Math 876: PDE Seminar 2022
Week 1: Spectral Methods for PDEs

January 25 & 27, 2022

General: This week we will look at numerical methods for solving parabolic PDEs. The chief difficulty in these PDEs is that both time and space is a continuum. In contrast computers only have finite memory and programs need to finish in finite time. Both time and space need to be discretized somehow in order to be numerically tractable. This leads us to the spectral/Galerkin method, whereby we approximate a PDE by an infinite dimensional ODE, and then approximate that by a high (but finite!) dimensional ODE.

A major theme in this course is investigating how the theory from finite dimensional dynamical systems / ODEs can be applied to study PDEs. The material this week builds a foundation for us to experimentally study the dynamics of PDEs. Furthermore, the 'non-rigorous' numerical approximations we make this week provide motivation for asking how this approach can be made rigorous on a theoretical level.

Required Reading: Section 1-3 and Appendix A from "A short ad hoc introduction to spectral methods for parabolic PDE and the Navier-Stokes equations" by Hannes Uecker (2009).

Supplementary Reading:

- Chapter 1 of the textbook (The Evolution of Evolutionary Equations by Sell and You).
- Orszag, SA "Numerical Simulation of Incompressible Flows Within Simple Boundaries. I. Galerkin (Spectral) Representations" (1971).

Important Concepts: Fourier transform; multiplication of Fourier series, discrete convolution; Galerkin/spectral methods for parabolic PDE; approximating PDEs by an infinite system of ODEs.

Discussion Questions: Email me at least 3 discussion questions at least an hour before class on Tuesday.

Presentations:

Fourier Transform (Tu): Tell us about the Fourier transform and the discrete Fourier transform (DFT). Talk about convergence of Fourier series, and some possible domains and co-domains of the Fourier transform. Is this an invertible transform? What is aliasing and anti-aliasing? The presentation of the Fourier transform in (Uecker 2009) decently self-contained, but feel free to use other sources.

Galerkin's Method (Th): Read pages 1-6 from (Orszag 1971) and do the Galerkin

Aproximation problem carefully. Tell us what Galerkin's method is, and talk about its advantages/disadvantages. Why is the Fast Fourier Transform important?

Problems:

MATLAB Examples: Run the MATLAB code from Uecker (2009) for the following example problems: 3.3, 3.4, 3.5, 3.15, 3.16.

Multiplication of Fourier Series: Consider 2π periodic functions $u, v \in C^2([0, 2\pi], \mathbb{R})$.

(a) Show there exist sequences $\{c_k\}_{k\in\mathbb{Z}}$, $\{d_k\}_{k\in\mathbb{Z}}$, with $c_k, d_k \in \mathbb{C}$ such that

$$u(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx}, \qquad v(x) = \sum_{k\in\mathbb{Z}} d_k e^{ikx}, \tag{1}$$

for all $x \in [0, 2\pi]$.

(b) Define sequences $\{a_k\}_{k\in\mathbb{Z}}$, $\{b_k\}_{k\in\mathbb{Z}}$ with $a_k, b_k \in \mathbb{R}$ and $c_k = a_k + ib_k$. Show that $a_k = a_{-k}$, $b_k = -b_{-k}$ and

$$u(x) = a_0 + 2\sum_{k=1}^{\infty} a_n \cos(kx) + b_k \sin(kx).$$

(c) Show that the product $w(x) = u(x)v(x)$ is in $C^2([0, 2\pi], \mathbb{R})$ and that

$$u(x)v(x) = \sum_{n\in\mathbb{Z}} (c * d)_k e^{inx}$$

where we define the **discrete convolution** $*$ by

$$(c * d)_k = \sum_{j\in\mathbb{Z}} c_{k-j} d_j.$$

(d) Suppose that:

$$u(x) = 5 + 2\cos x + \sin 2x$$

(i) Write u as a complex Fourier series, and compute the complex Fourier series of $u(x)^2$ by hand.

(ii) Use Matlab's FFT (with anti-aliasing!) to compute the Fourier series of $u(x)^2$.

(iii) Plot $u(x)^2$ and your Fourier series for $u(x)^2$. They should agree.

Symmetries and Boundary conditions: One advantage of using Fourier series is that they can allow one to incorporate the symmetries of a problem to reduce dimensionality.

(a) Suppose that $u : [0, \pi] \to \mathbb{R}$ satisfies Neumann boundary conditions

$$0 = \partial_x u(0) = \partial_x u(\pi) \tag{2}$$

What are sufficient conditions on u for which it can be expressed as a uniformly convergent cosine series?

$$u(x) = a_0 + 2\sum_{k=1}^{\infty} a_k \cos(kx), \qquad a_k \in \mathbb{R},\ \forall x \in [0, \pi]. \tag{3}$$

Suppose u is given by (3). What are sufficient conditions on the sequence $\{a_k\}_{k=0}^{\infty}$ so that both u satisfies (2) and $u \in C^1((0, \pi), \mathbb{R})$?

(b) Suppose $u : [0, 2\pi] \to \mathbb{R}$ satisfies odd and periodic boundary conditions

$$u(x) = -u(x), \qquad u(x) = u(x + 2\pi) \tag{4}$$

What are sufficient conditions on u for which it can be expressed as a uniformly convergent sine series?

$$u(x) = 2\sum_{k=1}^{\infty} b_k \sin(kx), \qquad b_k \in \mathbb{R},\ \forall x \in [0, 2\pi]. \tag{5}$$

Suppose u is given by (5). What are sufficient conditions on the sequence $\{b_k\}_{k=0}^{\infty}$ so that both u satisfies (4) and $u \in C^1((0, 2\pi), \mathbb{R})$?

Infinite system of ODEs: Consider a function $u \in C^\infty(\mathbb{R} \times [0, \pi], \mathbb{R})$ given by

$$u(t, x) = a_0(t) + 2\sum_{k \in \mathbb{N}} a_k(t) \cos(kx). \tag{6}$$

(a) Show that if u satisfies the Fisher-KPP equation:

$$\partial_t u = \partial_{xx} u + \lambda u(1 - u), \qquad 0 = \partial_x u(t, 0) = \partial_x u(t, \pi) \tag{7}$$

then the coefficients $a_k : \mathbb{R} \to \mathbb{R}$ satisfy the infinite system of differential equations

$$\begin{aligned} \partial_t a_k &= (\lambda - k^2) a_k - \lambda \sum_{\substack{k_1 + k_2 = k \\ k_1, k_2 \in \mathbb{Z}}} a_{k_1} a_{k_2} \\ &= (\lambda - k^2) a_k - \lambda \left(a * a\right)_k \end{aligned} \tag{8}$$

(b) Suppose there is a sequence of functions $a_k \in C^1(\mathbb{R}, \mathbb{R})$, and suppose there exists some $C > 0$ such that $|a_k(t)| \leq \frac{C}{(1+|k|)^5}$ for all $t \in \mathbb{R}$ and all $k \in \mathbb{N}$. Show that if each function a_k satisfies the ODE in (8), then the function u defined by (6) satisfies the PDE in (7).

Galerkin Approximation: In the context of the Fisher-KPP equation, Galerkin's method is to seek an approximate solution of the form

$$\bar{u}(t,x) = a_0(t) + 2\sum_{k=1}^{N} a_k(t)\cos(kx)$$

for some finite N. In another sense, this is making an ansatz that $a_k(t) = 0$ for $|k| > N$. Plugging this ansatz into (8) we obtain the Galerkin approximation, which is a finite dimensional ODE:

$$\partial_t a_k = (\lambda - k^2)a_k - \lambda \sum_{\substack{k_1+k_2=k \\ |k_1|,|k_2|\le N}} a_{k_1}a_{k_2} \tag{9}$$

For instance, $N = 1$ leads to the two-dimensional ODE

$$\dot{a} = f(a,\lambda) = \begin{pmatrix} \lambda a_0 - \lambda(a_0^2 + 2a_1^2) \\ (\lambda-1)a_1 - \lambda(2a_0a_1) \end{pmatrix}$$

$N = 2$ leads to the three-dimensional ODE

$$\dot{a} = f(a,\lambda) = \begin{pmatrix} \lambda a_0 - \lambda(a_0^2 + 2a_1^2 + 2a_2^2) \\ (\lambda-1)a_1 - \lambda(2a_0a_1 + 2a_1a_2) \\ (\lambda-4)a_2 - \lambda(2a_0a_2 + a_1^2) \end{pmatrix}$$

and $N = 3$ leads to the four-dimensional ODE

$$\dot{a} = f(a,\lambda) = \begin{pmatrix} \lambda a_0 - \lambda(a_0^2 + 2a_1^2 + 2a_2^2 + 2a_3^2) \\ (\lambda-1)a_1 - \lambda(2a_0a_1 + 2a_1a_2 + 2a_2a_3) \\ (\lambda-4)a_2 - \lambda(2a_0a_2 + a_1^2 + 2a_1a_3) \\ (\lambda-9)a_3 - \lambda(2a_0a_3 + 2a_1a_2) \end{pmatrix}$$

Note that since $a_k = a_{-k}$, we do not need to solve for the negative modes, however we do need to consider them when computing discrete convolution products. For more details, see the supplemental reading (Orszag 1971).

Consider a solution $u : \mathbb{R} \times [0,\pi] \to \mathbb{R}$ to the Allen-Cahn equation:

$$\partial_t u = \nu \partial_{xx} u + u - u^3, \qquad 0 = \partial_x u(t,0) = \partial_x u(t,\pi); \nu > 0. \tag{10}$$

(a) For arbitrary N, compute the Galerkin approximation for the Allen-Cahn equation.

(b) Write out the Galerkin approximation for $N = 1$ and $N = 2$.

(c) Consider Example 3.4 in (Uecker 2009) and the code for `ac.m`. If one changes line 15 so that the cubic product is instead computed with anti-aliasing, then their numerical approach would be using a Galerkin approximation. Explain why.