Math 876: PDE Seminar 2022
Week 2: Dynamical Systems: Basic Theory

February 1 & 3, 2022

General: This week we start the Sell & You book. In contrast to the intentionally informal and *ad hoc* reading from last week, Sell and You give a very mathematically precise presentation of the material. One of their goals in this book is to give a unified theory for the dynamics of evolutionary equations. A downside of this, however, is that the book can sometimes come off as overly abstract, and one can be left wanting for examples.

Chapter 2 introduces many many definitions. The first time you are reading the chapter, I would recommend skipping the proofs of the theorems and lemmas. Not a ton of examples are covered in this chapter (or rather they are delayed until later). To compensate, the homework focuses on concrete examples. The last two problems are numerical, and continue with the Galerkin approximation, and you'll explore the dynamics arising in these finite dimensional ODEs.

Required Reading: Sell & You §2.1- §2.3.4

Supplementary Reading: Sell & You, §2.3.7, and Appendix A.6

Important Concepts: Semiflows and semi-groups; Continuity Lemma; Invariant sets; Alpha and omega limit sets; Compact semi-flows; Attractors; Point dissipative; Lyapunov stability

Reading Questions: Email me at least 3 questions on the reading at least an hour before class on Tuesday.

Presentations:

Semiflows (Tu): Tell us about §2.1.1-§2.1.2 on semiflows, invariant sets and limit sets. What is the take-away message from the Characterization Lemma 21.4? What dynamical-systems issues are there when working with semi-flows that aren't present when working with flows? A lot of definitions are introduced in these sections. Many should seem familiar from finite dimensional dynamical systems, but adjustments have been made to work with semi-flows. Prepare HW 1 for Thursday, which looks at examples for several of these definitions.

Attractors (Tu): Tell us about §2.3.0-§2.3.4 on attractors and their properties. Again, a lot of definitions are introduced. Try to summarize all the named theorems/lemmas, such as Stability Theorem 23.10. A thumbnail sketch of §2.3 is given in §2.3.7 although that goes a bit beyond what we cover this week.

Numerics (Th): Do the Dynamics of the Galerkin Approximation problems for the Allen-Cahn equation.

Problems:

1. Semi-flow for the heat equation: For a periodic function $u : [0, 2\pi] \to \mathbb{R}$ consider a solution to the heat equation

$$\partial_t u = \partial_{xx} u$$

with initial data $u \in L^2([0, 2\pi], \mathbb{R})$ given by

$$u_0(x) = \sum_{k \in \mathbb{Z}} a_k e^{ikx}$$

where $\{a_k\}_{k \in \mathbb{Z}} \in \ell^2$. As we saw in Uecker (2009), this has solution

$$u(t, x) = \sum_{k \in \mathbb{Z}} e^{-t|k|^2} a_k e^{ikx}, \qquad t \geq 0.$$

Define the mapping $\hat{\sigma} : \ell^2 \times [0, \infty) \to \ell^2$ by

$$\hat{\sigma}(a, t) = \left\{ e^{-t|k|^2} a_k \right\}_{k \in \mathbb{Z}}, \tag{1}$$

where $a \in \ell^2$ and $t \geq 0$.

(a) Show that $\hat{\sigma}$ is a continuous semi-flow on ℓ^2. Is it a semi-group? Can it be extended to be a flow (and not just a semi-flow)?

(b) Let $\mathcal{F}$ denote the Fourier transform. Define the map $\sigma(u_0, t) : L^2 \times [0, \infty) \to L^2$ by

$$\sigma(u_0, t) = \mathcal{F}^{-1} \hat{\sigma}(\mathcal{F}(u_0), t). \tag{2}$$

Show that S is a continuous semi-group on L^2.

(c) Define a set $B = \{a \in \ell^2 : \|a\|_{\ell^2} \leq 1\}$. Compute the alpha limit set and the omega limit set of B. Is B an invariant set? A positively invariant set? What are some points in B with a globally defined motion? Does B contain an invariant set?

2. Compact imbeddings of weighted spaces

When working with partial differential equations it is common to consider function spaces with varying degrees of regularity. The space continuously differentiable functions are a prototypical example:

$$C^s(\mathbb{T}^d, \mathbb{R}) := \left\{ f : \mathbb{T}^d \to \mathbb{R} : (\partial_x)^s f \text{ is continuous } \right\}, \qquad \|f\|_{C^s} := \sum_{r=0}^{s} \sup_{x \in \mathbb{T}^d} \left| f^{(r)}(x) \right| \tag{3}$$

where $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$. Sobolev spaces generalize this concept in the context of L^p spaces. When working with the discrete Fourier transform, one can work with weighted sequence spaces[1] such as

$$\ell^1_{s,d} := \left\{ \{a_n\}_{n\in\mathbb{Z}^d} : a_n \in \mathbb{C}, \|a\|_{\ell^1_{s,d}} < \infty \right\}, \qquad \|a\|_{\ell^1_{s,d}} := \sum_{n\in\mathbb{Z}^d} (1+|n|)^s |a_n|,$$

where for $n = (n_1, \ldots n_d) \in \mathbb{Z}^d$ we define $|n| = |n_1| + \cdots + |n_d|$. More generally, we may define a norm

$$\|a\|_{\ell^p_{s,d}} = \begin{cases} \left(\sum_{n\in\mathbb{Z}^d} (1+|n|^p)^s |a_n|^p\right)^{1/p} & \text{if } p \neq \infty \\ \sup_{n\in\mathbb{Z}^d} (1+|n|)^s |a_n| & \text{if } p = \infty \end{cases}$$

These spaces of varying degrees of regularity imbed inside of each other.

(a) Prove the following statements

  (i) The Fourier transform $\mathcal{F} : C^s(\mathbb{T}^d, \mathbb{R}) \to \ell^\infty_{s,d}$ is a continuous imbedding.

  (ii) The inverse Fourier transform $\mathcal{F}^{-1} : \ell^1_{s,d} \hookrightarrow C^s(\mathbb{T}^d, \mathbb{R})$ is a continuous imbedding.

  (iii) The inclusion $C^{s+1}(\mathbb{T}^d, \mathbb{R}) \hookrightarrow C^s(\mathbb{T}^d, \mathbb{R})$ is a compact imbedding.

  (iv) If $\epsilon > 0$ and $d = 1$ then the inclusion $\ell^p_{s+\epsilon,d} \hookrightarrow \ell^p_{s,d}$ is a compact imbedding. What about other values of d?

  (v) If $\delta > 1$ and $d = 1$ then the inclusion $\ell^\infty_{s+\delta,d} \hookrightarrow \ell^1_{s,d}$ is a compact imbedding. What about other values of d?

(b) Show that $\hat{\sigma}$ defined in the previous problem is a compact semi-flow on $\ell^2_{0,1}$. Find some t_0 so that $\hat{\sigma}$ is compact for $t > t_0$.

3. More examples of semi-flows:

For each property below, construct a semi-flow on $\ell^2 \equiv \ell^2_{0,1}$ with the given property.

(a) The semi-flow can be extended to be a flow.

(b) The semi-flow is nonlinear.

(c) The exists a set $\mathfrak{A} \subseteq \ell^2$ which is a global attractor for the semi-flow.

(d) The semi-flow is point dissipative.

NOTE: You don't have to come up with a different semi-flow for each property, nor do you need to come up with a single semi-flow satisfying every property.

4. Dynamics of the Galerkin Approximation: Fisher-KPP equation

[1] To my knowledge, the arrangement of subscripts and superscripts on ℓ isn't completely standardized in the literature.

Consider the Fisher-KPP equation with Neumann boundary conditions:

$$\partial_t u = \partial_{xx} u + \lambda u(1-u), \qquad\qquad 0 = \partial_x u(t,0) = \partial_x u(t,\pi). \tag{4}$$

Using the Galerkin method, we look for approximate solutions of the form

$$\bar{u}(t,x) = a_0(t) + 2\sum_{k=1}^{N} a_k(t)\cos(kx), \tag{5}$$

yielding an N+1 dimensional ODE (c.f. the reading and homework from Week 1).

(a) Fix $\lambda = 0.5$, and study the dynamics of the Galerkin approximation to Fisher's equation with $N = 1$ (a 2 dimensional ODE). For example: What are the invariant sets? Are there local attractors? Is there a global attractor?

NOTE: Just use numerics to make conjectures about the dynamics. Don't worry about constructing a mathematical proof. I would recommend Mathematica's `StreamPlot` function, but use whatever software you feel most comfortable with.

(b) Fix some random initial condition (a_0, a_1). Numerically compute the solution with this initial condition and plot it three different ways:

(i) Plot the graph of the solution given by the Galerkin approximation in (5).

(ii) Plot the trajectory through (a_0, a_1) in phase space, a graph with coordinate axes $\{a_i\}_{i=0}^N$.

(iii) Plot the solution's coordinates vs time, with the horizontal axis being time, and each a_i being plotted on the vertical axis.

(c) The philosophy behind the Galerkin/spectral approach is that as $N \to \infty$, the dynamics of the ODE and its invariant sets will converge to the dynamics and invariant sets of the PDE.[2]

Investigate this hypothesis for the Fisher-KPP equation. How do the dynamics in the $N = 1$ Galerkin approximation compare to the $N = 2$ case? What about much larger values of N?

NOTE: When dealing with the Galerkin approximation for large N, I would highly recommend defining the ODE using the FFT methods we discussed in Week 1. Also for large N the ODE is "stiff". For numerically solving stiff ODEs you should use an implicit method, like the one-step method from Uecker (2009), or MATLAB's `ode15s`.

(d) Repeat parts (a)-(c), but this time use the parameter $\lambda = 2$.

5. Dynamics of the Galerkin Approximation: Allen-Cahn equation

Consider the Allen-Cahn equation we discussed in Week 1:

$$\partial_t u = \nu \partial_{xx} u + u - u^3, \qquad\qquad 0 = \partial_x u(t,0) = \partial_x u(t,\pi). \tag{6}$$

Repeat the previous problem, but instead use the Galerkin approximation of the Allen-Cahn equation with $\nu = 0.5$ and $\nu = 2$.

[2] What "converge" means here is somewhat ambiguous. In later weeks we'll study how to give this a more precise definition.