s \subseteq C^{s,1}$. Furthermore the space A_s is a Banach algebra, and is isomorphic via the Fourier transform to our familiar space ℓ^1_s.

In anycase, let W be your favorite Banach space between the choices of $L^2_{per,sym}(-\pi,\pi)$ and A_0. Define the sectorial operator $A = -\nu\partial_{xx}$ for $\nu > 0$ defined on W, whereby $\sigma(A) = \{\nu k^2\}_{k=1}^\infty$ and $\|e^{-At}\|_{\mathcal{L}(W,W)} \leq e^{-\nu t}$.

Furthermore, let $V^{2\alpha}$ denote the fractional power spaces generated by A. If you chose $W = L^2$, then we'd have $V^{2\alpha} \cong H^\alpha_{per,sym}((-\pi,\pi),\mathbb{R}) \cong \ell^2_\alpha$. Alternatively, if you chose $W = A_0$, then we'd have $V^{2\alpha} \cong A_\alpha \cong \ell^1_\alpha$.

Consider the forced, viscous Burgers-Sivashinsky equation:

$$u_t - \nu\partial_{xx}u = \underbrace{\lambda u + (\partial_x u)u - f(x)}_{=:F(u)}, \tag{2}$$

where we define the forcing term:

$$f(x) = \beta\,(\lambda - \nu)\sin(x) + \tfrac{1}{2}\beta^2 \sin(2x).$$

There are three parameters to the problem: λ, β, and ν. The λ term is adding instability to the dynamics about zero; ν is the viscosity which can make the higher modes decay slower or faster, and β determines how far away from the "flat" dynamics about 0 that we will be looking.

1. Show that functions represented by sine series are an invariant subspace of solutions in (2). That is, show that if $u \in V^1$ satisfies the boundary conditions (1), then it can be represented by a sine series $u(t,x) = 2\sum_{k\in\mathbb{Z}} a_k(t)\sin(kx)$ where $a_k : [0,T) \to \mathbb{R}$ and $a_k(t) = -a_{-k}(t)$. Furthermore, show that $\partial_{xx}u + F(u)$ can be represented by a sine series.

2. Show that (2) has an equilibrium point $u_0 \in V^1$ given by

$$u_0(x) = \beta \sin(x) = \beta \sum_{k=-\infty}^{\infty} i a_k e^{ikx}, \tag{3}$$

   where $a_{\pm 1} = \mp(2)^{-1}$, and $a_k = 0$ otherwise.

3. Define the maps $B = DF(u_0) \in \mathcal{L}(V^1, W)$ and $L = A - B$. Then localize the equation so that it is in the form like equation (71.7) in the book:

$$\partial_t v + Lv = E(u_0, v) \tag{4}$$

   In the case where $W = A_0$, explicitly describe the map L and the function E. *This will have a nice formula in terms of discrete convolutions, although things get a bit messy when you look at each component.*

4. Suppose that $\sqrt{\lambda/\nu} \notin \mathbb{N}$. Show that if $\beta > 0$ is sufficiently small then e^{-Lt} has an exponential dichotomy on W.

Problems for Next Week:

The following problems continue our study of the Burgers-Sivashinski equation. If you've finished all the previous problems, you can start working on these ones.

1. Show that $E \in C^1_{Lip}(V^1, W)$; for $\delta > 0$, there exists constants $C_0(\delta), C_1(\delta) > 0$ such that

$$\sup_{\|v\|\le\delta} \|E(v)\|_W \le C_0 \|v\|_{V^1} \tag{5}$$

$$\sup_{\|v_1\|,\|v_2\|\le\delta} \|E(v_1) - E(v_2)\|_W \le C_1 \|v_1 - v_2\|_{V^1} \tag{6}$$

   How does this change if we replace E by the prepared nonlinearity E^a (cf Lemma 47.10).

2. Suppose $\nu^{-1}, \lambda, \beta > 0$ are small. Use Gronwall's Lemma and a preparation of the nonlinearity as in Lemma 47.10 to show that there are some $\delta, K, \mu > 0$, such that

$$\|v_1(t)\|_{V^1} \leq K e^{-\mu t} \|v_1(0)\|_{V^1}, \qquad t \geq 0 \tag{7}$$

$$\|v_1(t) - v_2(t)\|_{V^1} \leq K e^{-\mu t} \|v_1(0) - v_2(0)\|_{V^1}, \qquad t \geq 0 \tag{8}$$

whenever v_1, v_2 are a mild solutions of (4) with $\|v_1(0)\|_{V^1}, \|v_2(0)\|_{V^1} \leq \delta$.

3. Compute a finite dimensional approximation of the operator $L = -A + B$ using the methods from the homework in week 3. In particular, for $M \in \mathbb{N}$, define the projection $\pi_M : \ell^2_\alpha \to \mathbb{R}^M$ and the inclusion $\iota_M : \mathbb{R}^M \to \ell^2_\alpha$ by

$$\pi_M(a) = (i^{-1}a_1, i^{-1}a_2, \ldots, i^{-1}a_M) \qquad \iota_M(a) = (\ldots 0, -ia_{-M}, \ldots ia_M, 0 \ldots)$$

Then define the finite dimensional operator $L_M : \mathbb{R}^M \to \mathbb{R}^M$ by

$$L_M = \pi_M(-A + B)\iota_M$$

Write computer code to compute this matrix L_M for given values of M, ν and β.

4. For parameters $\lambda = 0.1$, $\beta = 0.2$, $\nu = 0.1$, and $M = 16$, compute L_M. What are the few eigenvalues with the largest real part? Plot the corresponding eigenfunctions. How do these features change if you increase M to 64 or 256? What happens if you change to the parameter $\lambda = 0.2$?