Math 876: PDE Seminar 2022
Week 3: Attractors and C_0-Semigroups

February 8 & 10, 2022

General: This week we finish up attractors, looking at conditions that will guarantee the existence of a global attractor. These conditions are still fairly abstract and at this point it is not clear how to verify them for a specific system. For example, how does one go from a PDE to defining a semiflow? How does one show that a semiflow is κ-contracting or point dissipative?

In Chapter 3 we begin to answer these questions, starting with C_0-semigroups defined by linear evolutionary equations $\partial_t u = Au$. The finite dimensional analog of this is the matrix exponential e^{At}. The main, and quite nontrivial, difference though is that we wish to consider unbounded, densely defined operators A. In part to address this we introduce different notions of solutions (classical vs mild).

Required Reading: Sell & You §2.3.5 - 2.3.7, §3.0 - 3.5

Supplementary Reading: Fejér's proof of a continuous function whose Fourier series diverges at a point, in "Trigonometric Series Vol. 1" by Zygmund .

Important Concepts: Existence and robustness of attractors; C_0-semigroup; classical solutions and mild solutions.

Reading Questions: Email me at least 3 questions on the reading at least an hour before class on Tuesday.

Presentations:

Existence of Global Attractors (Tu): Tell us about the Existence Theorem 23.12 and sketch its proof.

An Illustrative Example (Tu): Walk us through the illustrative example in §3.2 of constructing the C_0-semigroup. The assumptions made on the operator A are somewhat restrictive; highlight where each property gets used. Can some of the hypotheses be relaxed? How would that affect the proof?

Problems:

FitzHugh-Nagumo: Consider the FitzHugh-Nagumo equation PDE:

$$\begin{aligned} \partial_t u &= \partial_{xx} u + f(u) - v \\ \partial_t v &= \delta \partial_{xx} v + \beta u - \gamma v \end{aligned} \tag{1}$$

taking $x \in [0, 2\pi L]$ with periodic boundary conditions. This PDE is a heuristic model of neurons and other excitable media, cf §5.1.3.

Typically one uses the function $f(u) = u - u^3/3$. However in this problem, we will be considering the linearization at 0. Throughout, let us fix $f(u) = u$.

(a) Let $X = L^2([0, 2\pi L], \mathbb{R}^2)$ and $Y = \ell^2_{sym} \times \ell^2_{sym}$ where

$$\ell^2_{sym} = \left\{ \{a_k\}_{k \in \mathbb{Z}} \in \ell^2 : a_{-k} = \bar{a}_k \right\}.$$

Let $\mathcal{F} : X \to Y$ denote the Fourier transform. Define a linear operator $A : Y \to Y$ so that

$$\partial_t c = Ac, \qquad c \in Y$$

corresponds to the PDE in (1).

(b) For parts (b)-(f), fix the parameters $\delta = 1/10$, $L = 10$, $\beta = 1/2$ and $\gamma = 1/4$.

Compute the eigenvalues and eigenvectors of A. (I recommend using *Mathematica*).

(c) Is A self-adjoint? Does A have a compact resolvant? What is $\mathcal{D}(A)$? Is A bounded below? i.e., is there an $a > 0$ such that $a < \inf_{u \in \mathcal{D}(A); c \neq 0} \|Ac\|/\|c\|$.

(d) Show that A is the infinitesimal generator of a C_0-semigroup $T(t)$, and describe the map $c \mapsto T(t)c$ for $c \in Y$.

(e) Is $T(t)$ a compact semigroup? Is it a κ-contracting semigroup?

(f) How many unstable eigenvalues does A have? (Be mindful of our boundary conditions.) What does this tell us about the dimension of the global attractor in the nonlinear FitzHugh-Nagumo PDE (supposing that it does exist)?

(g) Answer (b)-(f) again, but this time take $\delta = 0$.

Numerically computing the spectrum of an operator:

For $x, y \in \mathbb{C}^{2N+1}$ where $x = (x_{-N}, \dots, x_N)$, recall the discrete convolution $x * y \in \mathbb{R}^{2N+1}$ defined component-wise by

$$(x * y)_n := \sum_{\substack{k=-N \\ |n-k| \leq N}}^{N} x_{n-k} y_k.$$

If we have some symmetries, (such as when working with a real cosine series where $x_{-k} = \bar{x}_k$ and $x_k \in \mathbb{R}$) then we may want to work with a lower dimensional space. For $x, y \in \mathbb{R}^{N+1}$, the even discrete convolution $x *_e y \in \mathbb{R}^{N+1}$ is defined component-wise by

$$(x *_e y)_n := \sum_{\substack{k=-N \\ |n-k| \leq N}}^{N} x_{|n-k|} y_{|k|}$$

for $n = 0, \dots N$. Note that $x *_e y = y *_e x$.

(a) Fix a vector $b \in \mathbb{R}^{N+1}$ and define the map $B : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ by

$$Bx := b *_e x$$

Show that B is a linear operator and write B as a finite dimensional matrix.

(b) Define $b \in \mathbb{R}^{N+1}$ by

$$b_n := |n|e^{-|n|/2}, \qquad n = 0, \ldots, N \tag{2}$$

and define a linear operator $A : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ by

$$(Ax)_n = -n^2 x_n + (b *_e x)_n, \qquad x \in \mathbb{R}^{N+1};\ n = 0, \ldots, N$$

for $n = 0 \ldots N$.

Using computer software, write a matrix representation of A for arbitrary N. For $N = 10$, compute the eigenvalues and plot them. What can you say about the C_0-semigroup generated by A? Do the eigenvalues/eigenvectors seem to converge as $N \to \infty$?

(c) Define the function $w : [0, 2\pi] \to \mathbb{R}$ by

$$w(x) = 2 \sum_{k=1}^{\infty} b_k \cos(kx)$$

where the b_k are given by (2). Define the linear operator C by

$$[Cu](x) = \partial_{xx} u(x) + w(x)u(x),$$

where $u \in L^2([0, \pi])$ and $0 = \partial_x u(0) = \partial_x u(\pi)$. Using your results from (b) and the arguments from §3.2, what can you say about the C_0-semigroup generated by C?

Book problems: 21.2; 23.1; 23.3; 23.10; 31.2; 31.3;

Hints:

- 23.1: A hint for this is given in the commentary of Chapter 2.
- 23.3: I think this one is hard, but maybe you can find a more elegant proof. I'd suggest thinking about something like the semiflow generated by $u_t = u_{xx}$ defined on $L^\infty([0, 2\pi])$, and also recall that there are continuous functions whose Fourier series diverge at a point (cf Fejér's proof in the supplemental reading).
- 31.3 (2). Show that for $\rho > 0$ small enough, then the infinitesimal generator is given by $(T(\rho) - I)\left(\int_0^\rho T(s)ds\right)^{-1}$. The proof for this is in Pazy (1983)