Math 876: PDE Seminar 2022
Week 12: Local Dynamics Near an Equilibrium

April 19 & 21, 2022

General:

The goal in this section is to separate the stable dynamics from the unstable dynamics in a nonlinear flow, and in order to obtain a positive/negative invariant set, we need to control the trajectories for a semi-infinite amount of time.

We did this in §4.5 using exponential dichotomies when the equations were linear (albeit potentially nonautonomous and inhomogeneous). In §4.6-§4.7 we were able able to deal general nonlinear equations for a short amount of time. If the nonlinearity was globally Lipschitz, we proved solutions existed for all forward time. In §5.1, if we obtained global existence with the help of energy estimates and Lyapunov functions.

To prove the stable/unstable manifold theorem for generic hyperbolic equilibria and generic nonlinearities, we will combine all of the tools we've learned this semester. Like in §4.6-4.7, we'll try to apply a contraction mapping theorem to a variation of constants formula which treats the nonlinearity like an inhomogeneous forcing. Then we will apply our results on exponential dichotomies from §4.5 to pick out the stable/unstable sets. (Note we only need to deal with the autonomous case; the set $\mathcal{K} = \{u_0\}$ is just a point!) There still remains a difficulty with the nonlinearity not being globally Lipschitz, but we'll take care of this using the preparation lemma.

There are a lot of moving parts but we've seen them all before. Now we just have to put them together!

Primary Reading: All of §7.1.1.

Secondary Reading: Preparation Lemma 47.10

Important Concepts: Saddle Point Property; Lyapunov-Perron method.

Reading Questions: Email me at least 3 questions on the reading at least an hour before class on Tuesday.

Presentations:

Saddle Point Property (Tu): Present the proof of Lemma 71.2 for the unstable manifold. (3 presenters, 15 minutes each?)

- Introduce us to the Lyapunov-Perron operator (71.20); pages 467-first paragraph of page 468. Introduce necessary notation/variables, and explain the intuition for why $\mathcal{T}$ is the right operator to define.
- Present the proof showing that $\mathcal{T}$ maps $\mathcal{F}_\rho$ into $\mathcal{F}_\rho$ and that it is a contraction mapping; pages 468-469.

- Present the rest of the proof for the unstable manifold; page 470-471.

Part #1. Burgers-Sivashinsky Equation: Foundations and Dichotomies

Fix the Hilbert space $H = L^2_{per,sym}(-\pi,\pi)$ subject to periodic and odd boundary conditions:

$$u(t,x) = -u(t,-x), \qquad u(t,x) = u(t,x+2\pi). \tag{1}$$

Remark 1. *Alternatively, we can work with a weighted Wiener algebra; for a number $s \geq 0$ and $u \in H$ we define a norm*

$$\|u\|_{A_s} = \sum_{n\in\mathbb{Z}} (1+|n|)^s |\hat{u}(n)|, \qquad \hat{u}(n) = \frac{1}{2\pi}\int_{[-\pi,\pi]} u(x)e^{-inx},$$

for $s \geq 0$. We define the weighted Wiener algebra $A_s \subseteq H$ to be the set of functions u with $\|u\|_{A_s} < \infty$.

For $s \in \mathbb{N}$ we have $C^{s,0} \subseteq A_s \subseteq C^{s,1}$. Furthermore the space A_s is a Banach algebra, and is isomorphic via the Fourier transform to our familiar space ℓ^1_s.

In anycase, let W be your favorite Banach space between the choices of $L^2_{per,sym}(-\pi,\pi)$ and A_0. Define the sectorial operator $A = -\nu\partial_{xx}$ for $\nu > 0$ defined on W, whereby $\sigma(A) = \{\nu k^2\}_{k=1}^{\infty}$ and $\|e^{-At}\|_{\mathcal{L}(W,W)} \leq e^{-\nu t}$.

Furthermore, let $V^{2\alpha}$ denote the fractional power spaces generated by A. If you chose $W = L^2$, then we'd have $V^{2\alpha} \cong H^{\alpha}_{per,sym}((-\pi,\pi),\mathbb{R}) \cong \ell^2_\alpha$. Alternatively, if you chose $W = A_0$, then we'd have $V^{2\alpha} \cong A_\alpha \cong \ell^1_\alpha$.

Consider the forced, viscous Burgers-Sivashinsky equation:

$$u_t - \nu\partial_{xx}u = \underbrace{\lambda u + \partial_x(u^2) - f(x)}_{=:F(u)}, \tag{2}$$

where we define the forcing term:

$$f(x) = \beta(\lambda-\nu)\sin(x) + \tfrac{1}{2}\beta^2\sin(2x).$$

There are three parameters to the problem: λ, β, and ν. The λ term is adding instability to the dynamics about zero; ν is the viscosity which can make the higher modes decay slower or faster, and β determines how far away from the "flat" dynamics about 0 that we will be looking.

1. Show that functions represented by sine series are an invariant subspace of solutions in (2). That is, show that if $u \in V^1$ satisfies the boundary conditions (1), then it can be represented by a sine series $u(t,x) = 2\sum_{k\in\mathbb{Z}} a_k(t)\sin(kx)$ where $a_k : [0,T) \to \mathbb{R}$ and $a_k(t) = -a_{-k}(t)$. Furthermore, show that $\partial_{xx}u + F(u)$ can be represented by a sine series.

2. Show that (2) has an equilibrium point $u_0 \in V^1$ given by

$$u_0(x) = \beta \sin(x) = \beta \sum_{k=-\infty}^{\infty} i a_k e^{ikx}, \tag{3}$$

where $a_{\pm 1} = \mp(2)^{-1}$, and $a_k = 0$ otherwise.

3. Define the maps $B = DF(u_0) \in \mathcal{L}(V^1, W)$ and $L = A - B$. Then localize the equation so that it is in the form like equation (71.7) in the book:

$$\partial_t v + Lv = E(u_0, v) \tag{4}$$

In the case where $W = A_0$, explicitly describe the map L and the function E. *This will have a nice formula in terms of discrete convolutions, although things get a bit messy when you look at each component.*

4. Suppose that $\sqrt{\lambda/\nu} \notin \mathbb{N}$. Show that if $\beta > 0$ is sufficiently small then e^{-Lt} has an exponential dichotomy on W.

Part #2. Burgers-Sivashinsky Equation: Linearization Far from 0

(a) For $M \in \mathbb{N}$, define the projection $\pi_M : \ell^1_\alpha \to \mathbb{R}^M$ and the inclusion $\iota_M : \mathbb{R}^M \to \ell^1_\alpha$ by

$$\pi_M(a) = (i^{-1}a_1, i^{-1}a_2, \ldots, i^{-1}a_M) \qquad \iota_M(a) = (\ldots 0, -ia_M, \ldots ia_M, 0 \ldots)$$

Use this to write a projected ODE for (4); $\partial_t x + L_M x = E_M(x)$ where $E_M : \mathbb{R}^M \to \mathbb{R}^M$ with $E_M(x) = \pi_M E(\iota_M(x))$. What is this in the case $M = 2$?

(b) Compute a finite dimensional approximation of the operator $L = -A + B$ using the methods from the homework in week 3. That is, define the finite dimensional operator $L_M : \mathbb{R}^M \to \mathbb{R}^M$ by

$$L_M = \pi_M(-A + B)\iota_M$$

Write computer code to compute this matrix L_M for given values of M, ν and β.

(c) For a large variety of parameters λ, β, ν, M compute L_M and its eigenvalues. How are the eigenvalues affected by the parameters?

Short Mathematica exercise

Consider the system

$$\dot{x} = -x + Ry - xy \tag{5}$$
$$\dot{y} = -4y + x^2 \tag{6}$$

The origin is a stable equilibrium for all values of $R > 0$. How does the basin of attraction of the origin change as you vary R? How should this affect the way we choose our prepared nonlinearity? (Explore the system using numerics; don't write a formal proof)

Part #3. Burgers-Sivashinsky Equation: Neighborhood of stability

For this problem we are going to set $\beta = 0$ and $\nu = 1$. In Fourier space, this becomes:

$$\partial_t a_k = (-k^2 + \lambda) a_k - \tfrac{k}{2}(a * a)_k \tag{7}$$

with $a \in \ell^1_1$, $a_k = -a_{-k} \in \mathbb{R}$ for $k \in \mathbb{Z}$.

1. Suppose $\lambda = 0$. Find **explicit values** $\rho, K, \mu > 0$ such that

$$\|u_1(t) - u_2(t)\|_{V^1} \leq K e^{-\mu t} \|u_1(0) - u_2(0)\|_{V^1}, \qquad t \geq 0 \tag{8}$$

   whenever u_1, u_2 are a mild solutions of (4) with $\|u_1(0)\|_{V^1}, \|u_2(0)\|_{V^1} \leq \rho$.

2. (Optional) Suppose $\lambda = 2$.[1] Write out what the Lyapunov-Perron operator is using the basis from ℓ^1_1. Explicitly describe the domain/range of $\mathcal{T}$ so that it is not just a formal definition. Note that $\mathcal{T}$ is great for applying estimates, but ugly for doing computations.

[1] I am not sure if my λ matches up exactly with what is in the book; it might be off by a sign.