# Math 876: PDE Seminar 2022
# Week 13: Bifurcations and Center Manifolds

April 26, 28, May 3, 2022

## General:

Understanding bifurcations and the dynamics on center manifolds is difficult; the taxonomy of bifurcations is quite varied, the objects of interest are fundamentally nonlinear, and have fewer nice properties (eg uniqueness, smoothness). All of these problems are present in ODEs. Generalizing the theory to PDEs is not significantly harder than how we generalized the theory for stable/unstable manifolds. All in all, there are a lot of complicated things to keep track of.

Rather than try and do a broad treatment of the subject, we are going to a deep dive into a pitchfork bifurcation in the Burgers-Sivashinski equation: first in the finite dimensional Galerkin projection, and then treating the infinite dimensional PDE.

**Primary Reading:** §7.1.2, §7.2.

**Secondary Reading:** §7.1.3 and Chapter 4.2 of Carmen Chicone's book "Ordinary Differential Equations with Applications".

**Tertiary Reading:** To round out our study through the book, I'd recommend taking a look at §7.3 on periodic orbits and §8.1, §8.6 on inertial manifolds.

**Important Concepts:** Center manifolds are not unique; pitchfork bifurcation; gradient systems; Lyapunov-Schmidt reduction; center manifold reduction; Morse index; Morse decomposition.

**Reading Questions:** Email me at least 3 questions on the reading at least an hour before class on Tuesday.

## Presentations:

**Center Manifolds:** Tell us about the the center manifold theorem §7.1.2. How do these results compare with what we get from the (un)stable manifold theorems? What is this reduction principle?

**Gradient Systems:** Tell us about gradient systems and Theorem 72.1. Can a gradient system have a periodic orbit?

**Morse Decompositions:** Tell us about Morse decompositions and Theorem 72.3. For an example, describe Morse decompositions of the system:

$$\dot{x} = x - x^3$$
$$\dot{y} = -2y$$

## Problems:

We continue our study of the Burgers-Sivashinsky Equation:

$$u_t - \partial_{xx} u = \lambda u + \tfrac{1}{2}\partial_x(u^2)$$

We can write odd periodic solutions in terms of a Fourier-sine series

$$u(t,x) = -2 \sum_{k=-\infty}^{\infty} a_k(t)\sin(kx) = \sum_{k=-\infty}^{\infty} i a_k(t) e^{ikx}, \tag{1}$$

where $a_k : [0,+\infty) \to \mathbb{R}$ and $a_k(t) = -a_{-k}(t)$. Plugging in we obtain an equation in Fourier space:

$$\partial_t a_k = \underbrace{-k^2 a_k}_{(-Aa)_k} + \underbrace{\lambda a_k - \tfrac{k}{2}(a*a)_k}_{F_\lambda(a)_k} \tag{2}$$

with $a \in C([0,T), \ell^1_1)$, $a_k = -a_{-k} \in \mathbb{R}$ for $k \in \mathbb{Z}$. Due to the odd-symmetry, we can restrict to an isomorphic space of one sided sequences. That is, for $\{a_k\}_{k=0}^{\infty}, \{b_k\}_{k=0}^{\infty} \in \ell^1_{s,\mathbb{N}} \cong \ell^1_{s,odd}$, we define

$$\|a\|_{\ell^1_{s,\mathbb{N}}} = 2\sum_{n=1}^{\infty} |k|^s |a_k|$$

(Note that $a_0 = 0$.) With this norm, the space of one sided sequence is a Banach algebra; that is $\|a*b\|_{\ell^1_{s,\mathbb{N}}} \leq \|a\|_{\ell^1_{s,\mathbb{N}}} \|b\|_{\ell^1_{s,\mathbb{N}}}$ where we define the discrete convolution as $a*b = \{(a*b)_n\}_{n=0}^{\infty}$

$$\begin{aligned}(a*b)_n = \sum_{\substack{k_1+k_2=n \\ k_1,k_2\in\mathbb{Z}}} a_{k_1} b_{k_2} &= \sum_{k=0}^{n} a_{n-k} b_k + \sum_{k=1}^{\infty} a_{n+k} b_{-k} + a_{-k} b_{n+k} \\ &= \sum_{k=0}^{n} a_{n-k} b_k - \sum_{k=1}^{\infty} a_{n+k} b_k + a_k b_{n+k}\end{aligned}$$

We will also be looking at the 2-mode Galerkin projection:

$$\begin{aligned}\partial_t a_1 &= (-1^2 + \lambda) a_1 - \tfrac{1}{2}\left(-2a_1 a_2\right) \\ \partial_t a_2 &= (-2^2 + \lambda) a_2 - \tfrac{2}{2}\left(a_1^2\right)\end{aligned} \tag{3}$$

## #1 Lyapunov–Schmidt reduction: ODEs

The **Morse index** of a hyperbolic equilibrium is defined as the dimension of the unstable eigenspace in its linearization. When a supercritical pitchfork bifurcation occurs, the Morse index of an equilibrium $u_0$ increases by one, and two new equilibria $u_\pm$ are born having the same Morse index that $u_0$ used to have. In general, if the bifurcation occurs at parameter $\lambda_0$ then the 'amplitude' grows like $\|u_\pm - u_0\| = \mathcal{O}(\sqrt{\lambda - \lambda_0})$. Since square roots are unpleasant to work, often one will **unfold** the bifurcation; fix the amplitude $\|u_\pm - u_0\| = \epsilon$, then solve for $\lambda - \lambda_0 = \mathcal{O}(\epsilon^2)$. This is the essence of the **Lyapunov–Schmidt reduction**.

Fix the space $X = \mathbb{R} \times (\ell^1_s/\mathbb{R})$ and consider an element $b = (\lambda_0, b_2, b_3, \dots) \in X$. Define the function $\mathcal{F}_\epsilon : X \to \ell^1_{s-2}$ by

$$\mathcal{F}_\epsilon(b) = \begin{cases} \frac{1}{\epsilon^3}(-A\tilde{a} + F_\lambda(\tilde{a}))_1 & \text{if } k = 1 \\ \frac{1}{\epsilon^2}(-A\tilde{a} + F_\lambda(\tilde{a}))_k & \text{if } k \neq 1 \end{cases}$$

where we define $\lambda = 1 + \epsilon^2\lambda_0$ and we define $\tilde{a} \in \ell^1_s$ as

$$(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \dots) = (\epsilon, \epsilon^2 b_2, \epsilon^2 b_3, \dots)$$

We can use $\mathcal{F}_\epsilon$ to study the equilibria of (2); if $\epsilon \neq 0$ then $\mathcal{F}_\epsilon(b) = 0 \iff F_\lambda(\tilde{a}) = 0$.

1. Define the projection map $\pi_M : X \to \mathbb{R}^M$ and an inclusion map $\iota_M : \mathbb{R}^M \to \ell^1_s$ by

$$\begin{aligned} \pi_M(\lambda_0, b_2, \dots b_M, \dots) &= (\lambda_0, b_2, \dots, b_M) \\ \iota_M(\lambda_0, b_2, \dots, b_M) &= (\lambda_0, b_2, \dots b_M, 0, \dots) \end{aligned}$$

   Write out the function $\mathcal{F}^M_\epsilon = \pi_M \mathcal{F}_\epsilon$ for the ODE system in (3). Then find some $\bar{b}_M \in \mathbb{R}^2$ such that $\mathcal{F}^M_\epsilon(\bar{b}_M) = \mathcal{O}(\epsilon)$.

2. Fix $M = 2$ (or more generally $M \geq 2$). Use the implicit function theorem to prove that there exists some $\epsilon_0 > 0$ and there exists some $\hat{b}_M = \hat{b}_M(\epsilon) \in \mathbb{R}^M$ with $\|\bar{b}_M - \hat{b}_M\| = \mathcal{O}(\epsilon)$ and $\mathcal{F}^M_\epsilon(\hat{b}_M) = 0$ for all $0 < \epsilon < \epsilon_0$. Thereby $\|\bar{a}_M - \hat{a}_M\| = \mathcal{O}(\epsilon^3)$ and $F^M_{\hat{\lambda}}(\hat{a}_M) = 0$ for all $0 < \epsilon < \epsilon_0$.

## #2: Center manifold reduction: ODEs

Consider the extended 2-mode Galerkin projection:

$$\begin{aligned} \partial_t \lambda &= 0 \\ \partial_t a_1 &= (-1^2 + \lambda)a_1 - \tfrac{1}{2}(-2a_1a_2) \\ \partial_t a_2 &= (-2^2 + \lambda)a_2 - \tfrac{2}{2}\left(a_1^2\right) \end{aligned} \tag{4}$$

In general, center manifolds are not unique. Nevertheless $\varphi$ will, in general, have a unique Talyor series, and the globally bounded orbits on the center manifold are unique.

Note that from the reduction principle we have

$$\partial_t \xi + L\xi = RE(u_0, \xi + \varphi(\xi)).$$

Writing $\lambda = 1 + \tilde{\lambda}_0$, for (4) this becomes

$$\begin{aligned} \partial_t \tilde{\lambda} &= 0 \\ \partial_t a_1 &= \tilde{\lambda} a_1 + a_1 \varphi(\tilde{\lambda}, a_1). \end{aligned}$$

By the chain rule, we have

$$\begin{aligned}\partial_t \varphi(\tilde{\lambda}, a_1) &= \begin{bmatrix} \partial_{\tilde{\lambda}} \varphi(\tilde{\lambda}, a_1) & \partial_{a_1} \varphi(\tilde{\lambda}, a_1) \end{bmatrix} \cdot \begin{bmatrix} \partial_t \tilde{\lambda} \\ \partial_t a_1 \end{bmatrix} \\ &= \partial_{a_1} \varphi \left( \tilde{\lambda} a_1 + a_1 \varphi \right) \end{aligned} \tag{5}$$

Since $a_2 = \varphi(\xi)$ then by plugging into (4) we also obtain

$$\partial_t \varphi(\tilde{\lambda}, a_1) = (-3 + \tilde{\lambda}) \varphi(\tilde{\lambda}, a_1) - a_1^2 \tag{6}$$

Equating (5) and (6), we obtain the following **conjugacy equation**, which is a first order PDE:

$$(-3 + \tilde{\lambda}) \varphi - a_1^2 = (\partial_{a_1} \varphi) \left( \tilde{\lambda} a_1 + a_1 \varphi \right) \tag{7}$$

This equation is enforcing an invariance condition. To solve this PDE, we can make an ansatz that we can write $\varphi$ as a power series plus another function:

$$\varphi(\tilde{\lambda}, a_1) = \left( \sum_{m+n=2}^{N} p_{m,n} \tilde{\lambda}^m a_1^n \right) + h(\tilde{\lambda}, a_1) \tag{8}$$

where $\partial_{\tilde{\lambda}}^m \partial_{a_1}^n h(\tilde{\lambda}, a_1) = 0$ for all $|(m, n)| \leq N$. By plugging (8) into the conjugacy equation (7), we can try to solve for the coefficients $p_{m,n}$ for increasing orders of $|(m, n)| = m + n$.

Warning! Without addition work this method is not going to lead us to a proof of a center manifold theorem. For that we need to use other techniques, like those in Theorem 71.4 for example. These two approaches should be seen as complementary tools that, when used together, paint a more detailed picture of the center manifold. Theorem 71.4 guarantees that a center manifolds does exist and there is hope to write it as a graph, and by solving for the coefficients $p_{m,n}$ we are able to approximation this graph and actually compute the dynamics on the center manifold.

(a) Show that the system in (4) has an exponential trichotomy with $P = 0$, $\dim R = 2$, and codim(Q) =2. Show that there exists a local center manifold $M_{loc}^o$ given as the graph of Lipschitz mapping $\xi = (\lambda, a_1) \mapsto \xi + \varphi(\xi)$.

(b) Solve for the coefficients $p_{m,n}$ for $|(m, n)| \leq 2$.

(c) Show that the reduced equation for the center manifold is

$$\begin{aligned}\partial_t \tilde{\lambda} &= 0 \\ \partial_t a_1 &= \tilde{\lambda} a_1 - \frac{1}{3} a_1^3 + \text{higher order terms}\end{aligned}$$

This is also referred to as the **normal form** of the bifurcation.

(d) For $\tilde{\lambda} > 0$, define the one dimensional submanifold

$$M_{loc}^{o,\tilde{\lambda}} = \{(\mu, a_1, a_2) \in M_{loc}^o : \mu = \tilde{\lambda}\}$$

Show that there exists some $\tilde{\lambda}_0$, such that for all $0 < \tilde{\lambda} < \tilde{\lambda}_0$, the set $M_{loc}^{o,\tilde{\lambda}}$ has a Morse decomposition

$$\mathcal{K}_0 = \{u_+, u_-\}, \qquad\qquad \mathcal{K}_1 = \{0\}$$

(e) Show that for $0 < \tilde{\lambda} < \tilde{\lambda}_0$ the set $\mathfrak{A}_{\tilde{\lambda}} = \omega(\mathcal{K}_1) \subseteq M_{loc}^{o,\tilde{\lambda}}$ is an attractor for (3). Then show that there is some neighborhood about $M_{loc}^{o,\tilde{\lambda}}$ for which $\mathcal{K}_0, \mathcal{K}_1$ is a Morse decomposition.

## #3 Lyapunov–Schmidt reduction: PDEs

Repeat problem #1 for the full Burgers-Sivashinsky Equation. If you have any questions, you are more than welcome to come talk to me or your classmates!

## #4: Center manifold reduction: PDEs

Repeat problem #2 for the full Burgers-Sivashinsky Equation. If you have any questions, you are more than welcome to come talk to me or your classmates!

**#5: Bifurcation Diagram** Make a sketch of what you think the bifurcation diagram for the Burgers-Sivashinsky equation looks like for $\lambda \in \mathbb{R}$. How would this compare with a bifurcation diagram where we fix $\lambda = 1$ and let $\nu \to 0$? Make a prediction of what you think the Morse decomposition looks like.