Math 876: PDE Seminar 2022
Week 4: Sectorial operators and analytic semigroups

February 15 & 17, 2022

General: This week we look at a special class of C_0-semigroups, namely those which are analytic and generated by sectorial operators. This class of semigroups is very important in the study of reaction diffusion equations, and are more well behaved than a plain old C_0-semigroup. To prove the fundamental theorem of sectorial operators §3.7 we introduce interpolation spaces and fractional powers of operators. One motivation for this section are PDEs where there are spatial derivatives included in the nonlinearity. A simple example of this would be the viscous Burger's equation $\partial_t u = \nu \partial_{xx} u + \partial_x (u)^2$ and a complicated example would be the Navier-Stokes equation.

Required Reading: Sell & You: §3.6 - §3.7

Supplementary Reading: §3.6.2 - The Lax-Milgram theorem is important, but it is not the focus of what we are covering this week.

Important Concepts: analytic semigroup; sectorial operator; positive operator; fractional powers of operators; interpolation spaces; fundamental theorem of sectorial operators; continuity lemma.

Reading Questions: Email me at least 3 questions on the reading at least an hour before class on Tuesday.

Presentations:

Sectorial Operators (Tu): Tell us about sectorial operators, and explain Figure 3.1. Give some examples of operators which are/aren't sectorial. Sketch out the different characterizations given in the fundamental theorem of analytic semigroups (Theorem 36.2).

Interpolation spaces (Tu): Tell us about interpolation spaces, fractional powers of positive sectorial operators, and the relations between the two. Use ℓ^2_s as an example of interpolation spaces and fractional powers. The Fundamental Theorem of Sectorial Operators (Theorem 37.5) is likely the most important theorem of the week. Tell us a bit about items (2) and (3) of the theorem.

Continuity Lemma (Th): The Continuity lemma (Lemma 37.9) gives sufficient conditions for when $u \in L^2[0,t;H)$ is in fact continuous. The argument uses interpolation spaces, and leads us to the notion of weak solutions in Chapter 4. Tell us about this lemma and sketch its proof.

Problems:

Sectorial Operators: Consider the PDE below defined on $H = L^2([0,\pi],\mathbb{C})$ with Dirichlet-0 boundary conditions,

$$u_t = \underbrace{(1 + e^{i\theta}\partial_{xx})}_{-A_\theta} u$$

Calculate the values of $\theta \in [-\pi, \pi]$ for which:

(a) The linear operator $-A_\theta$ generates a C_0-semigroup.

(b) The map A_θ is a sectorial operator.

(c) The C_0-semigroup $(T_\theta(t), -A_\theta)$ is a differentiable semigroup.

Compactness and Sectorial Operators

(a) Construct a positive linear operator A such that e^{-At} is a compact C_0-semigroup, yet A is not sectorial.

(b) Construct a positive sectorial operator A such that e^{-At} is not compact.

Interpolation spaces: Recall our definition of $\ell^p_s \equiv \ell^p_{s,1}$ from Week 2, where

$$\ell^p_s := \{\{a_n\}_{n\in\mathbb{Z}} : a_n \in \mathbb{C}, \|a\|_{\ell^p_s} < \infty\}$$

and we define the norm

$$\|a\|_{\ell^p_{s,d}} = \begin{cases} \left(\sum_{n\in\mathbb{Z}}(1+|n|^p)^s|a_n|^p\right)^{1/p} & \text{if } p \neq \infty \\ \sup_{n\in\mathbb{Z}}(1+|n|)^s|a_n| & \text{if } p = \infty. \end{cases}$$

Note that $(\ell^2_s)^* \cong \ell^2_{-s}$ and $(\ell^1_s)^* \cong \ell^\infty_{-s}$.

(a) For $p = 1, 2, \infty$, show that ℓ^p_s is a family of interpolation spaces for $s \in \mathbb{R}$.

(b) Motivated from the PDE $u_t = -(1+\partial_{xx})^2 u$, consider the densely defined linear operator A on ℓ^p_0 given by

$$(Ac)_k = (1 - 2k^2 + k^4)c_k, \qquad c \in \ell^p_0.$$

Show that A is a sectorial operator on ℓ^p_0 for $p = 1, 2, \infty$.

(c) Is A a positive sectorial operator? If so, describe the map A^α for $\alpha \in \mathbb{R}$.

(d) Show that the spaces $V^{2\alpha} = \mathcal{D}(A^\alpha)$, the fractional power spaces generated by the sectorial operator A, are isomorphic to $\ell^p_{4\alpha}$, for $p = 1, 2, \infty$.

Fundamental Theorem of Sectorial Operators: Consider the PDE below defined on $L^2([0,\pi],\mathbb{R})$ with Dirichlet-0 boundary conditions,

$$u_t = \underbrace{(-2+\partial_{xx})}_{-\tilde{A}} u$$

and define the Hilbert spaces

$$\ell^2_{s,Dir} = \{c \in \ell^2_s : Re(c_k) = 0, c_k = c^*_{-k}\}.$$

(a) Define a linear operator A on V^0 which is conjugate to the operator $\tilde{A}$; ie $Ac = \mathcal{F}\tilde{A}\mathcal{F}^{-1}c$ where $\mathcal{F}$ is the Fourier transform.

(b) Show that A is a positive sectorial operator, and then calculate A^α for $\alpha > 0$.

(c) For which s is $\ell^2_{s,Dir}$ isomorphic to $V^{2\alpha} = \mathcal{D}(A^\alpha)$ with $V^0 = \ell^2_{0,Dir}$?

(d) For the operator A, calculate the constants M_r , K_α and C_r from items (2), (3) and (4) from Theorem 37.5.